PDEs and Geometric Measure Theory
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Harmonic measure for low dimensional sets

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The following are equivalent (co-dimension 1):

Geometry:

(G1) E is uniformly rectifiable

(G2) P. Jones β, X. Tolsa α coefficient characterizations

Analysis:

(H1) all singular integral operators are bounded in $L^2(E)$

(H2) usual square function estimates for the Cauchy kernel

(H3) the Riesz transform is bounded in $L^2(E)$

PDEs:

(P1) harmonic measure ω is A^∞ (absolutely continuous) w.r.t. the Lebesgue measure (but this, and only this, requires some a priori topology!)

(P2) all bounded solutions satisfy Carleson measure estimates

(P3) all bounded solutions are ε-approximable

(P4) uniform square function/non-tan. max function estimates

The following are equivalent (co-dimension bigger than 1):

Geometry:

(G1) E is uniformly rectifiable

(G2) P. Jones β, X. Tolsa α coefficient characterizations

Analysis:

(H1) all singular integral operators are bounded in $L^2(E)$

PDEs:

NEW IDEAS: (non)-Harmonic measure and ADR black holes

Harmonic measure

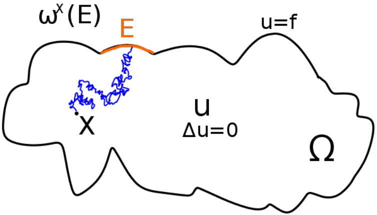
- for $E \subset \partial\Omega$, $X \in \Omega$, $\omega^X(E)$ is a solution to

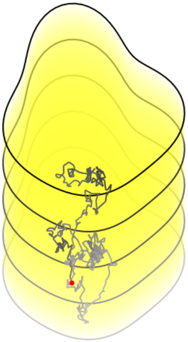
$$-\Delta u = 0 \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = \mathbf{1}_E$$

evaluated at point X, that is, $u(X)$.

- $\omega^X(E)$ is the probability for a Brownian motion starting at $X \in \Omega$ to exit through the set $E \subset \partial\Omega$

- the solution to
$-\Delta u = 0 \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = f$
is realized as $u(X) = \int_{\partial\Omega} f \, d\omega^X$





(pictures of M. Badger)

Harmonic measure

Key issues

- What is the dimension and the structure of the support of ω?
- Is ω absolutely continuous with respect to Lebesgue measure?

A^∞ condition (quantitative abs continuity):
$\forall\, Q \subseteq \partial\Omega$ and every Borel set $F \subset Q$, we have

$$\omega^{X_Q}(F) \le C \left(\frac{|F|}{|Q|}\right)^\theta \omega^{X_Q}(Q),$$

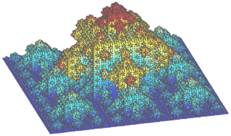
where X_Q is the "corkscrew point" relative to Q.
In other words, we want Brownian travelers to "see" portions of the boundary proportionally to their Lebesgue size. That is, nothing is shielded and nothing receives unfair attention.

You can guess that dimension and connectivity will be in the center of attention

Dimension of harmonic measure

- Carleson, 1973: in $\mathbb{R}^2$, for a simply connected domain bounded by a continuum, $dim\,\omega > 1/2 + \varepsilon$
- Makarov, 1985: in $\mathbb{R}^2$, for a simply connected domain bounded by a continuum, $dim\,\omega = 1$
- Jones-Wolff, 1988: in $\mathbb{R}^2$, for any planar domain (no connectivity), $dim\,\omega \leq 1$

- Bourgain, 1987: in $\mathbb{R}^n$, $n \geq 2$, $dim\,\omega < n$
- Wolff, 1991: but even for connected domains we can have $dim\,\omega > n - 1$ (Wolff snowflake)



(Filoche et al., PNAS 2008)

Somewhere between 2 and 3 there is a number giving the dimension of harmonic measure in $\mathbb{R}^3$...

