From fluid flow in cones to boundry Harnack with RHS[1]

Henrik Shahgholian

KTH Royal Inst. of Tech.

PDEs and Geometric Measure Theory
ETH, Zurich,

October 29–November 2, 2018

Details with proofs will appear at:
https://www.scilag.net/profile/henrik-shahgholian

[1] Based on joint work with Mark Allen (Brigham Young University)

From fluid flow in cones to bdry Harnack with RHS

- Flow inside cones,
- Boundary Harnack Principle with RHS in cones
- Application to FB regularity
- Proof of BHP with RHS (sketch)

Flow inside a cone

Hele-Shaw flow

The standard model problem

Hele-Shaw flow concerns geometric motion of an initial interface (boundary of fluid region) caused by pressure (in the system) such as injection of more fluid.

This is a toy model of various flow problems in industrial processes: Plastic industry (injection moulding), Reservoir simulation (flow in porous medium), Thin film (lubrication) ...

Hele-Shaw flow

The standard model problem

Hele-Shaw flow concerns geometric motion of an initial interface (boundary of fluid region) caused by pressure (in the system) such as injection of more fluid.

This is a toy model of various flow problems in industrial processes: Plastic industry (injection moulding), Reservoir simulation (flow in porous medium), Thin film (lubrication) ...

Hele-Shaw flow

The standard model problem

Specifically we consider pressure from the Green's function

$$\Delta G^t = -\delta_z \quad \text{in } \Omega^t, \qquad G^t = 0 \quad \text{on } \partial\Omega^t,$$

where

$$z \in \Omega^0 \subset \mathbb{R}^n, \qquad \text{and} \qquad \Omega^0 = \text{given initial state.}$$

Ω^0 evolves with time through injection of more fluid (or pressure in the system), with speed $V = |\nabla G|$ in the outward normal direction.

Hele-Shaw flow

The standard model problem

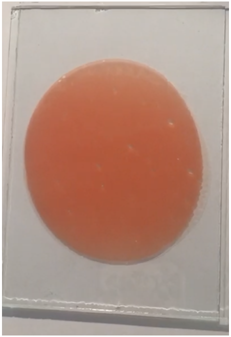
Specifically we consider pressure from the Green's function

$$\Delta G^t = -\delta_z \quad \text{in } \Omega^t, \qquad G^t = 0 \quad \text{on } \partial\Omega^t,$$

where

$$z \in \Omega^0 \subset \mathbb{R}^n, \qquad \text{and} \qquad \Omega^0 = \text{given initial state.}$$

Ω^0 evolves with time through injection of more fluid (or pressure in the system), with speed $V = |\nabla G|$ in the outward normal direction.



Klick to see the Video on Youtube

Hele-Shaw flow

Flow on a table

Consider now the Hele-Shaw flow on a table, where the top D of the table has an arbitrary shape with edges and corners, and

$$\boxed{\text{Injection point } z \in \Omega^0 \subset D.}$$

The liquid falls from the table when it reaches the edges and the corners of the table.

Hele-Shaw flow

Flow on a table

Consider now the Hele-Shaw flow on a table, where the top D of the table has an arbitrary shape with edges and corners, and

$$\boxed{\text{Injection point } z \in \Omega^0 \subset D.}$$

The liquid falls from the table when it reaches the edges and the corners of the table.

Hele-Shaw flow

Flow on a table

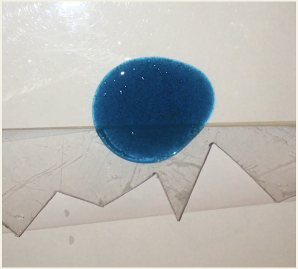


Figure: Flow on a table

Does the table get completely wet in finite time?

Hele-Shaw flow

Flow on a table

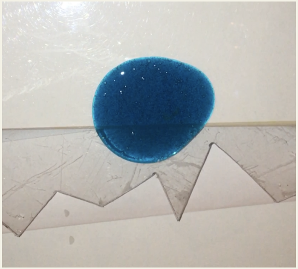


Figure: Flow on a table

Does the table get completely wet in finite time?

Hele-Shaw flow

Reformulation of the PDE

Reformulate the problem by integrating in time

$$u^t(x) = u^t(x,t) = \int_0^t G^\tau d\tau,$$

and obtain a new function u^t, that solves[2]

$$\Delta u^t = \chi_{\Omega^t} - \chi_{\Omega^0} - t\delta_z\,, \qquad \Omega^t = \{u^t(x,t) > 0\}.$$

This admits a variational formulation, and has a weak solution.

$$\boxed{\text{It is easier to work with } u^t\text{, hence the reformulation.}}$$

[2]Formally $\Delta u^t = \int_0^t \Delta G^\tau d\tau = \int_0^t (|\nabla G| d\sigma_x \lfloor_{\partial\Omega_\tau} - \delta_z) d\tau = \chi_{\Omega^t} - \chi_{\Omega^0} - t\delta_z$.

Hele-Shaw flow

Reformulation of the PDE

Reformulate the problem by integrating in time

$$u^t(x) = u^t(x,t) = \int_0^t G^\tau d\tau,$$

and obtain a new function u^t, that solves[2]

$$\Delta u^t = \chi_{\Omega^t} - \chi_{\Omega^0} - t\delta_z \ , \qquad \Omega^t = \{u^t(x,t) > 0\}.$$

This admits a variational formulation, and has a weak solution.

It is easier to work with u^t, hence the reformulation.

[2] Formally $\Delta u^t = \int_0^t \Delta G^\tau d\tau = \int_0^t (|\nabla G| d\sigma_x \lfloor_{\partial\Omega_\tau} - \delta_z) d\tau = \chi_{\Omega^t} - \chi_{\Omega^0} - t\delta_z$.

Hele-Shaw flow

Reformulation of the PDE

Reformulate the problem by integrating in time

$$u^t(x) = u^t(x,t) = \int_0^t G^\tau d\tau,$$

and obtain a new function u^t, that solves[2]

$$\Delta u^t = \chi_{\Omega^t} - \chi_{\Omega^0} - t\delta_z\ , \qquad \Omega^t = \{u^t(x,t) > 0\}.$$

This admits a variational formulation, and has a weak solution.

$$\boxed{\text{It is easier to work with } u^t\text{, hence the reformulation.}}$$

---

[2]Formally $\Delta u^t = \int_0^t \Delta G^\tau d\tau = \int_0^t (|\nabla G| d\sigma_x \lfloor_{\partial\Omega_\tau} - \delta_z) d\tau = \chi_{\Omega^t} - \chi_{\Omega^0} - t\delta_z$.

