How a minimal surface leaves a thin obstacle

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The thin obstacle problem for nonparametric minimal surfaces

Notation: $x = (x_1, \dots, x_{n+1}) = (x', x_{n+1}) \in B_1 \subset \mathbb{R}^{n+1}$,

$B_1' = B_1 \cap \{x_{n+1} = 0\}$, $B_1^+ = B_1 \cap \{x_{n+1} > 0\}$

The problem: $\min_{v \in \mathcal{A}_g} \int_{B_1} \sqrt{1 + |\nabla v|^2}\, dx$

where

- $g \in C^2(B_1)$ is given such that

$$g(x', x_{n+1}) = g(x', -x_{n+1}) \quad \text{and} \quad g|_{B_1'} \geq 0;$$

- $\mathcal{A}_g := \Big\{ v \in \mathrm{Lip}(B_1) : v|_{\partial B_1} = g|_{\partial B_1}, v(x', x_{n+1}) = v(x', -x_{n+1}), v|_{B_1'} \geq 0 \Big\}.$

Thin vs Classical obstacle problem

Classical obstacle problem

$$\min_{v \in \mathcal{B}_{\phi,g}} \int_{B_1} \sqrt{1 + |\nabla v|^2}\, dx$$

where

- $\phi : B_1 \to \mathbb{R}$ is a given obstacle;
- $g \in C^2(B_1)$ is given such that $g|_{\partial B_1} \geq \phi$;
- $\mathcal{B}_{\phi,g} := \Big\{ v \in \mathrm{Lip}(B_1) : v|_{\partial B_1} = g|_{\partial B_1},\ v \geq \phi \Big\}$.

Thin obstacle because the unilateral constraint is imposed only on a lower dimensional space:

$$v|_{B_1'} \geq 0.$$

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"How to fashion a cheap hat for Giacometti's brother"

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"One may be inclined to hold that there are no fellow creatures with shapes such as to motivate our present investigation.
Alberto Giacometti's superb portrait of his brother Diego, which is depicted in the figure below, provides evidence to the contrary."

The thin obstacle problem

$$\min_{v\in\mathcal{A}_g} \quad \int_{B_1} \sqrt{1+|\nabla v|^2}\, dx$$

$$\mathcal{A}_g = \left\{ v \in \mathrm{Lip}(B_1) \,:\, v|_{\partial B_1} = g|_{\partial B_1},\ v(x', x_{n+1}) = v(x', -x_{n+1}),\ v|_{B_1'} \geq 0 \right\}$$

Main questions

- EXISTENCE AND UNIQUENESS

- REGULARITY:
  - of the solution u;
  - of the *free boundary* $\Gamma(u)$: the boundary of the contact set

    $$\Lambda(u) := \left\{ (x', 0) \in B_1' \,:\, u(x', 0) = 0 \right\}$$

    in the relative topology of B_1', i.e. $\Gamma(u) = \partial_{B_1'} \Lambda(u)$.

Broader context: the scalar Signorini problem

The scalar Signorini problem, introduced in the '50s, is a simplified model for elastic bodies at rest on a surface and it consists in minimizing the linearized Dirichlet energy

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It arises in several contexts in applied mathematics and it is related to nonlocal operators, because $v|_{B_1'}$ can be interpreted as the localization of the solution to the obstacle problem for $(-\Delta)^{\frac{1}{2}}$.

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More generally, one can consider the weighted energies

$$\min_{v \in \mathcal{A}_g} \int_{B_1} |\nabla v|^2 |x_{n+1}|^a \, dx, \quad a \in (-1,1)$$

$v|_{B_1'}$ is the localization of the solution to the obstacle problem for $(-\Delta)^s$ with $s = \frac{1-a}{2}$.

Previous results

Signorini problem

- Existence: direct method.
- Regularity of u:
  - $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1^+ \cup B_1')$ for some $\alpha > 0$: Caffarelli '79;
  - $u \in C^{1,\frac{1}{2}}_{\mathrm{loc}}(B_1^+ \cup B_1')$ optimal: Athanasopoulos and Caffarelli '04

$$h(x) = \mathrm{Re}\left[(x_n + i\,|x_{n+1}|)^{\frac{3}{2}}\right] \text{ with } h|_{B_1'} \geq 0$$

- Regularity of $\Gamma(u)$:
  - *Regular points*: Athanasopoulos, Caffarelli & Salsa '08;
  - *Singular points*: Garofalo & Petrosyan '09; Colombo, Spolaor & Velichkov '18.
  - *Other points*: Focardi – S. '18.

