Theme & variations on div $\mu = \sigma$

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Theme

Consider the PDE:

$$\operatorname{div} \mu = \sigma \qquad \text{in } \mathscr{D}',$$

where

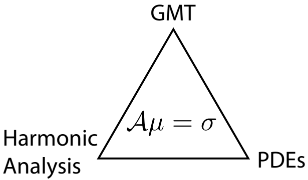
- $\mu \in \mathscr{M}(\Omega; \mathbb{R}^{d \times d})$ is a vector measure on $\Omega \subset \mathbb{R}^d$ with values in $\mathbb{R}^m$
- "div" is the row-wise divergence
- $\sigma \in \mathscr{M}(\Omega; \mathbb{R}^d)$

Prototype for $\mathscr{A}\mu = \sigma$ with $\mathscr{A}$ general constant-coefficient linear PDE operator

Central question: What can be said about the *singular part* μ^s of solutions

$$\mu = g\mathscr{L}^d + \mu^s \quad ?$$

μ^s = jumps, fractals, Cantor measures, ...?



Restrictions on the polar

Theorem (*De Philippis & R. '16*)

Let $\operatorname{div}\mu = \sigma$ *in* $\mathscr{D}'$. *Then,*

$$\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) \leq d-1 \qquad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

Corollary

Let $\operatorname{div}\mu = \sigma$ *in* $\mathscr{D}'$. *Assume that*

$$\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) = d \qquad \text{for } |\mu|\text{-a.e. } x.$$

Then, $|\mu| \ll \mathscr{L}^d$.

Remark: This result is "dual" to Alberti's Rank-One Theorem '93
($u \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R}^m)$, then $\operatorname{rank}\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right) = 1$ almost everywhere w.r.t. $|D^s u|$).

Singular Density Theorem

Let

$$\mathscr{A}\mu := \sum_{|\alpha|\leq k} A_\alpha \partial^\alpha \mu = \sigma \qquad \text{in } \mathscr{D}',$$

where $A_\alpha \in \mathbb{R}^{n\times m}$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$ for each $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$.

Examples: $\mathscr{A} = \operatorname{curl}$ (BV), $\mathscr{A} = \operatorname{curl}\operatorname{curl}$ (BD), $\mathscr{A} = \operatorname{div}$

Tartar wave cone:

$$\Lambda_{\mathscr{A}} := \bigcup_{|\xi|=1} \ker \mathbb{A}^k(\xi), \qquad \mathbb{A}^k(\xi) := \sum_{|\alpha|=k} (2\pi \mathrm{i})^k A_\alpha \xi^\alpha.$$

Theorem (*De Philippis & R. '16*)

Let $\mathscr{A}\mu = \sigma$ in $\mathscr{D}'$. Then,

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in \Lambda_{\mathscr{A}} \qquad \textit{for } |\mu^s|\textit{-a.e. } x \in \Omega.$$

Remark: Strong restriction on singularities!
(e.g.: $\mathscr{A} = \operatorname{curl}$: $\Lambda_{\mathscr{A}} = \{a \otimes b\}$, so Alberti's Rank-One Theorem follows)

$\mathscr{A} = \operatorname{div}$

Theorem (from above)

Let $\operatorname{div} \mu = \sigma$ *in* $\mathscr{D}'$*. Then,*

$$\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) \leq d-1 \qquad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

Proof: Let $\mu = (\mu_j^k) \in \mathscr{M}(\Omega; \mathbb{R}^{d\times d})$ and let

$$\operatorname{div} \mu = \left(\sum_{j=1}^{d} \partial_j \mu_j^k\right)_{k=1,\dots,d} = \sigma.$$

Then,

$$\mathbb{A}(\xi)M = (2\pi\mathrm{i})M\xi, \qquad \xi \in \mathbb{R}^d,\ M \in \mathbb{R}^{d\times d},$$

so that

$$\begin{aligned}\Lambda_{\operatorname{div}} &= \bigcup_{|\xi|=1} \ker \mathbb{A}(\xi) \\ &= \bigcup_{|\xi|=1} \{\, M \in \mathbb{R}^{d\times d} \ :\ M\xi = 0 \,\} \\ &= \{\, M \in \mathbb{R}^{d\times d} \ :\ \operatorname{rank} M \leq d-1 \,\}.\end{aligned}$$

The conclusion follows from the theorem on the previous slide. □

Partial ellipticity

- Assume $\mathscr{A}$ is a first-order operator:

$$\mathscr{A} = \sum_{\ell=1}^{d} A_\ell \partial_\ell \qquad \text{and} \qquad \mathbb{A}(\xi) = \mathbb{A}^1(\xi) = (2\pi \mathrm{i}) \sum_{\ell=1}^{d} A_\ell \xi_\ell .$$

- Let $\mathscr{A}\mu = 0$ ($\sigma \neq 0$: just lower-order terms)

- Let μ have the special structure

$$\mu = P_0\, \nu, \qquad P_0 \in \mathbb{R}^m \text{ fixed}, \quad \nu \in \mathscr{M}^+_{\mathrm{loc}}(\mathbb{R}^d) \text{ scalar measure.}$$

  $\rightsquigarrow$ This is approximately true locally around $|\mu|$-a.e. $x_0 \in \Omega$.