Corner points

Theorem: Results in 2-dim. (Sh. 2004)

Suppose the origin is a corner point of the table, with interior angle θ_0.

The following hold.

- **(a)** If $\theta_0 \leq \pi/2$ then the fluid does not reach the origin in finite time.
- **(b)** If $\theta_0 > \pi/2$, then the origin can be reached by the fluid in finite time.

Higher dimensional results: recent work with Mark Allen.

Corner points

Theorem: Results in 2-dim. (Sh. 2004)

Suppose the origin is a corner point of the table, with interior angle θ_0.

The following hold.

- **(a)** If $\theta_0 \leq \pi/2$ then the fluid does not reach the origin in finite time.
- **(b)** If $\theta_0 > \pi/2$, then the origin can be reached by the fluid in finite time.

Higher dimensional results: recent work with Mark Allen.

Corner points

Theorem: Results in 2-dim. (Sh. 2004)

Suppose the origin is a corner point of the table, with interior angle θ_0.

The following hold.

- **(a)** If $\theta_0 \leq \pi/2$ then the fluid does not reach the origin in finite time.
- **(b)** If $\theta_0 > \pi/2$, then the origin can be reached by the fluid in finite time.

Higher dimensional results: recent work with Mark Allen.

Corner points

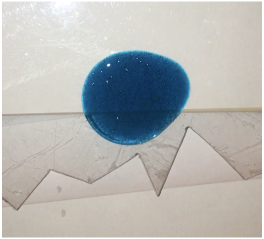
Theorem: Results in 2-dim. (Sh. 2004)

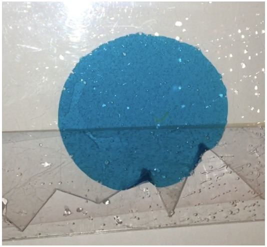
Suppose the origin is a corner point of the table, with interior angle θ_0.

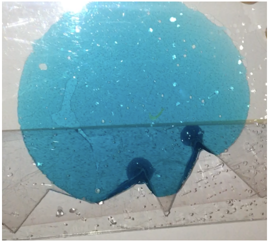
The following hold.

- **(a)** If $\theta_0 \leq \pi/2$ then the fluid does not reach the origin in finite time.
- **(b)** If $\theta_0 > \pi/2$, then the origin can be reached by the fluid in finite time.

$\boxed{\text{Higher dimensional results: recent work with Mark Allen.}}$







Klick to see the Video on Youtube

Boundary Harnack with RHS

BHP with RHS

Rephrasing the above discussion

That the table gets wet in finite time means for large values of t

$$u^t > 0 \qquad \text{in } D.$$

We localize the problem close to $z^1 \in \partial D$, since any interior point obviously gets wet in finite time.

BHP with RHS

Rephrasing the above discussion

That the table gets wet in finite time means for large values of t

$$u^t > 0 \qquad \text{in } D.$$

We localize the problem close to $z^1 \in \partial D$, since any interior point obviously gets wet in finite time.

BHP with RHS

Rephrasing the above discussion

Define h^t, and k as

$$\Delta h^t = 0 \qquad \text{in } D \cap B_r(z^1),$$

with boundary values $h^t = u^t$. Define also

$$\Delta k = -1 \quad \text{in } B_r(z^1) \cap D,$$

with zero boundary values.

Obviously $u^t > h^t - k$. Hence it suffices to show that for large t

$$h^t \geq k \qquad \text{in } B_r(z^1) \cap D, \qquad \text{for } t > t_0.$$

BHP with RHS

Rephrasing the above discussion

Define h^t, and k as

$$\Delta h^t = 0 \qquad \text{in } D \cap B_r(z^1),$$

with boundary values $h^t = u^t$. Define also

$$\Delta k = -1 \quad \text{in } B_r(z^1) \cap D,$$

with zero boundary values.

Obviously $u^t > h^t - k$. Hence it suffices to show that for large t

$$h^t \geq k \qquad \text{in } B_r(z^1) \cap D, \qquad \text{for } t > t_0.$$

BHP with RHS

Rephrasing the above discussion

Define h^t, and k as

$$\Delta h^t = 0 \qquad \text{in } D \cap B_r(z^1),$$

with boundary values $h^t = u^t$. Define also

$$\Delta k = -1 \quad \text{in } B_r(z^1) \cap D,$$

with zero boundary values.

Obviously $u^t > h^t - k$. Hence it suffices to show that for large t

$$h^t \geq k \qquad \text{in } B_r(z^1) \cap D, \qquad \text{for } t > t_0.$$

BHP with RHS

Rephrasing the above discussion

Now suppose for some t_0 there exists C_{t_0} such that[3]

$$C_{t_0} h^{t_0} \geq k. \qquad (1)$$

Next, by the (standard) boundary Harnack principle

$$\sup_{B_{r/2}(z^1)\cap D} \frac{h^{t_0}}{h^t} \leq C \frac{h^{t_0}(z^2)}{h^t(z^2)},$$

for a fixed interior point $z^2 \in B_r(z^1) \cap D$.

[3] This is a kind of boundary Harnack!

BHP with RHS

Rephrasing the above discussion

Now suppose for some t_0 there exists C_{t_0} such that[3]

$$C_{t_0} h^{t_0} \geq k. \qquad (1)$$

Next, by the (standard) boundary Harnack principle

$$\sup_{B_{r/2}(z^1)\cap D} \frac{h^{t_0}}{h^t} \leq C \frac{h^{t_0}(z^2)}{h^t(z^2)},$$

for a fixed interior point $z^2 \in B_r(z^1) \cap D$.

[3] This is a kind of boundary Harnack!