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Nitsche's problem

- Existence: Giusti '71-'72;
- Regularity of u:
  - $\partial_i u \in C^1_{\text{loc}}(B_1^+ \cup B_1')$ if $i \in \{1, \dots, n\}$, : Frehse '77.
  - $\nabla u \in C^1_{\text{loc}}(B_1^+ \cup B_1')$ if $n = 1$: Frehse '77.
- Regularity of $\Gamma(u)$?

Main results

Theorem (Focardi and S. '18)

Let $u \in \mathrm{Lip}(B_1)$ be a solution to the thin obstacle problem for nonparametric minimal surfaces. Then,

(i) $u \in C^{1,\frac{1}{2}}_{\mathrm{loc}}(B_1^+ \cup B_1')$;

(ii) *the same regularity of $\Gamma(u)$ as for the Signorini problem holds: e.g.,*

$(\mathrm{ii})_1$ $\Gamma(u)$ *is has locally $\mathcal{H}^{n-1}$ finite measure and it is countably $\mathcal{H}^{n-1}$-rectifiable;*

$(\mathrm{ii})_2$ *the regular part of the free boundary $\mathrm{Reg}(u)$ is a $C^{1,\alpha}$ submanifold of dimension $n-1$ in B_1'.*

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- Regularity of u:

(A1) blowup analysis and C^1 regularity;
(B1) penalized problem and $W^{2,2}$ regularity;
(C1) De Giorgi's metheod and $C^{1,\alpha}$ regularity;
(D1) two valued minimal graphs and optimal regularity.

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  (D1) two valued minimal graphs and optimal regularity.

- Regularity of the free boundary $\Gamma(u)$:

  (A2) the frequency function and the classification of free boundary points;
  (B2) classification and rigidity of homogeneous solutions;
  (C2) spatial oscillation of the frequency;
  (D2) Naber–Valtorta's technique.

Regularity of u

Regularity of u: (A1) C^1-regularity

Proposition

Let $u \in W^{1,\infty}(B_1)$ be a solution to the thin obstacle problem. Then, $u|_{B_1^+ \cup B_1'} \in C^1_{\mathrm{loc}}(B_1^+ \cup B_1')$.

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Proof.

1. We show that for every $z_k \in \Gamma(u)$ and $t_k \downarrow 0$ it holds

$$u_k(x) := \frac{u(z_k + t_k x)}{t_k} \to 0 \quad \text{in } L^\infty.$$

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2. $\forall\, a > 0\ \exists\, \varepsilon_0 > 0$ s.t. $w_\varepsilon : B_1 \to \mathbb{R}$ solution to the thin obstacle problem with $w_\varepsilon|_{\partial B_1} = g_\varepsilon(x) = -a|x_{n+1}| + \varepsilon$ with $\varepsilon \in (0, \varepsilon_0)$ satisfies

$$w_\varepsilon|_{B'_{1/2}} \equiv 0.$$

Remark: for both points Frehse's result (1977) plays a crucial role. □

Regularity of u: (B1) the penalized problems

Let $\beta, \chi \in C^\infty(\mathbb{R})$ be s.t. $\forall\, t \in \mathbb{R}$

$$|t| - 1 \le |\beta(t)| \le |t| \quad \forall\, t \le 0, \quad \beta(t) = 0 \quad \forall\, t \ge 0, \quad \beta'(t) \ge 0\,,$$

$$\chi(t) = \begin{cases} 0 & \text{for } \ t \le \mathrm{Lip}(u), \\ \frac{1}{2}\,(t - 2\mathrm{Lip}(u))^2 & \text{for } \ t > 3\mathrm{Lip}(u), \end{cases} \qquad \chi''(t) \ge 0\,.$$

Consider

$$\beta_\varepsilon(t) := \varepsilon^{-1}\beta(t/\varepsilon), \quad F_\varepsilon(t) := \int_0^t \beta_\varepsilon(s)\, ds$$

and set

$$\mathscr{E}_\varepsilon(v) := \int_{B_1} \left(\sqrt{1 + |\nabla v|^2} + \chi(|\nabla v|)\right) dx + \int_{B_1'} F_\varepsilon(v(x', 0))\, dx'\,.$$

The unique minimizer $u_\varepsilon \in g + H_0^1(B_1)$ of $\mathscr{E}_\varepsilon$ satisfies

$$\int_{B_1^+} \Big(\frac{\nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}} + \chi'(|\nabla u_\varepsilon|)\,\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\Big) \cdot \nabla \eta\, dx + \int_{B_1'} \beta_\varepsilon(u_\varepsilon)\,\eta\, dx' = 0 \quad \forall\, \eta \in H_0^1(B_1).$$

Lemma

$u_\varepsilon \to u$ converge weakly in $H^1(B_1)$ as $\varepsilon \downarrow 0$

Regularity of u: (B1) $W^{2,2}$-regularity

Proposition

If either $v = u_\varepsilon$ or $v = u$

$$\int_{B_r^+(x_0)} |\nabla^2 v|^2 \, dx \leq C \int_{B_{2r}^+(x_0)} |\nabla' v|^2 \, dx \tag{1}$$

$\forall\, x_0 \in B_1^+ \cup B_1'$, $\forall\, 0 < r < \frac{1-|x_0|}{2}$, *for some* $C = C(n, g)$.