- Formally, via the Fourier transform,

$$\mathscr{A}\mu = 0 \qquad \Leftrightarrow \qquad \mathbb{A}(\xi) P_0 \widehat{\nu}(\xi) = 0 \quad \forall \xi \in \mathbb{R}^d .$$

- Hence, at each $\xi \neq 0$: either $P_0 \in \ker \mathbb{A}(\xi)$ or $\widehat{\nu}(\xi) = 0$.

- If $P_0 \notin \Lambda_{\mathscr{A}} = \bigcup_{|\xi|=1} \ker \mathbb{A}(\xi)$, then

$$\widehat{\nu}(\xi) = 0 \qquad \forall \xi \neq 0,$$

  hence $\nu \ll \mathscr{L}^d$. So, μ "locally" around x_0 is like $\mathscr{L}^d$ $\rightsquigarrow$ no singularity!

- Thus: $P_0 \in \Lambda_{\mathscr{A}}$ at singularities.

Variation I: Converses to Rademacher-type theorems

Differentiability of Lipschitz functions

Theorem (Rademacher)

Every Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$ is differentiable $\mathscr{L}^d$-almost everywhere.

Question: Given a positive measure ν on $\mathbb{R}^d$ such that every Lipschitz $f\colon \mathbb{R}^d \to \mathbb{R}$ is differentiable ν-almost everywhere, what can be said about ν? Are there singularities that are "not seen" by Lipschitz functions?

Conjecture (ACP conjecture, *Alberti, Csörnyei & Preiss '04*, but older)

$\nu \ll \mathscr{L}^d$.

- **Preiss '90:** There exists a null set $E \subset \mathbb{R}^2$ such that every Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable at some point of E.
- **Preiss & Speight '15:** There exists a null set $E \subset \mathbb{R}^d$ such that every Lipschitz map $f\colon \mathbb{R}^d \to \mathbb{R}^m$ with $m < d$ is differentiable at some point of E.
- **Alberti, Csörnyei & Preiss, announced in '04:** Every null set in $\mathbb{R}^2$ is contained in the non-differentiability set of some Lipschitz map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (also true for $\mathbb{R}^1 \to \mathbb{R}^1$ by an easier argument of Zahorski '46).
- **Csörnyei & Jones, announced in '11:** Every null set in $\mathbb{R}^d$ is contained in the non-differentiability set of some Lipschitz map $f\colon \mathbb{R}^d \to \mathbb{R}^d$.

Lipschitz differentiability spaces (LDS)

- (X, ρ) separable, complete metric space
- μ positive Radon measure on X
- (U, φ) with $U \subset X$, $\varphi \colon U \to \mathbb{R}^d$ Lipschitz is a d-**chart**
- $f \colon X \to \mathbb{R}$ is **differentiable** with respect to a d-chart (U, φ) at $x_0 \in U$ if there is $df(x_0) \in \mathbb{R}^d$ such that

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{\rho(x, x_0)} = 0$$

Definition (*Cheeger '99, Keith '04*)

(X, ρ, μ) is a **Lipschitz differentiability space (LDS)** if there is a countable family $\{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$ of $d(i)$-charts with

$$X = \bigcup_i U_i$$

and every Lipschitz function $f \colon X \to \mathbb{R}$ is differentiable with respect to every (U_i, φ_i) at μ-a.e. $x_0 \in U_i$ ("Rademacher's Theorem holds").

Example of LDS: First Heisenberg group

$\mathbb{H} := (\mathbb{R}^3; \cdot)$, where

$$(x, y, t) \cdot (x', y', t') := \big(x + x', y + y', t + t' - 2(xy' - yx')\big).$$

Then:

$$\mathbb{R}^3 = V_1 \oplus V_2, \qquad V_1 := \text{span}\{X, Y\}, \qquad V_2 := \text{span}\{T\},$$

where

$$X := \partial_x + 2y\partial_t, \qquad Y := \partial_y - 2x\partial_t, \qquad T := \partial_t$$

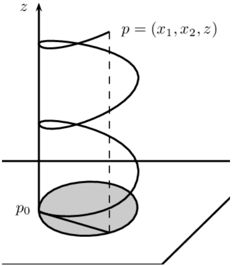
...has Haar measure $\mu_{\mathbb{H}}$ (just $\mathscr{L}^3$ in coordinates).