BHP with RHS

Rephrasing the above discussion

Since

$$\lim_{t\to\infty} h^t(z^2) = \infty,$$

we choose t large enough so that

$$\frac{h^{t_0}(z^2)}{h^t(z^2)} < \frac{C^{-1}}{C_t}.$$

Then

$$\sup_{B_{r/2}(z^1)\cap D} \frac{h^{t_0}}{h^t} \le C\frac{h^{t_0}(z^2)}{h^t(z^2)} \le \frac{CC^{-1}}{C_{t_0}} = \frac{1}{C_{t_0}},$$

so that

$$h^t \ge C_{t_0} h^{t_0} \quad (\ge k) \qquad \text{by (1)},$$

which in turn implies $u^t > 0$ in $B_{r/2}(z^1) \cap D$, for t large.

Rephrasing the above discussion

Since

$$\lim_{t\to\infty} h^t(z^2) = \infty,$$

we choose t large enough so that

$$\frac{h^{t_0}(z^2)}{h^t(z^2)} < \frac{C^{-1}}{C_t}.$$

Then

$$\sup_{B_{r/2}(z^1)\cap D} \frac{h^{t_0}}{h^t} \leq C\frac{h^{t_0}(z^2)}{h^t(z^2)} \leq \frac{CC^{-1}}{C_{t_0}} = \frac{1}{C_{t_0}},$$

so that

$$h^t \geq C_{t_0} h^{t_0} \quad (\geq k) \qquad \text{by (1)},$$

which in turn implies $u^t > 0$ in $B_{r/2}(z^1) \cap D$, for t large.

BHP with RHS

Rephrasing the above discussion

Since

$$\lim_{t\to\infty} h^t(z^2) = \infty,$$

we choose t large enough so that

$$\frac{h^{t_0}(z^2)}{h^t(z^2)} < \frac{C^{-1}}{C_t}.$$

Then

$$\sup_{B_{r/2}(z^1)\cap D} \frac{h^{t_0}}{h^t} \le C\frac{h^{t_0}(z^2)}{h^t(z^2)} \le \frac{CC^{-1}}{C_{t_0}} = \frac{1}{C_{t_0}},$$

so that

$$h^t \ge C_{t_0} h^{t_0} \quad (\ge k) \qquad \text{by (1)},$$

which in turn implies $u^t > 0$ in $B_{r/2}(z^1) \cap D$, for t large.

BHP with RHS

Rephrasing the above discussion

Since

$$\lim_{t\to\infty} h^t(z^2) = \infty,$$

we choose t large enough so that

$$\frac{h^{t_0}(z^2)}{h^t(z^2)} < \frac{C^{-1}}{C_t}.$$

Then

$$\sup_{B_{r/2}(z^1)\cap D} \frac{h^{t_0}}{h^t} \leq C\frac{h^{t_0}(z^2)}{h^t(z^2)} \leq \frac{CC^{-1}}{C_{t_0}} = \frac{1}{C_{t_0}},$$

so that

$$h^t \geq C_{t_0} h^{t_0} \quad (\geq k) \qquad \text{by (1)},$$

which in turn implies $u^t > 0$ in $B_{r/2}(z^1) \cap D$, for t large.

BHP with RHS

What we need to show

To make the previous argument intact we need to show (1)

$$C_t h^t \geq k.$$

This is a boundary Harnack principle between solutions h^t and supersolutions k.

Smooth boundary case

If ∂D is $C^{1,Dini}$, then we may invoke Hopf's boundary point lemma to conclude the above inequality.

BHP with RHS

What we need to show

To make the previous argument intact we need to show (1)

$$C_t h^t \geq k.$$

This is a boundary Harnack principle between solutions h^t and supersolutions k.

Smooth boundary case

If ∂D is $C^{1,Dini}$, then we may invoke Hopf's boundary point lemma to conclude the above inequality.

Main Result

Harmonic functions in cones (basics)

Let $\mathcal{C}$ be any open cone in $\mathbb{R}^n$, with vertex at the origin such that $\mathcal{C} \cap \mathbb{S}^{n-1}$ is connected.

For u harmonic on $\mathcal{C}$ with $u = 0$ on $\partial\mathcal{C}$, we have

$$u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta), \tag{2}$$

where f_k are the eigenfunctions to the Laplace-Beltrami on $\mathcal{C} \cap \partial B_1$.

If $u \geq 0$ and harmonic on $\mathcal{C}$ then for some $C > 0$

$$u = C r^{\alpha_1} f_1(\theta).$$

Main Result

Harmonic functions in cones (basics)

Let $\mathcal{C}$ be any open cone in $\mathbb{R}^n$, with vertex at the origin such that $\mathcal{C} \cap \mathbb{S}^{n-1}$ is connected.

For u harmonic on $\mathcal{C}$ with $u = 0$ on $\partial\mathcal{C}$, we have

$$u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta), \tag{2}$$

where f_k are the eigenfunctions to the Laplace-Beltrami on $\mathcal{C} \cap \partial B_1$.

If $u \geq 0$ and harmonic on $\mathcal{C}$ then for some $C > 0$

$$u = C r^{\alpha_1} f_1(\theta).$$

Main Result

Harmonic functions in cones (basics)

Let $\mathcal{C}$ be any open cone in $\mathbb{R}^n$, with vertex at the origin such that $\mathcal{C} \cap \mathbb{S}^{n-1}$ is connected.

For u harmonic on $\mathcal{C}$ with $u = 0$ on $\partial\mathcal{C}$, we have

$$u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta), \tag{2}$$

where f_k are the eigenfunctions to the Laplace-Beltrami on $\mathcal{C} \cap \partial B_1$.

If $u \geq 0$ and harmonic on $\mathcal{C}$ then for some $C > 0$

$$u = C r^{\alpha_1} f_1(\theta).$$

Main Result

Theorem (Allen, Sh.)

