Isoperimetric inequalities for spectrum of Laplacian on surfaces

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Let M be a closed surface and g be a Riemannian metric on M. Let us consider the Laplace-Beltrami operator

$$\Delta f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right),$$

and its eigenvalues

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \lambda_3(M,g) \leqslant \dots \tag{1}$$

Let us denote by $m(M,g,\lambda_i)$ the multiplicity of the eigenvalue $\lambda_i(M,g)$, Let M be a closed surface and g be a Riemannian metric on M. Let us consider the Laplace-Beltrami operator

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Let us denote by $m(M,g,\lambda_i)$ the multiplicity of the eigenvalue $\lambda_i(M,g)$, Let us consider a functional

$$\bar{\lambda}_i(M,g) = \lambda_i(M,g) Area(M,g),$$

where $Area(M,g)$ is the area of M with respect to metric g.

Isoperimetric inequalities for Λ_i

Hersch (1970)

$$\bar{\lambda}_1(\mathbb{S}^2, g) \leq 8\pi.$$

Neumann spectrum

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary,
$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$ be the eigenvalues of the Neumann problem in Ω.

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4. Polya conjecture:

$$\mu_k \leq 4\pi k / Area\, \Omega$$

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It follows that the functionals $\bar{\lambda}_i(M,g)$ are bounded from above and it is a natural question to find for a given compact surface M and number $i \in \mathbb{N}$ the quantity

$$\Lambda_i(M) = \sup_g \bar{\lambda}_i(M,g),$$

where the supremum is taken over the space of all Riemannian metrics g on M.

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$\Lambda_1(M)$ is also known for $\mathbb{T}^2$ (equilateral torus) N.N., $\mathbb{K}^2$ (Lawson torus), Jakobson, N.N., Polterovich; ElSoufi, Giacomini, Jazar, $\mathbb{T}^2 \# \mathbb{T}^2$ (Bolza surface): Jakobson, Levitin, N.N., Nigam, Polterovich; Nayatani, Shoda,

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Theorem (M. Karpukhin, N.N., A. Penskoi, I. Polterovich)

The following holds

$$\Lambda_k(\mathbb{S}^2) = 8\pi k,$$

for $k = 1, 2, \ldots$.

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Corollary

For the Neumann problem

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Theorem (N.N., A. Penskoi)

$$\Lambda_2(\mathbb{R}P^2) = 20\pi.$$

Multiplicity of eigenvalues

For eigenvalue λ_i of Δ on (M, g) denote by m_i its multiplicity

Cheng proved (1976)

$$m_i \leq C(i, \chi(M))$$

on $\mathbb{S}^2$

$$m_i \leq 2i + 1$$

Sharp bounds for m_1 are known for $\chi(M) \geq -3$:
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Theorem (Colin de Verdière)

$$\sup_g m_i \geq chr(M) - 1 = [\frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})] - 1$$

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Theorem (N.N., A.Penskoi)

On $\mathbb{R}P^2$ the inequality holds

$$m_2 \leq 5$$

Conjecture.

$$m_i \leq C(\chi(M))\sqrt{i}$$

Theorem (Courant Nodal Domain Theorem)

An eigenfunction corresponding to the eigenvalue λ_i has at most $i+1$ nodal domains.

Theorem (Bers, 1955)

Let (M, g) be a compact 2-dimensional closed Riemannian manifold and x_0 is a point on M. Then there exist its neighborhood chart U with coordinates $x = (x^1, x^2) \in U \subset \mathbb{R}^2$ centered at x_0 such that for any eigenfunction u of the Laplace-Beltrami operator on M there exists an integer $n \geqslant 0$ and a non-trivial homogeneous harmonic polynomial $P_n(x)$ of degree n on the Euclidean plane $\mathbb{R}^2$ such that $u(x) = P_n(x) + O(|x|^{n+1})$

Proposition Let u be an eigenfunction corresponding to the eigenvalue λ_i. Let x_j, $j = 1, \ldots, n$, be zeroes of u of order $m_j > 1$. Then

$$i + 1 \geqslant \chi(M) - n + \sum_{j=1}^{n} m_j.$$

Maximizing metrics and bubbling phenomena

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Theorem (N.N., 1996, El Soufi, Ilias, 2000)

Let (M, g) be a sufficiently regular Riemannian surface with a metric g maximizing (or extremalizing) $\bar{\lambda}_k$. Then (M, g) is isometric to a minimal submanifold Euclidian sphere.