Carnot–Carathéodory distance on $\mathbb{H}$:

$$d_{CC}(P, Q) := \inf\Big\{ \mathrm{HL}(\gamma) \ : \ \gamma \text{ horizontal curve joining } P, Q \Big\},$$

where γ horizontal if $\dot{\gamma} \in \text{span}\{X, Y\}$ and $\mathrm{HL}(\gamma)$ is the horizontal length.

Geodesics in $\mathbb{H}$:



(image by Lerario–Rizzi '14)

Theorem (*Chow–Rashevskii*)

d_{CC} is a metric on $\mathbb{H}$ that is not Lipschitz-equivalent to the Eucliden metric.

Cheeger's conjecture

Conjecture (*Cheeger '99*)

Let (X, ρ, μ) be a Lipschitz differentiability space and let (U, φ) be a d-chart. Then,

$$\varphi_{\#}(\mu \llcorner U) \ll \mathscr{L}^d$$

("the measure $\varphi_{\#}(\mu \llcorner U)$ is d-rectifiable").

Structure theory of LDS, in particular questions on whether bi-Lipschitz (finite distortion) embeddings exist.

Applications in complexity theory:

- Many graph-cut problems (like SparsestCut) are NP-hard
- ℓ_2^2-optimization problem $\rightsquigarrow$ yields LDS
- Can embed this metric into ℓ^1 via a theorem by Bourgain '85 with distortion $O(\log n)$ (n = number of vertices)
- **Goemans–Linial Conjecture:** Can also do embedding with distortion $O(1)$ (bi-Lipschitz embedding with L-constant independent of n)
- A positive solution would give a P-algorithm for approximation of SparsestCut and other applications in polynomial approximation algorithms
- Disproved by Khot–Vishnoi '05 and Lee–Naor '05

Lipschitz differentiability & currents

Normal 1-current: $T = \vec{T}\|T\| \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ with $\partial T := d^* T \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d)$

Theorem (*Alberti & Marchese '16*)

If all Lipschitz functions $f : \mathbb{R}^d \to \mathbb{R}$ are ν-a.e. differentiable, then then there are d normal 1-currents $T_1, \dots, T_d$ with

(i) $\nu \ll \|T_i\|$ *for* $i = 1, \dots, d$,

(ii) *for ν-a.e. x,* $\mathrm{span}\{\vec{T}_1(x), \dots, \vec{T}_d(x)\} = \mathbb{R}^d$.

Theorem (*Bate '16*)

In a Lipschitz differentiability space (X, ρ, μ) with a d-chart (U, φ), then there are d normal 1-currents $T_1, \dots, T_d$ with

(i) $\varphi_\# \mu \ll \|T_i\|$ *for* $i = 1, \dots, d$,

(ii) *for $(\varphi_\# \mu)$-a.e. x,* $\mathrm{span}\{\vec{T}_1(x), \dots, \vec{T}_d(x)\} = \mathbb{R}^d$.

Remark: Proofs via Alberti Representations:

$$\nu = \int_{\text{curve fragments } \gamma \colon K \to \mathbb{R}^d} \nu_\gamma \, \mathrm{d}\pi(\gamma), \qquad \nu_\gamma \ll \mathscr{H}^1 \llcorner (\mathrm{Im}\,\gamma).$$

Structure of normal 1-currents

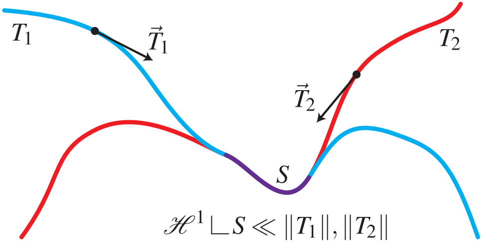
Theorem (Structure Theorem for 1D Normal Currents, *De Philippis & R. '16*)

Let $T_1 = \vec{T}_1 \|T_1\|, \dots, T_d = \vec{T}_d \|T_d\|$ be normal 1-currents such that there exists a positive Radon measure $\nu \in \mathscr{M}_+(\mathbb{R}^d)$ with the following properties:

(i) $\nu \ll \|T_i\|$ *for* $i = 1, \dots, d$,

(ii) *for* $|\nu|$*-a.e.* x, $\operatorname{span}\{\vec{T}_1(x), \dots, \vec{T}_d(x)\} = \mathbb{R}^d$.

Then, $\nu \ll \mathscr{L}^d$.



Proof: $\partial T \in \mathscr{M} \iff \operatorname{div} T = \sigma$. Then, the theorem is (essentially) a corollary to:

Theorem (from earlier)

Let $\operatorname{div} \mu = \sigma$ *in* $\mathscr{D}'$. *Assume that* $\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) = d$ *for* $|\mu|$*-a.e.* x. *Then,* $|\mu| \ll \mathscr{L}^d$.