Let $\mathcal{C}$ be as above. Let u be a positive harmonic function in $\mathcal{C}$ with zero boundary values on $(\partial\mathcal{C}) \cap B_1$ and v satisfy

$$\begin{aligned} 0 \geq \Delta v(x) \geq -C_0|x|^\gamma \quad & \text{in } \mathcal{C} \cap B_1, \\ v = 0 \quad & \text{on } \partial\mathcal{C} \cap B_1, \\ |v| \leq C_0 \quad & \text{in } \mathcal{C} \cap B_1, \end{aligned}$$

with $2 - \alpha_1 + \gamma > 0$. If $x^0 \in \mathcal{C} \cap B_1$, then $\exists$ C: [4]

$$\frac{v(x)}{u(x)} \leq C \frac{v(x^0)}{u(x^0)} \text{ for any } x \in \mathcal{C} \cap B_{1/2}.$$

[4] depending only on $\mathcal{C}$, $2 - \alpha_1 + \gamma$, dimension n, and dist$(x^0, \partial(\mathcal{C} \cap B_1))$

Main Result

Theorem (Allen, Sh.)

Let $\mathcal{C}$ be as above. Let u be a positive harmonic function in $\mathcal{C}$ with zero boundary values on $(\partial\mathcal{C}) \cap B_1$ and v satisfy

$$\begin{aligned} 0 \geq \Delta v(x) \geq -C_0|x|^\gamma \quad & \text{in } \mathcal{C} \cap B_1, \\ v = 0 \quad & \text{on } \partial\mathcal{C} \cap B_1, \\ |v| \leq C_0 \quad & \text{in } \mathcal{C} \cap B_1, \end{aligned}$$

with $2 - \alpha_1 + \gamma > 0$. If $x^0 \in \mathcal{C} \cap B_1$, then $\exists\ C$:[4]

$$\frac{v(x)}{u(x)} \leq C \frac{v(x^0)}{u(x^0)} \quad \text{for any } x \in \mathcal{C} \cap B_{1/2}.$$

[4] depending only on $\mathcal{C}$, $2 - \alpha_1 + \gamma$, dimension n, and $\mathrm{dist}(x^0, \partial(\mathcal{C} \cap B_1))$

Main Result

Theorem (Allen, Sh.)

Let $\mathcal{C}$ be as above. Let u be a positive harmonic function in $\mathcal{C}$ with zero boundary values on $(\partial\mathcal{C}) \cap B_1$ and v satisfy

$$\begin{aligned} 0 \geq \Delta v(x) \geq -C_0|x|^{\gamma} &\quad \text{in } \mathcal{C} \cap B_1, \\ v = 0 &\quad \text{on } \partial\mathcal{C} \cap B_1, \\ |v| \leq C_0 &\quad \text{in } \mathcal{C} \cap B_1, \end{aligned}$$

with $2 - \alpha_1 + \gamma > 0$. If $x^0 \in \mathcal{C} \cap B_1$, then $\exists\, C$: [4]

$$\frac{v(x)}{u(x)} \leq C \frac{v(x^0)}{u(x^0)} \text{ for any } x \in \mathcal{C} \cap B_{1/2}.$$

[4] depending only on $\mathcal{C}, 2 - \alpha_1 + \gamma$, dimension n, and $\mathrm{dist}(x^0, \partial(\mathcal{C} \cap B_1))$

Main Result

Theorem (Sharpness)

Let $\mathcal{C}$ be a cone in $\mathbb{R}^n$ with $2 - \alpha_1 + \gamma \leq 0$. Then the boundary Harnack principle with right hand side does not hold.

Lipschitz domains

The result holds for general Lipschitz domains, with small Lipschitz constant, that is given by the same condition as above, replacing $|x|$ with $dist(x, \partial D)$, and $2 - \alpha_1 + \gamma > 0$, for every boundary point.

Main Result

Theorem (Sharpness)

Let $\mathcal{C}$ be a cone in $\mathbb{R}^n$ with $2 - \alpha_1 + \gamma \leq 0$. Then the boundary Harnack principle with right hand side does not hold.

Lipschitz domains

The result holds for general Lipschitz domains, with small Lipschitz constant, that is given by the same condition as above, replacing $|x|$ with $dist(x, \partial D)$, and $2 - \alpha_1 + \gamma > 0$, for every boundary point.

Main Result

General cases

The proof employs the following standard techniques:

- Compactness methods
- Behavior of a nonnegative harmonic functions at the boundary
- A Liouville type result which is slightly non-standard
- Properties of the domain should be invariant under scaling
- PDEs converge to clean case

The approach seems plausible to be applied to more general settings: Various PDEs, Parabolic problems, as well as more general domains.

Main Result

General cases

The proof employs the following standard techniques:

- Compactness methods
- Behavior of a nonnegative harmonic functions at the boundary
- A Liouville type result which is slightly non-standard
- Properties of the domain should be invariant under scaling
- PDEs converge to clean case

The approach seems plausible to be applied to more general settings: Various PDEs, Parabolic problems, as well as more general domains.

Main Result

General cases

The proof employs the following standard techniques:

- Compactness methods
- Behavior of a nonnegative harmonic functions at the boundary
- A Liouville type result which is slightly non-standard
- Properties of the domain should be invariant under scaling
- PDEs converge to clean case

The approach seems plausible to be applied to more general settings: Various PDEs, Parabolic problems, as well as more general domains.

Main Result

General cases

The proof employs the following standard techniques:

- Compactness methods
- Behavior of a nonnegative harmonic functions at the boundary
- A Liouville type result which is slightly non-standard
- Properties of the domain should be invariant under scaling
- PDEs converge to clean case

The approach seems plausible to be applied to more general settings: Various PDEs, Parabolic problems, as well as more general domains.

Main Result

General cases

The proof employs the following standard techniques:

- Compactness methods
- Behavior of a nonnegative harmonic functions at the boundary
- A Liouville type result which is slightly non-standard
- Properties of the domain should be invariant under scaling
- PDEs converge to clean case

The approach seems plausible to be applied to more general settings: Various PDEs, Parabolic problems, as well as more general domains.

Main Result

General cases

The proof employs the following standard techniques:

- Compactness methods
- Behavior of a nonnegative harmonic functions at the boundary
- A Liouville type result which is slightly non-standard
- Properties of the domain should be invariant under scaling
- PDEs converge to clean case

The approach seems plausible to be applied to more general settings: Various PDEs, Parabolic problems, as well as more general domains.