Maximizing metrics and bubbling phenomena

Theorem. (N.N., Y.Sire, R.Petrides, 2010 - 2015) *Let* (M, g) *be a Riemannian surface. For any* $k \geqslant 1$ *and a sequence of metrics* $\{g_i'\}_{i\geqslant 1} \in [g]$ *of the form* $g_i' = \mu_i' g$ *such that*

$$\lim_{i\to\infty} \lambda_k(g_i') = \Lambda_k(M, [g])$$

there exists a subsequence of metrics $\{g_n\}_{n\geqslant 1} = \{g_{i_n}'\}_{n\geqslant 1} \in [g]$, *where* $g_n = \mu_n g$, *such that*

$$\lim_{n\to\infty} \lambda_k(g_n) = \Lambda_k(M, [g])$$

and a probability measure μ *such that*

$$\mu_n \rightharpoonup^* \mu \quad \textit{weakly in measure as } n \to +\infty.$$

Moreover, the following decomposition holds,

$$\mu = \mu_r + \mu_s$$

where μ_r *is a nonnegative* C^∞ *function and* μ_s *is the singular part given, if not trivial, by the formula*

$$\mu_s = \sum_{i=1}^{K} c_i \delta_{x_i}$$

for some $K \geqslant 1$, $c_i \geqslant 0$ *and some "bubbling points"* $x_i \in M$. *Furthermore, the number* K *satisfies the bound*

$$K \leqslant k - 1.$$

If we denote by U *the eigenspace of the Laplace-Beltrami operator on* $(M, \mu_r g)$ *associated to the eigenvalue* $\Lambda_k(M, [g])$, *then there exists a family of eigenvectors* $\{u_1, \ldots, u_l\} \subset U$ *such that the map*

$$\varphi = (u_1, \ldots, u_l) : M \to \mathbb{R}^l$$

is a harmonic immersion into the sphere $\mathbb{S}^{l-1}$.

Definition

Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $f : M \longrightarrow N$ is called harmonic if f is an extremal for the energy functional $E[f] = \int_M |df(x)|^2 \, dVol_g$.

Proposition A harmonic map $\mathbb{S}^2 \looparrowright \mathbb{S}^n$ is automatically conformal and hence minimal in the induced metric.

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Proposition A harmonic map $\mathbb{S}^2 \looparrowright \mathbb{S}^n$ is automatically conformal and hence minimal in the induced metric.

Corollary

For any $k \in \mathbb{N}$ there is a disconnected union $\Sigma = (\mathbb{S}^2, g_1) \cup \cdots \cup (\mathbb{S}^2, g_m)$, Area $\Sigma = 1$ such that $(\mathbb{S}^2, g_i)$, $i = 1 \ldots m$, are isometric to a minimal submanifolds of Euclidian spheres $\mathbb{S}^n_{r_i}$ and $\lambda_k(\Sigma) = \Lambda_k(\mathbb{S}^2)$.

Corollary

For any $k \in \mathbb{N}$ there is a union $\Sigma = (\mathbb{R}P^2, g_0) \cup (\mathbb{S}^2, g_1) \cup \cdots \cup (\mathbb{S}^2, g_m)$, Area $\Sigma = 1$ such that $(\mathbb{S}^2, g_i)$, $i = 1 \ldots m$, are isometric to a minimal submanifolds of Euclidian spheres $\mathbb{S}^n_{r_i}$ and $\lambda_k(\Sigma) = \Lambda_k(\mathbb{R}P^2)$.

Proposition Let $M = (\mathbb{R}P^2, g)$ be a smooth connected Riemannian surface with a metric g maximizing λ_2. Then either M is isometric to a minimal surface with branch points $F : \mathbb{R}P^2 \longrightarrow \mathbb{S}^n$, or M is a union $\Sigma = (\mathbb{R}P^2, g_0) \cup (\mathbb{S}^2, g_1)$, where $(\mathbb{R}P^2, g_0)$ and $(\mathbb{S}^2, g_1)$ are minimal submanifolds of Euclidean spheres.