Solution of ACP & Cheeger conjectures

Theorem (Lipschitz Differentiability Theorem, *De Philippis & R. '16*)

Let $\nu \in \mathscr{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable ν-almost everywhere. Then, $\nu \ll \mathscr{L}^d$.

Theorem (*De Philippis & Marchese & R. '17*)

Let (X, ρ, μ) be a Lipschitz differentiability space and let (U, φ) be a d-chart. Then,

$$\varphi_\#(\mu \llcorner U) \ll \mathscr{L}^d.$$

Variation II: Dimensions and rectifiability

Co-cancelling operators

$\mathscr{A}$: general constant-coefficient linear PDE operator

Definition (*van Schaftingen '13*)

The operator $\mathscr{A}$ is called **co-cancelling** if

$$\Lambda^1_{\mathscr{A}} := \bigcap_{\xi \in \mathbb{R}^d \setminus \{0\}} \ker \mathbb{A}^k = \{0\}.$$

Theorem (*van Schaftingen '13*)

Assume that $\mathscr{A}$ is homogeneous and co-cancelling. If

$$\mathscr{A}(P_0 \delta_0) = 0 \qquad \text{for some } P_0 \in \mathbb{R}^m,$$

then $P_0 = 0$.

Corollary

Let $\mathscr{A}\mu = 0$ in $\mathscr{D}'$. If μ is "0-rectifiable", then $\mu = 0$.

Conclusion: Other wave cones might give information about the dimension of μ...

Hierarchy of wave cones

$\mathscr{A}$: general constant-coefficient linear PDE operator; recall $\Lambda_{\mathscr{A}} := \bigcup_{\xi \neq 0} \ker \mathbb{A}^k(\xi)$.

Definition

Let $\mathrm{Gr}(\ell, d)$ be the Grassmanian of ℓ planes in $\mathbb{R}^d$. For $\ell = 1, \ldots, d$ we define the ℓ-**dimensional wave cone** as

$$\Lambda^{\ell}_{\mathscr{A}} := \bigcap_{\pi \in \mathrm{Gr}(\ell,d)} \bigcup_{\xi \in \pi \setminus \{0\}} \ker \mathbb{A}^k(\xi),$$

where $\mathbb{A}^k$ is the principal symbol of $\mathscr{A}$.

Equivalently:

$$P_0 \notin \Lambda^{\ell}_{\mathscr{A}} \quad \Longleftrightarrow \quad (\mathscr{A} \llcorner \pi) P_0 \text{ is elliptic for some } \pi \in \mathrm{Gr}(\ell, d),$$

where $(\mathscr{A} \llcorner \pi)(\varphi) := \mathscr{A}(\varphi \circ \mathbf{p}_\pi)$ with $\mathbf{p}_\pi$ the orthogonal projection onto π.

Inclusions:

$$\Lambda^1_{\mathscr{A}} = \bigcap_{\xi \in \mathbb{R}^d \setminus \{0\}} \ker \mathbb{A}^k(\xi) \subset \Lambda^j_{\mathscr{A}} \subset \Lambda^\ell_{\mathscr{A}} \subset \Lambda^d_{\mathscr{A}} = \Lambda_{\mathscr{A}}, \qquad 1 \leq j \leq \ell \leq d.$$

Dimensional estimates

Theorem (*Arroyo-Rabasa & De Philippis & Hirsch & R. '18*, on arXiv imminently...)

Let $\mathscr{A}\mu = \sigma$ in $\mathscr{D}'$. If $\mathscr{H}^\ell(E) = 0$ for some $\ell \in \{0, \ldots, d\}$, then

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in \Lambda^\ell_{\mathscr{A}} \qquad \text{for } |\mu|\text{-a.e. } x \in E.$$

Remark: For $\ell = d$, this recovers the '16 Singular Density Theorem.

Corollary

Let $\mathscr{A}\mu = \sigma$ in $\mathscr{D}'$. Define

$$\ell_{\mathscr{A}} := \max\left\{ \ell \in \mathbb{N} \ : \ \Lambda^\ell_{\mathscr{A}} = \{0\} \right\}.$$

Then,

$$\mu \ll \mathscr{H}^{\ell_{\mathscr{A}}}.$$

Remark: For $\ell = 1$, this also improves the result of van Schaftingen '13.