Remark:

- topology matters: connectivity, continuum
- our knowledge is $n > 2$ is notoriously incomplete: $dim\,\omega$ somewhere strictly between $n - 1$ and n

Structure of the support of harmonic measure, co-dim 1

Let's say that $0 < H^{n-1}(E) < \infty$, $\omega \approx \sigma$. What do we know about E? Can every set of dimension $n-1$ host the harmonic measure?

Looking towards absolute continuity of ω w.r.t. σ

A closed set E is d-Ahlfors-David regular (ADR) if the measure of E is any ball $B(x, r)$,

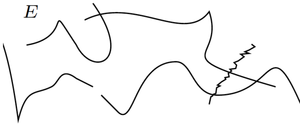
$$H^d(B(x,r) \cap E) \approx r^d$$

E is uniformly d-dimensional (very far from regularity in the sense of smoothness!)

- Part I: $d = n - 1$
- Part II: $d < n - 1$ (possibly fractional)

A set $E \subset \mathbb{R}^n$ is rectifiable if it can be covered by a countable union of Lipschitz graphs, modulo a set of measure zero

A set E is uniformly rectifiable if in any $B(x, r)$ 1% of E lies on a Lipschitz image, with relevant constants uniform in x, r



When ω is absolutely continuous w.r.t. H^{n-1}? (A^∞)

Dimension 2:

- F.&M.Riesz, 1916: $\Omega \subset \mathbb{R}^2$, simply connected, rectifiable
- Lavrent'ev, 1936: quantifiable analogue
- Bishop, Jones, 1990: "local F. & M. Riesz" $\Omega \subset \mathbb{R}^2$, $\partial\Omega$ on a continuum, $E \subset \partial\Omega$ is rectifiable

Connectivity is important:

- Bishop, Jones, 1990: counterexample
  $\partial\Omega$ is rectifiable, yet ω is singular w.r.t. σ

Higher dimension:

- Dahlberg, 1977: Lipschitz domain
- David, Jerison; Semmes 1990; Badger, 2012, NTA domain
- Hofmann, Martell, 2013; Azzam, Hofmann, Martell, Nyström, Toro, 2014; UR+1-sided NTA

Bottom line:

- Unif Rectifiability + some connectivity $\Longrightarrow \omega \in A^\infty$
  Unif Rectifiability $\not\Longrightarrow \omega \in A^\infty$

Structure of harmonic measure, codim 1: PDE $\rightarrow$ geom

"Free boundary problem":

Theorem (Azzam, Hofmann, Martell, S.M., Mourgoglou, Tolsa, Volberg, 2016)

For any open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, any $E \subset \partial\Omega$, $0 < H^{n-1}(E) < \infty$, if ω is abs continuous w.r.t. H^{n-1} then $\omega|_E$ is rectifiable.

"Free boundary problem":
necessary conditions for absolute continuity of ω

- Kenig-Toro'1997–2003: if $\partial\Omega$ is Reifenberg flat, $k = \frac{d\omega}{d\sigma}$, $\log k \in VMO \iff \nu \in VMO$
- Hofmann-Martell'2015: if $\partial\Omega$ is Ahlfors-David regular and satisfies interior cork-screw condition, $\omega \in A^\infty \implies \partial\Omega$ is uniformly rectifiable

Theorem (Azzam, Hofmann, Martell, S.M., Mourgoglou, Tolsa, Volberg, 2016)

For any open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, any $E \subset \partial\Omega$, $0 < H^{n-1}(E) < \infty$, if ω is abs continuous w.r.t. H^{n-1} then $\omega|_E$ is rectifiable.