Theorem (Calabi1967, Barbosa1975)

Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a harmonic immersion with branch points such that the image is not contained in a hyperplane. Then

(i) *the area of $\mathbb{S}^2$ with respect to the induced metric $(\mathbb{S}^2, F^*g)$ is an integer multiple of 4π;*

(ii) *n is even, $n = 2m$, and*

$$Area(\mathbb{S}^2, F^*g) \geq 2\pi m(m+1).$$

Definition

If $Area(\mathbb{S}^2, F^*g) = 4\pi d$, then we say that F is of harmonic degree d.

We obtain immediately a lower bound for the harmonic degree.
Proposition Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^{2m}$ be a harmonic immersion with branch points such that the image is not contained in a hyperplane. Then $d \geqslant \frac{m(m+1)}{2}$.

Proposition Let $M = (\mathbb{R}P^2, g)$ be a smooth connected Riemannian surface with a metric g maximizing λ_2. Then M is given by a harmonic immersion with branch points $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ (such that the image is not contained in a hyperplane) of harmonic degree $d > 3$.

Proposition Let $M = (\mathbb{R}P^2, g)$ be a smooth connected Riemannian surface with a metric g maximizing λ_2. Then M is given by a harmonic immersion with branch points $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ (such that the image is not contained in a hyperplane) of harmonic degree $d > 3$.

Definition (Penrose twistor map)

$$T : \mathbb{CP}^3 \longrightarrow \mathbb{HP}^1 \cong \mathbb{S}^4, \quad T([z_0 : z_1 : z_2 : z_3]) = [z_0 + z_1 j : z_2 + z_3 j].$$

Let z be a conformal parameter on $\mathbb{S}^2$.

Definition

Let us call a curve

$$f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3, \quad f(z) = [f_0(z) : f_1(z) : f_2(z) : f_3(z)],$$

horizontal if

$$f_1' f_2 - f_1 f_2' + f_3' f_4 - f_3 f_4' = 0.$$

Theorem (Bryant 1982)

For each harmonic immersion with branch points $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *there exist either holomorphic or antiholomorphic horizontal curve* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$, *such that* $T \circ f = F$,
For each (anti)holomorphic horizontal curve $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$ *the map* $F = T \circ f : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *is a harmonic immersion with branched points.*
If a harmonic immersion $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *has a holomorphic (antiholomorphic) horizontal curve* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$, *then* $A \circ F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *has an antiholomorphic (holomorphic) horizontal curve.*

Definition

An (anti)holomorphic horizontal curve f appearing in Bryant Theorem is called the lift of an harmonic immersion F.

Theorem (Bryant 1982)

Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ be a harmonic immersion with branched points of harmonic degree d with holomorphic lift $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$. Then $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$ is an algebraic curve of degree d.

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Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *be a harmonic immersion with branched points of harmonic degree* d *with holomorphic lift* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$*. Then* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$ *is an algebraic curve of degree* d*.*

Theorem (Bolton-Woodward 2001)

For a linearly full holomorphic horisontal curve in $\mathbb{CP}^3$ *of degree* d
if $d = 3$ *then* F *does not have neither branch points nor umbilics,*
if $d > 3$ *then* F *has at least one branch point or umbilic.*

Definition

Let (M, g) and (N, h) be Riemannian manifolds and ∇^M and ∇^N be the corresponding Levi-Civita connections.
Let $F : M \longrightarrow N$ be an immersion. Then a) the second fundamental form $\mathbf{II}^F$ of F is defined by the formula

$$\nabla^N_{dF(X)} dF(Y) = dF(\nabla^M_X Y) + \mathbf{II}^F(X, Y);$$

b) a point $p \in M$ is called an umbilic point if there exists a vector $v \in T_{F(p)}N$ such that at the point p one has

$$\mathbf{II}^F_p(X, Y) = g_p(X, Y) \cdot v. \tag{3}$$

Corollary

A point $p \in \mathbb{S}^2$ *is an umbilic of a harmonic immersion* $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *if and only if for a conformal parametr* z

$$\mathbf{II}_p^F(\partial/\partial z, \partial/\partial z) = 0. \tag{4}$$

and hence $f_{zz}(p)$ *is a linear combination of* $f_z(p)$ *and* $f_{\bar{z}}(p)$*, where* f *is a Bryant lift of* F*.*

Theorem (Ejiri 1998)

Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^2 m$ *be a linearly full harmonic immersion with branched points of harmonic degree* $d > 1$ *Then*

$$N(2) \geq d + 1$$