Rectifiability

Define the **upper ℓ-density** of $|\mu|$:

$$\theta^*_\ell(|\mu|)(x) := \limsup_{r\to 0} \frac{|\mu|(B_r(x))}{(2r)^\ell}.$$

Theorem (*Arroyo-Rabasa & De Philippis & Hirsch & R. '18*)

Let $\mathscr{A}\mu = \sigma$ in $\mathscr{D}'$ and assume

$$\Lambda^\ell_{\mathscr{A}} = \{0\}.$$

Then, $\mu \llcorner \{\theta^*_\ell(|\mu|) > 0\}$ is concentrated on an ℓ-rectifiable set R and

$$\mu \llcorner R = P(x)\, \mathscr{H}^\ell_x \llcorner R,$$

where

$$P(x_0) \in \bigcap_{\xi \in (T_{x_0} R)^\perp} \ker \mathbb{A}^k(\xi) \qquad \text{for } \mathscr{H}^\ell\text{-a.e. } x_0 \in R \text{ (or } |\mu|\text{-a.e. } x_0 \in R\text{).}$$

Here, $T_{x_0} R$ is the the approximate tangent plane to R at x_0.

Proof: Via the Besicovitch–Federer rectifiability theorem.

Remark: Recovers rectifiability results for BV-maps ($\mathscr{A} = \operatorname{curl}$) and for BD-maps ($\mathscr{A} = \operatorname{curl}\operatorname{curl}$).

Rectifiability of varifolds and of defect measures

Corollary

Let $\operatorname{div}\mu = \sigma$ *in* $\mathscr{D}'$. *Assume that*

$$\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) \geq \ell \qquad \text{for } |\mu|\text{-a.e. } x.$$

Then, $|\mu| \ll \mathscr{H}^{\ell}$ *and there exist an* ℓ*-rectifiable set* $R \subset U$ *such that*

$$\mu \llcorner \{\theta^*_\ell(|\mu|) > 0\} = P(x)\, \mathscr{H}^{\ell}_x \llcorner R, \qquad \operatorname{rank} P(x) = \ell.$$

Remark: Recovers several known rectifiability criteria for varifolds (Allard '72, Ambrosio–Soner '97, Lin '99, Moser '03, De Philippis–De Rosa–Ghiraldin '18).

Proof: Let $\widetilde{\mu} := (\mu, \sigma)$ and $\mathscr{A}(\widetilde{\mu}) := \operatorname{div}\mu - \sigma$. Then,

$$\begin{aligned}
\Lambda^{\ell}_{\mathscr{A}} &= \bigcap_{\pi \in \mathrm{Gr}(\ell,d)} \left\{ M \in \mathbb{R}^d \otimes \mathbb{R}^d \ : \ \ker M \cap \pi \neq \{0\} \right\} \times \mathbb{R}^d \\
&= \left\{ M \in \mathbb{R}^d \otimes \mathbb{R}^d \ : \ \dim\ker M > d - \ell \right\} \times \mathbb{R}^d \\
&= \left\{ M \in \mathbb{R}^d \otimes \mathbb{R}^d \ : \ \operatorname{rank} M < \ell \right\} \times \mathbb{R}^d.
\end{aligned}$$

□

Variation III: Liftings

BV-maps with jumps: Relaxation

Let $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain, $d, m > 1$, and

$$\mathscr{F}[u] := \int_\Omega f(x, u(x), \nabla u(x)) \,\mathrm{d}x, \qquad u \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^m),$$

where $f \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to [0, \infty)$ with

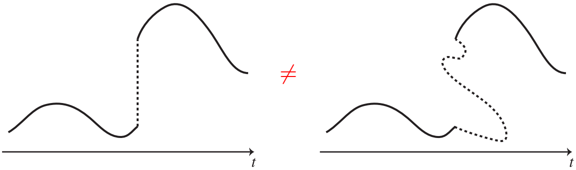
$$0 \leq f(x, y, A) \leq C(1 + |y|^{d/(d-1)} + |A|).$$

Relaxation of $\mathscr{F}$ at $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$:

$$\mathscr{F}_{**}[u] := \inf \left\{ \liminf_{j \to \infty} \mathscr{F}[u_j] : \ (u_j)_j \subset \mathrm{W}^{1,1}(\Omega; \mathbb{R}^m), \ u_j \rightsquigarrow u \right\}$$

with "$u_j \rightsquigarrow u$" meaning BV-weak* or L^1-strong convergence.

Q: What is $\mathscr{F}_{**}$? Does it have an integral representation? Jump paths matter!



Previous work: Fonseca–Müller '93, Ambrosio–Dal Maso '92 and many other works (Leoni, Bouchitté, Mascarenhas, ...).