Structure of harmonic measure, codim 1: PDE $\to$ geom

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- 1st full converse to F. & M. Riesz
- no dimension restriction, no topological restriction, no connectivity restriction, local/global...
- uses David-Semmes Conjecture (now Nazarov-Tolsa-Volberg'2014 theorem): finiteness of the Riesz transform

$$R\mu(x) = \int_E \frac{x-y}{|x-y|^{n+1}}\, d\mu(y)$$

implies rectifiability.
$n = 1$: Melnikov, Verdera, 1990's; David-Léger, 1999

Structure of harmonic measure, codim 1: PDE $\to$ geom

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- Nazarov-Tolsa-Volberg'2014 theorem:
  bounds on the Riesz transform imply rectifiability.

$$R\mu(x) = \int_E \nabla \mathcal{E}(x-y)\, d\mu(y)$$

- $$\nabla G(x,y) = \nabla \mathcal{E}(x-y) - \int_{\partial\Omega} \nabla \mathcal{E}(x-z)\, d\omega^y(z)$$

  Here $\mathcal{E}(x) = c_n |x|^{1-n}$ is the fundamental solution for $-\Delta$.
- Apply the Riesz transform characterization with $d\mu = d\omega$!

Necessary and sufficient

Theorem (Azzam, Hofmann, Martell, Mourgoglou, Tolsa, 2018)

Let $\Omega \subset \mathbb{R}^n$ be an open set with $n-1$-AD-regular boundary. The weak-A^∞ condition for harmonic measure holds if and only if $\partial\Omega$ is uniformly $n-1$-rectifiable and the weak local John condition is satisfied.

What about higher co-dimension?

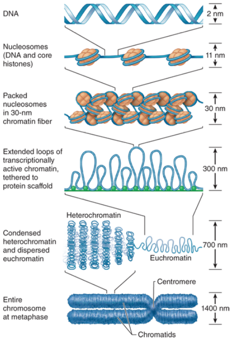
$0 < H^d(E) < \infty$, $E \subset \mathbb{R}^n$, $d < n-1$ (integer)

Think: a curve in $\mathbb{R}^3$, DNA, "big data"

- harmonic functions do not "see" sets of higher co-dimension
- harmonic measure makes no sense

Need "harmonic functions"

- how to not miss small E?
- you need to attract Brownian travelers!



Source: Anthony L. Mescher: Junqueira's Basic Histology, 14th Edition.
www.accessmedicine.com

Idea

build harmonic measure from $L = -\operatorname{div} \frac{1}{\operatorname{dist}_E^{n-d-1}} \nabla$

NB: $L = -\Delta$ when $n = d+1$

What about higher co-dimension?

$0 < H^d(E) < \infty$, $E \subset \mathbb{R}^n$, $d < n-1$ (integer)
Think: a curve in $\mathbb{R}^3$

Idea

build harmonic measure from $L = \operatorname{div} \frac{1}{\operatorname{dist}_E^{n-d-1}} \nabla$

Question (G. David, S.M.)

Prove or disprove that our harmonic measure is absolutely continuous w.r.t. Lebesgue measure (in fact, A^∞) if and only if the set is uniformly rectifiable (of dimension $d < n-1$)

- topology is now a friend: note the difference with $d = n-1$ when Bishop-Jones give counterexamples to $\omega \in A^\infty$ on uniformly rectifiable sets. We do not expect topological difficulties of access of $d = n-1$ case.
- equation is now an enemy or at least a mystery...

There are two big parts of what is coming:

- Basic elliptic theory: traces, extensions, weak solutions, Poincare, Caccioppoli, Harnack, maximum principle...
  There are many forerunners and alternative approaches: for degenerate PDEs: Fabes-Kenig-Serapioni; Jerison-Kenig, and others; for p-Laplacian and other quasilinear PDEs: Lewis, Vogel, Nystrom, and others; for higher order PDEs: S.M., Maz'ya and others; function spaces: Maz'ya, Jonsson-Wallin.
  We did it from scratch, in the full generality of

$$L = -\operatorname{div} \frac{1}{\operatorname{dist}_E^{n-d-1}} A(x)\nabla,$$

  where A is an elliptic matrix and for all ADR sets, of possibly fractional dimension, but all this is fairly predictable.