Main Result

Hölder regularity of the quotient

For Lipschitz domains, with small Lip. norm from inside, there exists $\beta > 0$ depending on (Lip.-norm) such that

$$\left\|\frac{v}{u}\right\|_{C^{0,\beta}(B_{1/2}\cap D)} \leq C\frac{\left(\|v\|_{L^\infty(D)} + \|f\|_{L^\infty(D)}\right)}{u(e_n/2)}. \tag{3}$$

Theorem (Sharpness)

Let $\mathcal{C}$ be a cone in $\mathbb{R}^n$ with $\alpha_1 \geq 2 + \gamma$. Then the boundary Harnack principle with right hand side does not hold.

Main Result

Hölder regularity of the quotient

For Lipschitz domains, with small Lip. norm from inside, there exists $\beta > 0$ depending on (Lip.-norm) such that

$$\left\| \frac{v}{u} \right\|_{C^{0,\beta}(B_{1/2} \cap D)} \leq C \frac{\left(\|v\|_{L^\infty(D)} + \|f\|_{L^\infty(D)} \right)}{u(e_n/2)}. \tag{3}$$

Theorem (Sharpness)

Let $\mathcal{C}$ be a cone in $\mathbb{R}^n$ with $\alpha_1 \geq 2 + \gamma$. Then the boundary Harnack principle with right hand side does not hold.

Higher order

A result of De Silva and Savin:

If $\partial\Omega \in C^{k,\beta}$ with $\Delta u = 0$ and $\Delta v = f$ with $u > 0$ and both u, v vanishing on $\partial\Omega \cap B_1$, then

$$\left\|\frac{v}{u}\right\|_{C^{k,\beta}}(\Omega \cap B_{1/2}) \leq C(\|v\|_{L^\infty} + \|f\|_{C^{k-1,\beta}}).$$

Application

Regularity of Free Boundaries: Obstacle problem

Obstacle problem: Definition

Let v be a solution to the obstacle problem

$$\Delta v = h\chi_{\{v>0\}}, \qquad v \geq 0 \qquad \text{in } B_1.$$

We assume $h \geq c_0 > 0$ is Lipschitz, and a Dirichlet data on ∂B_1 has been prescribed.

$\boxed{\text{The domain } D \text{ is now } \{v > 0\}.}$

Regularity of Free Boundaries: Obstacle problem

Obstacle problem: Definition

Let v be a solution to the obstacle problem

$$\Delta v = h\chi_{\{v>0\}}, \qquad v \geq 0 \qquad \text{in } B_1.$$

We assume $h \geq c_0 > 0$ is Lipschitz, and a Dirichlet data on ∂B_1

has been prescribed.

$\boxed{\text{The domain } D \text{ is now } \{v > 0\}.}$

Regularity of Free Boundaries: Obstacle problem

Obstacle problem: Definition

Let v be a solution to the obstacle problem

$$\Delta v = h\chi_{\{v>0\}}, \qquad v \geq 0 \qquad \text{in } B_1.$$

We assume $h \geq c_0 > 0$ is Lipschitz, and a Dirichlet data on ∂B_1 has been prescribed.

$\boxed{\text{The domain } D \text{ is now } \{v > 0\}.}$

Regularity of Free Boundaries: Obstacle problem

Lipschitz FB implies $C^{1,\alpha}$

If $z \in \partial\{v > 0\} \cap B_{1/2}$ is not a cusp point, then for some $r > 0$ and direction e, $v_e > 0$ in the set $\{v > 0\} \cap B_r(z)$, and that the free boundary is Lipschitz in $B_r(z)$.

The BHP with r.h.s. allows us to deduce $C^{1,\alpha}$-regularity of the free boundary for the obstacle problem, in an elementary way.[5]

[5] For constant h, this is due to Athanasopoulos-Caffarelli (1985)

Regularity of Free Boundaries: Obstacle problem

Lipschitz FB implies $C^{1,\alpha}$

If $z \in \partial\{v > 0\} \cap B_{1/2}$ is not a cusp point, then for some $r > 0$ and direction e, $v_e > 0$ in the set $\{v > 0\} \cap B_r(z)$, and that the free boundary is Lipschitz in $B_r(z)$.

The BHP with r.h.s. allows us to deduce $C^{1,\alpha}$-regularity of the free boundary for the obstacle problem, in an elementary way.[5]

[5] For constant h, this is due to Athanasopoulos-Caffarelli (1985)

Regularity of Free Boundaries: Obstacle problem

How does it work?

Set $H(x) := v_{e_1} - Cv$, which satisfies[6]

$$H > 0, \qquad \Delta H = h_{e_1} - Ch \leq 0 \qquad \text{in } \{v > 0\} \cap B_r(z).$$

Apply our BHP to $H = v_{e_1} - Cv$, and v_e,[7] where $e \perp e_1$, and $\gamma = 0$. This implies, for r small

$$\frac{v_e}{v_{e_1} - Cv} = \frac{v_e}{H} \in C^\alpha(B_r(z) \cap \{v > 0\}). \tag{4}$$

[6] This conclusion is part of proving the Lipschitz regularity of the free boundary.

[7] Actually we apply BHP to harmonic minorant $\tilde{H} \leq H$ and and to $\tilde{v}_e \geq v_e$ solving the PDE $\Delta \tilde{v}_e = -f_e^+$.

How does it work?

Set $H(x) := v_{e_1} - Cv$, which satisfies[6]

$$H > 0, \qquad \Delta H = h_{e_1} - Ch \leq 0 \qquad \text{in } \{v > 0\} \cap B_r(z).$$

Apply our BHP to $H = v_{e_1} - Cv$, and v_e,[7] where $e \perp e_1$, and $\gamma = 0$. This implies, for r small

$$\frac{v_e}{v_{e_1} - Cv} = \frac{v_e}{H} \in C^\alpha(B_r(z) \cap \{v > 0\}). \tag{4}$$

---

[6] This conclusion is part of proving the Lipschitz regularity of the free boundary.

[7] Actually we apply BHP to harmonic minorant $\tilde{H} \leq H$ and and to $\tilde{v}_e \geq v_e$ solving the PDE $\Delta \tilde{v}_e = -f_e^+$.

Regularity of Free Boundaries: Obstacle problem

How does it work?

Next fix a level surface[8] $v = l$, and denote this surface by $x_1 = G(x')$.