Liftings

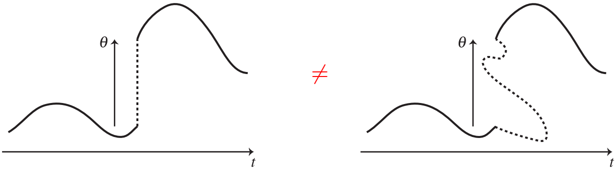
$$\mathrm{BV}_{\#}(\Omega;\mathbb{R}^m) := \left\{ u \in \mathrm{BV}(\Omega;\mathbb{R}^m) \ : \ \fint_\Omega u(x)\,\mathrm{d}x = 0 \right\}.$$

Definition (*Jung & Jerrard '04*)

A **lifting** $\gamma \in \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ is a measure $\gamma \in \mathcal{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m\times d})$ for which there exists a (unique) $u \in \mathrm{BV}_{\#}(\Omega;\mathbb{R}^m)$ such that the **chain rule** holds:

$$\int_\Omega \nabla_x \varphi(x, u(x))\,\mathrm{d}x + \int_{\Omega\times\mathbb{R}^m} \nabla_y \varphi(x,y)\,\mathrm{d}\gamma(x,y) = 0 \quad \text{for all } \varphi \in \mathrm{C}^1_0(\Omega\times\mathbb{R}^m).$$

This u is called the **barycenter** $[\gamma]$ of γ.
Weak* convergence of liftings means weak* convergence in $\mathcal{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m\times d})$.



Lemma

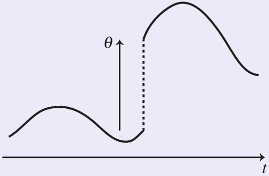
$\pi_{\#}\gamma = Du$ in $\mathcal{M}(\Omega;\mathbb{R}^{m\times d})$ and $\pi_{\#}|\gamma| \geq |Du|$ in $\mathcal{M}^+(\Omega)$.

Elementary liftings

Definition (Elementary/Minimal Liftings)

Given $u \in \mathrm{BV}_{\#}(\Omega; \mathbb{R}^m)$, the associated **elementary lifting** $\gamma[u] \in \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ is

$$\gamma[u] := Du \otimes \int_0^1 \delta_{u^\theta} \, \mathrm{d}\theta,$$



where u^θ is the jump interpolant,

$$u^\theta(x) := \begin{cases} \theta u^-(x) + (1-\theta)u^+(x) & \text{if } x \in J_u, \\ \widetilde{u}(x) & \text{otherwise.} \end{cases}$$

So,

$$\langle \varphi, \gamma[u] \rangle = \int_\Omega \int_0^1 \varphi(x, u^\theta(x)) \, \mathrm{d}\theta \, \mathrm{d}Du(x) \quad \text{for all } \varphi \in \mathrm{C}_0(\Omega \times \mathbb{R}^m).$$

Chain rule

The liftings chain rule for the elementary lifting

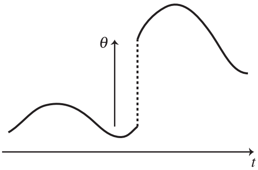
$$\gamma[u](\mathrm{d}x, \mathrm{d}y) := Du(\mathrm{d}x) \otimes \int_0^1 \delta_{u^\theta(x)}(\mathrm{d}y)\, \mathrm{d}\theta,$$

follows from usual BV-chain rule:

For $\varphi \in \mathrm{C}_0^1(\Omega \times \mathbb{R}^m)$:

$$\begin{aligned}
&\int_\Omega \nabla_x \varphi(x, u(x))\, \mathrm{d}x + \int_{\Omega \times \mathbb{R}^m} \nabla_y \varphi(x, y)\, \mathrm{d}\gamma(x, y) \\
&\quad = \int_\Omega \nabla_x \varphi(x, u(x))\, \mathrm{d}x + \int_\Omega \int_0^1 \nabla_y \varphi(x, u^\theta(x))\, \mathrm{d}\theta\, \mathrm{d}Du(x) \\
&\quad = \int_\Omega \nabla_x \big[\varphi(x, u(x))\big]\, \mathrm{d}x \\
&\quad = 0.
\end{aligned}$$

Non-elementary liftings



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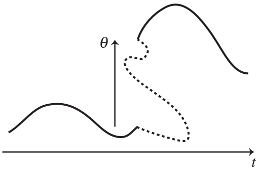
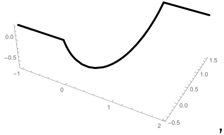


$\gamma_1 := Du \otimes \int_0^1 \delta_{u^\theta_{\mathrm{affine}}} \, \mathrm{d}\theta$ $\qquad$ $\gamma_2 := Du \otimes \int_0^1 \delta_{u^\theta_{\mathrm{squiggle}}} \, \mathrm{d}\theta$

$u_j =$



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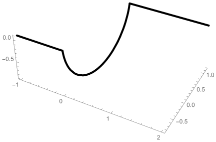

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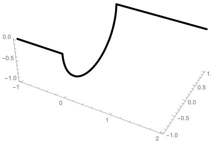

Example:

$\gamma[u_j] \xrightarrow{*} \gamma \neq \gamma[u]$ for some $\gamma \in \mathbf{Lift}((-1,1) \times \mathbb{R}^2)$.