Our big goal was different: find one "elliptic" operator (one A) for which harmonic measure is absolutely continuous with respect to the Hausdorff measure on lower dimensional sets. Ours is the first result of this type, even for a Lipschitz curve.

Caffarelli-Silvestre extension

A big portion of the forthcoming discussion applies to

$$L = -\operatorname{div} \frac{1}{\operatorname{dist}_E^{n-d-1+\beta}} A(x)\nabla,$$

where $\beta \in (0,1)$.

In the particular case $A = I$, $d = n-1$, $E = \mathbb{R}^d = \mathbb{R}^{n-1}$, this is the Caffarelli-Silvestre extension operator, and the corresponding Dirichlet-to-Neumann operator on E is the fractional Laplacian $(-\Delta)^\gamma$, $2\gamma = 1+\beta$.

Thus, our results are likely to further yield a new fractional Laplacian, on extremely rough sets, including lower dimensional ones – to be discussed further.

Theorem I = Theorems 1-25...

Γ is d - Ahlfors-David regular of some dimension $d < n-1$:

$$C_0^{-1} r^d \le H^d(\Gamma \cap B(x,r)) \le C_0 r^d \quad \text{for } x \in \Gamma \text{ and } r > 0.$$

"quantifiably d-dimensional" (d possibly not integer for now)

Define a divergence form operator $L = -\mathrm{div} A \nabla$ on $\Omega = \mathbb{R}^n \setminus \Gamma$ with the ellipticity condition of a different homogeneity:

$$\mathrm{dist}(x,\Gamma)^{n-d-1} A(x)\xi \cdot \zeta \le C_1 |\xi|\,|\zeta| \quad \text{for } x \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n,$$

$$\mathrm{dist}(x,\Gamma)^{n-d-1} A(x)\xi \cdot \xi \ge C_1^{-1} |\xi|^2 \quad \text{for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

This yields a comprehensive elliptic theory:

Theorem I = Theorems 1-25...

Γ is $d-ADR$, $d<n-1$, $\Omega=\mathbb{R}^n\setminus\Gamma$, $L=-\mathrm{div}A\nabla$

Set $\delta(x)=\mathrm{dist}(x,\Gamma)$, $w(x)=\delta(x)^{-n+d+1}$,
and $W=\dot{W}^{1,2}_w(\Omega)$ the weighted Sobolev space with

$$||u||_W=\Big\{\int_\Omega|\nabla u(x)|^2w(x)dx\Big\}^{1/2}$$

and $H=\dot{H}^{1/2}(\Gamma)$ with

$$||g||_H^2=\int_\Gamma\int_\Gamma\frac{|g(x)-g(y)|^2}{|x-y|^{d+1}}d\sigma(x)d\sigma(y)$$

- Trace/Extension theorems:
  we construct two bounded linear operators $T:W\to H$ (a trace operator) and $E:H\to W$ (an extension operator), such that $T\circ E=I_H$.
- Existence and uniqueness of weak solutions for $g\in H$ there is a unique weak solution $u\in W$ of $Lu=0$ such that $Tu=g$.

Theorem I = Theorems 1-25...

Γ is $d-ADR$, $d<n-1$, $\Omega=\mathbb{R}^n\setminus\Gamma$, $L=-\mathrm{div}A\nabla$

- quantitative boundedness of solutions (Moser estimates), interior and at the boundary
- quantitative Hölder continuity (De Giorgi-Nash estimates), interior and at the boundary

Theorem I = Theorems 1-25...