Differentiating both side of

$$v(x_1 - G(x'), x') = l$$

gives

$$G_e = \frac{v_e}{v_{e_1}}.$$

We want to show G_e is C^α for all directions $e \in \mathbb{R}^{n-1}$.

[8] The level surface is smooth since $v_{e_1} > 0$ there.

Regularity of Free Boundaries: Obstacle problem

How does it work?

Next fix a level surface[8] $v = l$, and denote this surface by $x_1 = G(x')$.

Differentiating both side of

$$v(x_1 - G(x'), x') = l$$

gives

$$G_e = \frac{v_e}{v_{e_1}}.$$

We want to show G_e is C^α for all directions $e \in \mathbb{R}^{n-1}$.

[8] The level surface is smooth since $v_{e_1} > 0$ there.

Regularity of Free Boundaries: Obstacle problem

How does it work?

Next fix a level surface[8] $v = l$, and denote this surface by $x_1 = G(x')$.

Differentiating both side of

$$v(x_1 - G(x'), x') = l$$

gives

$$G_e = \frac{v_e}{v_{e_1}}.$$

We want to show G_e is C^α for all directions $e \in \mathbb{R}^{n-1}$.

[8] The level surface is smooth since $v_{e_1} > 0$ there.

How does it work?

Rephrasing (4) and inserting this gives us

$$\frac{v_e}{v_{e_1} - Cv} = \frac{\frac{v_e}{v_{e_1}}}{1 - \frac{Cv}{v_{e_1}}} = \frac{G_e}{1 - \frac{Cl}{v_{e_1}}},$$

is C^α, independent of l.

Since $v_{e_1} \approx \sqrt{l}$ we have that

$$\frac{v_e}{v_{e_1} - Cv} = \frac{G_e}{1 - \frac{Cl}{v_{e_1}}} \to G_e, \qquad \text{as } l \to 0.$$

$\boxed{\text{Hence } G_e \text{ is } C^\alpha.}$

Regularity of Free Boundaries: Obstacle problem

How does it work?

Rephrasing (4) and inserting this gives us

$$\frac{v_e}{v_{e_1} - Cv} = \frac{\frac{v_e}{v_{e_1}}}{1 - \frac{Cv}{v_{e_1}}} = \frac{G_e}{1 - \frac{Cl}{v_{e_1}}},$$

is C^α, independent of l.

Since $v_{e_1} \approx \sqrt{l}$ we have that

$$\frac{v_e}{v_{e_1} - Cv} = \frac{G_e}{1 - \frac{Cl}{v_{e_1}}} \to G_e, \qquad \text{as } l \to 0.$$

Hence G_e is C^α.

Regularity of Free Boundaries: Obstacle problem

How does it work?

Rephrasing (4) and inserting this gives us

$$\frac{v_e}{v_{e_1} - Cv} = \frac{\frac{v_e}{v_{e_1}}}{1 - \frac{Cv}{v_{e_1}}} = \frac{G_e}{1 - \frac{Cl}{v_{e_1}}},$$

is C^α, independent of l.

Since $v_{e_1} \approx \sqrt{l}$ we have that

$$\frac{v_e}{v_{e_1} - Cv} = \frac{G_e}{1 - \frac{Cl}{v_{e_1}}} \to G_e, \qquad \text{as } l \to 0.$$

$$\boxed{\text{Hence } G_e \text{ is } C^\alpha.}$$

Speculations

Obstacle problem with continuous RHS

Let h^ϵ and v^ϵ be smooth approximation of RHS and the solution, respectively.

Assume[9] the approximate FB is as Lipschitz as the original problem.

Hence $C^{1,\alpha}$-regularity of FB for each ϵ will follow from our theorem, and the norm is independent of ϵ. So the limit problem has the same property!

Speculation: Continuous RHS, and FB Lip. implies FB $C^{1,\alpha}$.

[9] This is the unclear and probably the hard part, that one needs to show.

Speculations

Obstacle problem with continuous RHS

Let h^ϵ and v^ϵ be smooth approximation of RHS and the solution, respectively.

Assume[9] the approximate FB is as Lipschitz as the original problem.

Hence $C^{1,\alpha}$-regularity of FB for each ϵ will follow from our theorem, and the norm is independent of ϵ. So the limit problem has the same property!

Speculation: Continuous RHS, and FB Lip. implies FB $C^{1,\alpha}$.

[9] This is the unclear and probably the hard part, that one needs to show.

Speculations

Obstacle problem with continuous RHS

Let h^ϵ and v^ϵ be smooth approximation of RHS and the solution, respectively.

Assume[9] the approximate FB is as Lipschitz as the original problem.

Hence $C^{1,\alpha}$-regularity of FB for each ϵ will follow from our theorem, and the norm is independent of ϵ. So the limit problem has the same property!

Speculation: Continuous RHS, and FB Lip. implies FB $C^{1,\alpha}$.

[9] This is the unclear and probably the hard part, that one needs to show.

Speculations

Obstacle problem with continuous RHS

Let h^ϵ and v^ϵ be smooth approximation of RHS and the solution, respectively.

Assume[9] the approximate FB is as Lipschitz as the original problem.

Hence $C^{1,\alpha}$-regularity of FB for each ϵ will follow from our theorem, and the norm is independent of ϵ. So the limit problem has the same property!

Speculation: Continuous RHS, and FB Lip. implies FB $C^{1,\alpha}$.

[9] This is the unclear and probably the hard part, that one needs to show.

Proof of the theorem (Sketch)

Recalling the Theorem

Theorem (Allen, Sh.)

Let $u = v = 0$ on $\partial\mathcal{C} \cap B_1$, with u positive harmonic and

$$\begin{aligned} 0 \geq \Delta v(x) \geq -C_0|x|^\gamma \qquad & \text{in } \mathcal{C} \cap B_1, \\ |v| \leq C_0 \qquad & \text{in } \mathcal{C} \cap B_1, \end{aligned}$$

with $2 - \alpha_1 + \gamma > 0$. If $x^0 \in \mathcal{C} \cap B_1$, then $\exists$ C: [10]

$$\frac{v(x)}{u(x)} \leq C \frac{v(x^0)}{u(x^0)} \text{ for any } x \in \mathcal{C} \cap B_{1/2}.$$

[10] depending only on $\mathcal{C}$, $2 - \alpha_1 + \gamma$, dimension n, and $\text{dist}(x^0, \partial(\mathcal{C} \cap B_1))$

Proof of Theorem

Non-negative harmonic functions on cones

Recall that if u is any non-negative harmonic function on $\mathcal{C}$ with $u = 0$ on $\partial\mathcal{C}$, then (up to a multiplicative constant)

$$u(x) := u(r,\theta) = r^{\alpha_1} f_1(\theta), \tag{5}$$

where f_1 is the first eigenfunction to the Laplace-Beltrami on $\mathcal{C} \cap \partial B_1$.