Lemma

Every lifting $\gamma \in \mathbf{Lift}(\Omega \times \mathbb{R})$ *is elementary:* $\gamma = \gamma[u]$ *for some* $u \in \mathrm{BV}_{\#}(\Omega; \mathbb{R})$.

Compactness for liftings

Lemma (Compactness)

Let $(\gamma_j)_j \subset \textbf{Lift}(\Omega \times \mathbb{R}^m)$ be such that $\sup_j |\gamma_j|(\Omega \times \mathbb{R}^m) < \infty$. Then there exists a subsequence $(\gamma_{j_k})_k \subset (\gamma_j)_j$ and a limit $\gamma \in \textbf{Lift}(\Omega \times \mathbb{R}^m)$ such that

$$\gamma_{j_k} \overset{*}{\rightharpoonup} \gamma \text{ in } \mathscr{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m\times d}) \text{ and } [\gamma_{j_k}] \overset{*}{\rightharpoonup} [\gamma] \text{ in } \mathrm{BV}_{\#}(\Omega; \mathbb{R}^m).$$

Corollary (Lifting generation from BV)

Let $(u_j)_j \subset \mathrm{BV}_{\#}(\Omega; \mathbb{R}^m)$ be a bounded sequence with $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}_{\#}(\Omega; \mathbb{R}^m)$. Then there exists a (non-relabelled) subsequence and a limit $\gamma \in \textbf{Lift}(\Omega \times \mathbb{R}^m)$ with $[\gamma] = u$ such that

$$\gamma[u_j] \overset{*}{\rightharpoonup} \gamma \text{ in } \textbf{Lift}(\Omega \times \mathbb{R}^m).$$

Structure theorem

Graph map: $\mathrm{gr}^u \colon x \mapsto (x, u(x))$ for $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$

Pushforward: If $\mu \in \mathscr{M}(\Omega)$ satisfying $|\mu| \ll \mathscr{H}^{d-1}$ and $|\mu|(J_u) = 0$, then the pushforward $\mathrm{gr}^u_\# \mu$ of μ under gr^u is well-defined as a measure on $\Omega \times \mathbb{R}^m$.
(we will usually take $\mu = |Du| \llcorner (\Omega \setminus J_u)$)

Theorem (Structure Theorem for Liftings, *R. & Shaw 2017*)

*If $\gamma \in$ **Lift**$(\Omega \times \mathbb{R}^m)$ with $u = [\gamma]$, then γ admits the following decomposition into mutually singular measures:*

$$\gamma = \gamma[u] \llcorner ((\Omega \setminus J_u) \times \mathbb{R}^m) + \gamma^{\mathrm{gs}}.$$

Moreover, $\gamma^{\mathrm{gs}} \in \mathscr{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m \times d})$ satisfies

$$\mathrm{div}_y \, \gamma^{\mathrm{gs}} = -|D^j u| \otimes \frac{n_u}{|u^+ - u^-|} (\delta_{u^+} - \delta_{u^-})$$

and it is **graph-singular** *with respect to u in the sense that γ^{gs} is singular with respect to all measures of the form $\mathrm{gr}^u_\# \lambda$ where $\lambda \in \mathscr{M}(\Omega)$ satisfies both $\lambda \ll \mathscr{H}^{d-1}$ and $\lambda(J_u) = 0$.*

(the Singular Density Theorem hence gives restrictions on γ^{gs})

Perspective functionals

Proposition

Let $\gamma \in \textbf{Lift}(\Omega \times \mathbb{R}^m)$ *with* $u = [\gamma]$ *be* **minimal** *in the sense that* $|\gamma|(\Omega \times \mathbb{R}^m) = |Du|(\Omega)$. *Then* γ *must be elementary,* $\gamma = \gamma[u]$. *In particular, if* $u_j \to u$ *in* $\mathrm{BV}_\#(\Omega; \mathbb{R}^m)$ *strictly, then* $\gamma[u_j] \to \gamma[u]$ *strictly in* $\textbf{Lift}(\Omega \times \mathbb{R}^m)$.

Define $\mathscr{F}_\mathrm{L} \colon \textbf{Lift}(\Omega \times \mathbb{R}^m) \to \mathbb{R}$ by

$$\mathscr{F}_\mathrm{L}[\gamma] = \int_\Omega f(x, [\gamma](x), \nabla[\gamma](x)) \,\mathrm{d}x + \int_{\Omega\times\mathbb{R}^m} f^\infty(x, y, \gamma^s) \,.$$

For $u \in \mathrm{BV}_\#(\Omega; \mathbb{R}^m)$ we have by the structure theorem

$$\mathscr{F}_\mathrm{L}[\gamma[u]] = \int_\Omega f(x, u, \nabla u) \,\mathrm{d}x + \int_{\Omega\times\mathbb{R}^m} f^\infty(x, u, D^s u) = \mathscr{F}[u].$$

Strategy: Study $\mathscr{F}$ via $\mathscr{F}_\mathrm{L}$ (via blowups / Young measures for liftings ...).