Γ is $d - ADR$, $d < n - 1$, $\Omega = \mathbb{R}^n \setminus \Gamma$, $L = -\mathrm{div} A \nabla$

- quantitative boundedness of solutions (Moser estimates), interior and at the boundary
- quantitative Hölder continuity (De Giorgi-Nash estimates), interior and at the boundary

Think about it: in co-dim one continuity (and even more so Hölder continuity) at the boundary requires fatness of the complement of the domain: a cusp can be bad.
Recall Wiener criterion: fatness, massiveness of the complement (capacity estimates) is necessary and sufficient for continuity.
Further Hölder continuity requires almost being a Lipschitz domain.

Here we have tiny complement: $\Omega = \mathbb{R}^n \setminus \Gamma$, e.g., a complement of a curve in $\mathbb{R}^3$ and yet it is perceived as very massive by solutions

Theorem I = Theorems 1-25...

Γ is $d - ADR$, $d < n-1$, $\Omega = \mathbb{R}^n \setminus \Gamma$, $L = -\mathrm{div} A \nabla$

- we always have Poincaré inequality:

$$\fint_{B(x,r)} |u(y)| dy \leq C r^{-d} \int_{B(x,r)} |\nabla u(y)| w(y) dy$$

  for $u \in W$, $x \in \Gamma$, and $r > 0$ such that $Tu = 0$ on $\Gamma \cap B(x,r)$, as well as its interior analogue
- we always have Harnack chains (there is plenty of access)
- Harnack inequality

All this and the maximum principle yield the definition of the harmonic measure $\omega = \omega_L$ so that $u(x) = \int_\Gamma g d\omega^x$ is the value at x of the solution of the Dirichlet problem

$$Lu = 0 \text{ in } \Omega \text{ with } \mathrm{Tr}\, u = g \text{ on } \Gamma$$

- doubling
- comparison principle

Theorem I = Theorems 1-25...

Lemma (Nondegeneracy)

For any $x \in E$, $r > 0$,

$$\omega^{A_{\Delta_r(x)}}(\Delta_r(x)) \geq C.$$

Lemma (Doubling)

For any $x \in E$, $r > 0$, *for any* $Y \in \Omega \setminus B_{2r}(x, 0)$,

$$\omega^Y(\Delta_{2r}(x)) \leq C\,\omega^Y(\Delta_r(x)).$$

Lemma (Change of Pole)

For any $x \in E$, $r > 0$, *for any* $Y \in \Omega \setminus B_{2r}(x, 0)$, *and any ball* $\Delta' \subset \Delta := \Delta_r(x)$ *we have*

$$\omega^{A_\Delta}(\Delta') \approx \frac{\omega^Y(\Delta')}{\omega^Y(\Delta)}.$$

Theorem I = Theorems 1-25...

All this and the maximum principle yield the definition of the harmonic measure $\omega = \omega_L$ so that $u(x) = \int_\Gamma g d\omega^x$ is the value at x of the solution of the Dirichlet problem

$$Lu = 0 \text{ in } \Omega \text{ with } \operatorname{Tr} u = g \text{ on } \Gamma$$

- non-degeneracy
- doubling
- change of pole
- comparison principle
- definition and estimates for the Green function
- connection between the harmonic measure and the gradient of the Green function

Theorem (G. David, J. Feneuil, S.M., 2017)

Γ is a d-dimensional Lipschitz graph with a small Lipschitz constant, $d < n-1$, integer, $\Omega = \mathbb{R}^n \setminus \Gamma$, $L = -\operatorname{div} D(x)^{d+1-n}\nabla$. Then $\omega \in A^\infty(\sigma)$.

Here, D is equivalent to the distance:

$$c_1 \operatorname{dist}(x,\Gamma) \leq D(x) \leq c_2 \operatorname{dist}(x,\Gamma)$$

but more intricately built: $D_\alpha(X) = \left\{ \int_\Gamma |X-y|^{-d-\alpha} d\sigma(y) \right\}^{-1/\alpha}$,
$D(x) = \operatorname{dist}(x,\Gamma)$ would not work except for $n=3$
for similar reasons as β_∞ coefficients.
You have to gently guide your Brownian travelers.