Proof of Theorem

Simplification

Fix $x^0 \in \mathcal{C} \cap B_{1/2}$.

1) Since $u \geq 0$, by standard BHP we replace u with $r^{\alpha_1} f(\theta)$.

2) By the comparison principle we also replace v by a solution to $\Delta v = -|x|^\gamma$.

3) It suffices to show: $v(rx^0) \leq Cu(rx^0)$ for all $0 < r \leq 1/2$.

4) Apply 3) to any ray emanating from a boundary point.

Proof of Theorem

Simplification

Fix $x^0 \in \mathcal{C} \cap B_{1/2}$.

1) Since $u \geq 0$, by standard BHP we replace u with $r^{\alpha_1} f(\theta)$.

2) By the comparison principle we also replace v by a solution to $\Delta v = -|x|^\gamma$.

3) It suffices to show: $v(rx^0) \leq Cu(rx^0)$ for all $0 < r \leq 1/2$.

4) Apply 3) to any ray emanating from a boundary point.

Proof of Theorem

Simplification

Fix $x^0 \in \mathcal{C} \cap B_{1/2}$.

1) Since $u \geq 0$, by standard BHP we replace u with $r^{\alpha_1} f(\theta)$.

2) By the comparison principle we also replace v by a solution to $\Delta v = -|x|^\gamma$.

3) It suffices to show: $v(rx^0) \leq Cu(rx^0)$ for all $0 < r \leq 1/2$.

4) Apply 3) to any ray emanating from a boundary point.

Proof of Theorem

Simplification

Fix $x^0 \in \mathcal{C} \cap B_{1/2}$.

1) Since $u \geq 0$, by standard BHP we replace u with $r^{\alpha_1} f(\theta)$.

2) By the comparison principle we also replace v by a solution to $\Delta v = -|x|^\gamma$.

3) It suffices to show: $v(rx^0) \leq Cu(rx^0)$ for all $0 < r \leq 1/2$.

4) Apply 3) to any ray emanating from a boundary point.

Proof of Theorem

Blow-up device and indirect argument

Consider the function

$$w_r(x) := \frac{v(rx) - \frac{v(rx^0)}{u(rx^0)} u(rx)}{\sup_{B_1 \cap \mathcal{C}} |v(rx) - \frac{v(rx^0)}{u(rx^0)} u(rx)|},$$

defined on $B_{1/r}$, and use indirect argument.

Proof of Theorem

Properties of w_r

i) $w_r(x^0) = 0$ (by inspection).

ii) $\sup_{B_1 \cap \mathcal{C}} |w_r| = 1$ (by inspection).

iii) The indirect argument implies $\exists\ r_k \to 0$ such that

$$\sup_{B_{2^j}} |w_{r_k}(x)| \leq Cj2^{j\alpha_1} \qquad \text{for } j = 1, 2, \ldots$$

iv) Use $2 - \alpha_1 + \gamma > 0$, and indirect argument to show

$$|\Delta w_{r_k}(x)| \leq Cr_k^{2-\alpha_1+\gamma}[\ln(1/r_k)]^2|x|^\gamma \quad \to \ 0.$$

Proof of Theorem

Properties of w_r

i) $w_r(x^0) = 0$ (by inspection).

ii) $\sup_{B_1 \cap \mathcal{C}} |w_r| = 1$ (by inspection).

iii) The indirect argument implies $\exists\ r_k \to 0$ such that

$$\sup_{B_{2^j}} |w_{r_k}(x)| \leq Cj2^{j\alpha_1} \qquad \text{for } j = 1, 2, \ldots$$

iv) Use $2 - \alpha_1 + \gamma > 0$, and indirect argument to show

$$|\Delta w_{r_k}(x)| \leq Cr_k^{2-\alpha_1+\gamma}[\ln(1/r_k)]^2|x|^\gamma \quad \to \quad 0.$$

Proof of Theorem

Properties of w_r

i) $w_r(x^0) = 0$ (by inspection).

ii) $\sup_{B_1 \cap \mathcal{C}} |w_r| = 1$ (by inspection).

iii) The indirect argument implies $\exists\ r_k \to 0$ such that

$$\sup_{B_{2^j}} |w_{r_k}(x)| \leq Cj2^{j\alpha_1} \qquad \text{for } j = 1, 2, \ldots$$

iv) Use $2 - \alpha_1 + \gamma > 0$, and indirect argument to show

$$|\Delta w_{r_k}(x)| \leq Cr_k^{2-\alpha_1+\gamma}[\ln(1/r_k)]^2|x|^\gamma \quad \to \quad 0.$$

Proof of Theorem

Properties of w_r

i) $w_r(x^0) = 0$ (by inspection).

ii) $\sup_{B_1 \cap \mathcal{C}} |w_r| = 1$ (by inspection).

iii) The indirect argument implies $\exists\ r_k \to 0$ such that

$$\sup_{B_{2^j}} |w_{r_k}(x)| \leq Cj2^{j\alpha_1} \qquad \text{for } j = 1, 2, \ldots$$

iv) Use $2 - \alpha_1 + \gamma > 0$, and indirect argument to show

$$|\Delta w_{r_k}(x)| \leq C r_k^{2-\alpha_1+\gamma} [\ln(1/r_k)]^2 |x|^\gamma \quad \to \quad 0.$$

Proof of Theorem

Properties of w_r

i) $w_r(x^0) = 0$ (by inspection).

ii) $\sup_{B_1 \cap \mathcal{C}} |w_r| = 1$ (by inspection).