Relaxation theorem in BV

Theorem (*R. & Shaw 2017*, weak* version)

Let $f : \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to [0,\infty)$ *where* $d \geq 2$ *and* $m \geq 1$ *be such that*

(i) f *is a Carathéodory function whose recession function* f^∞ *exists as a limit,*

$$f^\infty(x,y,A) = \lim_{\substack{(x,y_k,A_k)\to(x,y,A)\\ t_k\to\infty}} \frac{f(x_k,y_k,t_kA_k)}{t_k};$$

(ii) $0 \leq f(x,y,A) \leq C(1 + |y|^{d/(d-1)} + |A|)$;

(iii) $f(x,y,\bullet)$ *is quasiconvex for every* $(x,y) \in \overline{\Omega} \times \mathbb{R}^m$.

Then the sequential **weak* relaxation** $\mathscr{F}_{**}$ *of* $\mathscr{F}$ *to* $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ *is*

$$\mathscr{F}^{w*}_{**}[u] = \int_\Omega f(x,u,\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\left(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right)\mathrm{d}|D^c u| + \int_J K_f[u]\,\mathrm{d}\mathscr{H}^{d-1}$$

where J *is the jump set of* u *and*

$$K_f[u](x) := \inf\left\{ \frac{1}{\omega_{d-1}} \int_{\mathbb{B}^d} f^\infty(x,\varphi(y),\nabla\varphi(y))\,\mathrm{d}y \;\; : \;\; \varphi \in \mathrm{C}^\infty(\mathbb{B}^d;\mathbb{R}^m),\ \varphi|_{\partial\mathbb{B}^d} = u^\pm(x) \text{ if } y\cdot n_u(x) \gtrless 0 \right\}$$

Remark: Improves classical weak*-relaxation theorem in BV by Fonseca & Müller '92.

Coda

Question

Does the distance of $\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}$ *to the wave cone* $\Lambda_{\mathscr{A}}$ *control (perhaps in a nonlinear way) how "close"* μ *is to being singular?*

Thank you for your attention!

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Encore: Pansu's Theorem (work in progress with De Philippis & Ghiraldin)

Let $\mathbb{H}$ be the 1st Heisenberg group.

Dilation:

$$\delta_\lambda(x, y, t) := (\lambda x, \lambda y, \lambda^2 t).$$

$f : \mathbb{H} \to \mathbb{H}$ is **Pansu-differentiable** at $x \in \mathbb{H}$ if there exists a homogeneous (dilation-invariant) group homorphism $L_x : \mathbb{H} \to \mathbb{H}$ such that

$$\lim_{y \to e} \frac{d_{\mathbb{H}}(f(x)^{-1} f(xy), L_x(y))}{d_{\mathbb{H}}(y, e)} = 0$$

where e is the identity element in $\mathbb{H}$.

Theorem (*Pansu '89*)

Every Lipschitz $f : \mathbb{H} \to \mathbb{H}$ is Pansu-differentiable $\mu_{\mathbb{H}}$-almost everywhere ($\mu_{\mathbb{H}}$ is the Haar measure on $\mathbb{H}$).

Theorem (*De Philippis & Ghiraldin & R. '17/'18*, to be written up...)

Let μ be a positive Radon measure on $\mathbb{H}$ (or any Carnot groups) such that every Lipschitz function $f : \mathbb{H} \to \mathbb{H}$ is Pansu-differentiable μ-almost everywhere. Then, $\mu \ll \mu_{\mathbb{H}}$.

Encore 2: Dimension conjecture

For $\ell \in \{0, \ldots, d-1\}$ let us then define the cone

$$\mathscr{N}_{\mathscr{A}}^{\ell} := \bigcup_{\pi \in \mathrm{Gr}(\ell, d)} \bigcap_{\xi \in \pi^{\perp}} \ker \mathbb{A}^{k}(\xi) = \bigcup_{\tilde{\pi} \in \mathrm{Gr}(d-\ell, d)} \bigcap_{\xi \in \tilde{\pi}} \ker \mathbb{A}^{k}(\xi).$$

Conjecture (also in *Raita '17*)

Let $\mathscr{A}\mu = \sigma$ in $\mathscr{D}'$. Then,

$$\dim_{\mathscr{H}}(\mu) \geq \ell_{\mathscr{A}}^{*} := \min\big\{\, \ell \in \mathbb{N} \;:\; \mathscr{N}_{\mathscr{A}}^{\ell} \neq \{0\} \,\big\}.$$

Sharpness: Our dimensional theorem is sharp for div, curl, curl curl, but the sharpness is unclear for $\partial_1^3 + \partial_2^3 + \partial_3^3$.