Conjecture, G. David, S.M.

Γ is a d-dimensional uniformly rectifiable set, $d < n-1$, integer, $\Omega = \mathbb{R}^n \setminus \Gamma$, $L = -\operatorname{div} D(x)^{d+1-n}\nabla$. Then $\omega \in A^\infty(\sigma)$.

No topological assumptions! – magic...

Is there a converse? More magic

joint with G. David, M. Engelstein

We discussed that having $\omega \in A^\infty$ on all uniformly rectifiable sets is magic (compared to co-dim 1). But there is more...

For $n > d - 2$ (at least co-dimension 2) and a particular value of a parameter $\alpha = n - d - 2$ our distance

$$D_\alpha(X) = \left\{ \int_\Gamma |X - y|^{-d-\alpha} d\sigma(y) \right\}^{-1/\alpha}$$

is a solution vanishing at the boundary: $LD_\alpha = 0$, and hence,

Our distance D is exactly the Green function with a pole at infty!

Think: how many formulas for Green function do you know?! - this is explicit on arbitrary domains

The only analogue for co-dimension 1 is x_{n+1} on $\mathbb{R}^{n+1}_+$ but that is a unique lucky strike, the situation is much murkier in more general domains.

Is there a converse? More magic

For $n > d - 2$ (at least co-dimension 2) and a particular value of a parameter $\alpha = n - d - 2$ our distance

$$D_\alpha(X) = \left\{ \int_\Gamma |X - y|^{-d-\alpha} d\sigma(y) \right\}^{-1/\alpha}$$

is a solution vanishing at the boundary: $LD_\alpha = 0$, and hence, distance is exactly the Green function with a pole at infty!
From here it is possible to prove that in the exceptional case $\alpha = n - d - 2$ we have $\omega = \omega_{L^\alpha} \in A^\infty$ on all Ahlfors regular sets! (Regardless of uniform rectifiability), $\frac{\partial \omega}{\partial \sigma} \approx 1$.

It could be that this is exceptional and for all other $\alpha > 0$ we have

$$\omega = \omega_{L^\alpha} \in A^\infty \quad \Longleftrightarrow \quad \text{uniform rectifiability}$$

Why do we have a hope?

Theorem (G. David, M. Engelstein, S.M., 2018)

Let $n > d+1$, and let Γ be d-ADR. Then Γ is rectifiable if and only if n.t. limits of ∇D_α exist a.e. on E

To compare: when $n = d+1$, Γ is rectifiable iff p.v. limits of $R1$ exist a.e. on E (Tolsa 2008) (or iff R is bounded on $L^2(E)$)

- formally taking $\alpha = n - d - 2$ and $d = n - 1$, we arrive at the the usual Riesz transform $R1$. It is, however, not allowed by our higher co-dim method.
- for $d < n - 1$ and any $\alpha > 0$ $\nabla D_\alpha =: \nabla D_\alpha 1$ yields a new rescaled hypersingular operator:

$$\nabla D_\alpha f(X) = \frac{d+\alpha}{\alpha} \int_\Gamma \frac{(X-y)}{|X-y|^{d+\alpha+2}} f(y) d\sigma_y \left(\int_\Gamma |X-y|^{-d-\alpha} d\sigma_y \right)^{-\frac{1}{\alpha}-1}$$

- Contrary to the usual Riesz transform, this operator is always bounded in $L^2(E)$, on all ADR sets, in the sense of $\sup_{\varepsilon>0}$ over ε-truncations, and yet the existence of $\lim_{\varepsilon\to 0}$ is necessary and sufficient for rectifiability (NB: n.t., NOT p.v.)