iii) The indirect argument implies $\exists\ r_k \to 0$ such that

$$\sup_{B_{2^j}} |w_{r_k}(x)| \leq Cj2^{j\alpha_1} \qquad \text{for } j = 1, 2, \ldots$$

iv) Use $2 - \alpha_1 + \gamma > 0$, and indirect argument to show

$$|\Delta w_{r_k}(x)| \leq C r_k^{2-\alpha_1+\gamma} [\ln(1/r_k)]^2 |x|^\gamma \quad \to \ 0.$$

Proof of Theorem

The blow-up limit

The limit function $w = \lim_k w_{r_k}$ will satisfy

i) $w(x^0) = 0$,

ii) $\sup_{B_1 \cap \mathcal{C}} |w| = 1$,

iii) $w(x) \leq C|x|^{\alpha_1} \ln(|x| + 1) \qquad \text{for } |x| \geq 1.$

iv) $\Delta w = 0.$

Proof of Theorem

The blow-up limit

The limit function $w = \lim_k w_{r_k}$ will satisfy

i) $w(x^0) = 0$,

ii) $\sup_{B_1 \cap \mathcal{C}} |w| = 1$,

iii) $w(x) \leq C|x|^{\alpha_1} \ln(|x| + 1) \qquad \text{for } |x| \geq 1.$

iv) $\Delta w = 0.$

Proof of Theorem

The blow-up limit

The limit function $w = \lim_k w_{r_k}$ will satisfy

i) $w(x^0) = 0$,

ii) $\sup_{B_1 \cap \mathcal{C}} |w| = 1$,

iii) $w(x) \le C|x|^{\alpha_1} \ln(|x| + 1) \qquad \text{for } |x| \ge 1.$

iv) $\Delta w = 0.$

Proof of Theorem

The blow-up limit

The limit function $w = \lim_k w_{r_k}$ will satisfy

i) $w(x^0) = 0$,

ii) $\sup_{B_1 \cap \mathcal{C}} |w| = 1$,

iii) $w(x) \leq C|x|^{\alpha_1} \ln(|x| + 1) \qquad \text{for } |x| \geq 1.$

iv) $\Delta w = 0$.

Proof of Theorem

The blow-up limit

The limit function $w = \lim_k w_{r_k}$ will satisfy

i) $w(x^0) = 0$,

ii) $\sup_{B_1 \cap \mathcal{C}} |w| = 1$,

iii) $w(x) \le C|x|^{\alpha_1} \ln(|x| + 1) \qquad \text{for } |x| \ge 1.$

iv) $\Delta w = 0.$

Proof of Theorem

Contradiction and conclusion

By property ii) we have that w is not identically zero.

By i), and iv) w changes sign, so that by (2) [11] we have

$$\sup_{B_R} |w| \geq CR^{\alpha_2}, \qquad \text{for } R \geq 1. \tag{6}$$

This, along with (iii) and that $\alpha_2 > \alpha_1$ gives us a contradiction:

$$CR^{\alpha_2} \leq C \sup_{B_R} w(x) \leq \sup_{B_R} |x|^{\alpha_1} \ln(|x|+1) = o(R^{\alpha_2}), \quad \text{for large } R.$$

Hence the theorem follows.

[11] Recall that (2) says $u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta)$.

Proof of Theorem

Contradiction and conclusion

By property ii) we have that w is not identically zero.

By i), and iv) w changes sign, so that by (2) [11] we have

$$\sup_{B_R} |w| \geq CR^{\alpha_2}, \qquad \text{for } R \geq 1. \tag{6}$$

This, along with (iii) and that $\alpha_2 > \alpha_1$ gives us a contradiction:

$$CR^{\alpha_2} \leq C \sup_{B_R} w(x) \leq \sup_{B_R} |x|^{\alpha_1} \ln(|x|+1) = o(R^{\alpha_2}), \quad \text{for large } R.$$

$\boxed{\text{Hence the theorem follows.}}$

[11] Recall that (2) says $u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta)$.

Proof of Theorem

Contradiction and conclusion

By property ii) we have that w is not identically zero.

By i), and iv) w changes sign, so that by (2) [11] we have

$$\sup_{B_R} |w| \geq CR^{\alpha_2}, \qquad \text{for } R \geq 1. \tag{6}$$

This, along with (iii) and that $\alpha_2 > \alpha_1$ gives us a contradiction:

$$CR^{\alpha_2} \leq C \sup_{B_R} w(x) \leq \sup_{B_R} |x|^{\alpha_1} \ln(|x|+1) = o(R^{\alpha_2}), \quad \text{for large } R.$$

$\boxed{\text{Hence the theorem follows.}}$

[11] Recall that (2) says $u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta)$.

Proof of Theorem

Contradiction and conclusion

By property ii) we have that w is not identically zero.

By i), and iv) w changes sign, so that by (2) [11] we have

$$\sup_{B_R} |w| \geq CR^{\alpha_2}, \qquad \text{for } R \geq 1. \tag{6}$$

This, along with (iii) and that $\alpha_2 > \alpha_1$ gives us a contradiction:

$$CR^{\alpha_2} \leq C \sup_{B_R} w(x) \leq \sup_{B_R} |x|^{\alpha_1} \ln(|x|+1) = o(R^{\alpha_2}), \quad \text{for large } R.$$

Hence the theorem follows.

[11] Recall that (2) says $u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta)$.

Proof of Theorem

Contradiction and conclusion

By property ii) we have that w is not identically zero.

By i), and iv) w changes sign, so that by (2) [11] we have

$$\sup_{B_R} |w| \geq CR^{\alpha_2}, \qquad \text{for } R \geq 1. \tag{6}$$

This, along with (iii) and that $\alpha_2 > \alpha_1$ gives us a contradiction:

$$CR^{\alpha_2} \leq C \sup_{B_R} w(x) \leq \sup_{B_R} |x|^{\alpha_1} \ln(|x|+1) = o(R^{\alpha_2}), \quad \text{for large } R.$$

$\boxed{\text{Hence the theorem follows.}}$

[11] Recall that (2) says $u(r,\theta) = \sum_{k=1}^{\infty} r^{\alpha_k} f_k(\theta)$.

Thank you for your attention

For open problems visit www.scilag.net