And finally, magic square (function)

In co-dimension 1 one of the major characterizations of uniform rectifiability at heart of many, many results connecting it to PDEs is the usual square function estimate (USFE), David-Semmes'94:

Let $Sf(x) := C_n \int_\Gamma |x-y|^{2-n} f(y)\, d\sigma_y$. Then Γ is $n-1$ uniformly rectifiable if and only if $|\nabla^2 S1(x)|^2 \operatorname{dist}(x)$ is a Carleson measure.

That is, for all balls

$$\iint_B |\nabla^2 S1(x)|^2 \operatorname{dist}(x)\, dx \le Cr^n.$$

What is so remarkable about it?

- this is one operator characterizing uniform rectifiability (nowadays we also know that the Riesz transform does, but this was way before - 1994)
- Sf is the harmonic single layer potential
- the kernel $\mathcal{E}(x,y) = C_n |x-y|^{2-n}$ is the harmonic fundamental solution
- $\nabla Sf = Rf$ is the Riesz transform

And finally, magic square (function)

Theorem (G. David, S. Semmes, 1994)

Let $Sf(x) := C_n \int_\Gamma |x-y|^{2-n} f(y)\, d\sigma_y$. *Then* Γ *is* $n-1$ *uniformly rectifiable if and only if* $|\nabla^2 S1(x)|^2 \,\mathrm{dist}(x)$ *is a Carleson measure.*

What about the higher co-dimension?

- there is no global fundamental solution analogue of $\mathcal{E}(x,y) = C_n |x-y|^{2-n}$ $(-\Delta_x \mathcal{E}(x-y) = \delta_x(y)$ in $\mathbb{R}^n)$
- the operator $L = -\,\mathrm{div}\, D(x)^{-n+d+1}\nabla$ does not make sense when there is no domain (D is the distance to the boundary of the domain)
- there is no single layer potential, no Riesz transform; the analogue with $Sf(x) := C_d \int_\Gamma |x-y|^{1-d} f(y)\, d\sigma_y$ fails

But there is

Theorem (G. David, M. Engelstein, S.M., 2018)

Let $n > d+1$, *and let* Γ *be* d*-ADR. Then* Γ *is uniformly rectifiable if and only if* $|\nabla |\nabla D(x)|^2|\,\mathrm{dist}(x)$ *is a Carleson measure.*

And finally, magic square (function)

Theorem (G. David, M. Engelstein, S.M., 2018)

Let $n > d + 1$, and let Γ be d-ADR. Then Γ is uniformly rectifiable if and only if $|\nabla|\nabla D(x)|^2| \operatorname{dist}(x)$ is a Carleson measure.

- in the special case $\alpha = n - d - 2$ this connects back to the solutions (because D_α is then a solution)
- in fact, if we choose a special case $\alpha = n - d - 2$ and then formally let $n = d + 1$, we will see exactly the usual USFE from co-dimension 1 (but these choices are incompatible)
- it is important to take $|\nabla|\nabla D(x)|^2|$ and not $\nabla^2 D$ (a more naive analogue of co-dimension 1) because the latter is not Carleson: our radial direction is very different from angular direction.

Theorem II

Theorem (G. David, J. Feneuil, S.M., 2017)

Γ is a d-dimensional Lipschitz graph with a small Lipschitz constant, $d < n-1$, integer, $\Omega = \mathbb{R}^n \setminus \Gamma$, $L = -\operatorname{div} D(x)^{d+1-n}\nabla$. Then $\omega \in A^\infty(\sigma)$.

How would you prove this?

- Make a change of variables to get from
  $L = -\operatorname{div} D(x)^{d+1-n}\nabla$ on $\mathbb{R}^n \setminus \Gamma$ to
  $L = -\operatorname{div} \widetilde{D}(x)^{d+1-n}A(x)\nabla$ on $\mathbb{R}^n \setminus \mathbb{R}^d$
  (A is a matrix depending on the change of variables)
- treat $L = -\operatorname{div} \widetilde{D}(x)^{d+1-n}A(x)\nabla$ on $\mathbb{R}^n \setminus \mathbb{R}^d$

The challenge: create a change of variables for which $L = -\operatorname{div} \widetilde{D}(x)^{d+1-n}A(x)\nabla$ gives rise to an absolutely continuous elliptic measure (not all A's are good!)

Theorem II

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Two main approaches in co-dimension 1:

- $t \mapsto t + \varphi(x)$ – requires t-independence of the matrix of coefficients. The distance, however, is not t-independent, in fact, $|t|$ is its major part!
- $t \mapsto ct + P_t * \varphi(x)$
  $t|\nabla A|^2$ is a Carleson measure – too much torsion in t in higher co-dimension.

Our approach:

- $\rho(x,t) = (x, P_{|t|} * \varphi(x)) + h(x,t)R_{x,|t|}(0,t)$ where $R_{x,r}$ is a linear isometry of $\mathbb{R}^n$ mapping $\mathbb{R}^d$ to the d-plane $P(x,r)$ tangent to a smoothened Γ_r and h is a positive (subtle, slowly varying, and vital) "corrector".

Conjecture, G. David, S.M.

Γ is a d-dimensional uniformly rectifiable set, $d < n-1$, integer, $\Omega = \mathbb{R}^n \setminus \Gamma$, $L = -\operatorname{div} D(x)^{d+1-n}\nabla$. Then $\omega \in A^\infty(\sigma)$.

The major steps:

- Prove the result on a Lipschitz graph with a small Lipschitz constant (G. David, J. Feneuil, S.M., 2017).
- Bite out from a uniformly rectifiable set bad parts and replace them with something better, which feels like a Lipschitz graph with a small Lipschitz constant.
  Corresponding to any relatively flat saw-tooth region there is a replacement set which is Reifenberg flat and which can be parametrized as a small Lipschitz graph.
- Compare harmonic measure of a replacement set to the original one.
- Use some (upgraded) extrapolation ideas to manage all scales and iterate flatness by induction until it is not flat any more.

Inspirations

- (G. David, J. Feneuil, S.M., 2017)
  Γ is a d-dimensional Lipschitz graph with a small Lipschitz constant. Then $\omega \in A^\infty(\sigma)$.

  This result for harmonic functions on usual Lipschitz domains (with co-dimension 1 boundary) is due to Dahlberg, 1977.

  Our approach is closer to later developments, in particular, due to Kenig-Koch (Kircheim)-Pipher-Toro in 2000, 2014, which brought together more systematically Carleson measure bounds, square functions, absolute continuity.

  We pioneer a new "change of variables" (even for co-dim 1) which "guides" the Brownian travelers and changes the game.

  The full version of this result for our generality of elliptic operators is new even in co-dimension 1.

Inspirations

- Corresponding to any relatively flat saw-tooth region there is a replacement set which is Reifenberg flat and which can be parametrized as a Lipschitz graph.

  This uses some ideas of David-Toro Hölder parametrization of Reifenberg flat sets.

  However, we need to carefully, "seamlessly", execute the replacement.

  And we need the parametrization to be very well-controlled, in particular, to be Lipschitz, with a very careful control of Lipschitz constants (depending on parameters of the saw-tooth).

Inspirations

- Compare harmonic measure of a replacement set to the original one.
- Use some (upgraded) extrapolation ideas to iterate flatness until it is not flat any more.

This goes back to Dahlberg-Jerison-Kenig 1984 (work through the saw-tooth), extrapolation (Carleson-Garnett, 1975; Lewis-Murray 1995), but really we use a much later and cleaner version perfected in the work of Hofmann-Martell, 2014. However:

- Our extrapolation is more delicate: we can only afford to work with very particular sawtooth regions built above.
- We are in higher co-dimension.
- Saw-tooth per se is a wild domain of mixed dimension, we never can work on it directly.
- Working instead with a replacement is a pain: how far is the impact of a replacement felt? Our operators are non-local!