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The Euler-Lagrange conditions for u hold in the sense of traces:

$$\begin{cases} \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{2\partial_{x_{n+1}} u(x',0)}{\sqrt{1+|\nabla u|^2}} \mathcal{H}^n \llcorner B_1' & \mathscr{D}'(B_1), \\ \partial_{x_{n+1}} u(x',0) \le 0 & B_1', \\ u(x',0)\, \partial_{x_{n+1}} u(x',0) = 0 & B_1'. \end{cases}$$

Regularity of u: (C1) $C^{1,\alpha}_{\text{loc}}$-regularity

Proposition

Let u be the solution to the Signorini problem, then for some constant $C = C(n, g) > 0$ the function $v = \pm\partial_i u$, $i = 1, \dots, n+1$, satisfies for all $k \geq 0$

$$\int_{B_r^+(x_0)\cap\{v>k\}} |\nabla v|^2 \, dx \leq \frac{C}{r^2} \int_{B_{2r}^+(x_0)} (v-k)_+^2 \, dx$$

$\forall \; x_0 \in B_1', \; 2r < 1 - |x_0|$.

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Proof.

We follow the approach developed by Ural'tseva '87:

1. the nonlinearity does not allowed to pass into the limit when testing the equation satisfied by u_ε with $\eta = \partial_{n+1}[(-\partial_{n+1}u - k)_+\phi^2]$;
2. we use the one-sided continuity of $\partial_{n+1}u$ to suitably regularize the test.

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De Giorgi's method can be then employed to get

Corollary (Ural'tseva '87)

Let u be the solution to the Signorini problem, then $u \in C^{1,\alpha}_{loc}(B_1^+ \cup B_1')$ for some $\alpha \in (0,1)$.

Regularity of u: (D1) optimal $C^{1,1/2}_{\mathrm{loc}}$-regularity

- <u>Minimal two-valued graphs:</u> consider $U = \{u, -u\}$ and

$$\mathsf{G}_U := \big\{(x, \pm u(x)) : x \in B_1\big\}$$

naturally inherits the structure of rectifiable varifold.

Proposition

Let u be a solution to the thin obstacle problem. Then, $U = \{u, -u\}$ is a minimal two-valued graph, i.e.

$$\int_{\mathsf{G}_U} \mathrm{div}_{\mathsf{G}_U} Y \, d\mathcal{H}^n = 0 \qquad \forall\ Y \in C^\infty_c(B_1 \times \mathbb{R}).$$

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- A regularity result by Simon and Wickramasekera

Theorem (Simon – Wickramasekera '16)

$C^{1,\alpha}$ minimal two-valued graphs are $C^{1,\frac{1}{2}}$-regular.

Regularity of the free boundary $\Gamma(u)$

Monotonicity of the frequency

- Frequency function: for $x_0 \in B_1'$ and $r < 1 - |x_0|$, set

$$I_u(x_0, r) := \frac{r \int_{B_r(x_0)} \vartheta(x) |\nabla u|^2 dx}{\int_{\partial B_r(x_0)} \vartheta(x) u^2(x) \, d\mathcal{H}^{n-1}}$$

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Remark: by Schauder estimates $\vartheta \in \mathrm{Lip}_{\mathrm{loc}}(B_1)$, moreover, $\frac{1}{\sqrt{1+\mathrm{Lip}(u)^2}} \leq \vartheta \leq 1$ and $\vartheta(x) = 1 \; \forall x \in \Gamma(u)$.

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- Monotonicity

Proposition

Let u be a solution to the thin obstacle problem in B_1. Then, $\exists \, C = C(n, \mathrm{Lip}(u)) > 0$ s.t. $\forall x_0 \in B_1'$,

$$(0, 1 - |x_0|) \ni t \mapsto e^{C\,t} I_u(x_0, t) \quad \textit{is non-decreasing}$$

and $\lambda(x_0) := \lim_{t \downarrow 0} I_u(x_0, t)$ exists finite.

Regular, singular & other points

- Def. of regular points: $x_0 \in \mathrm{Reg}(u)$ if $\lambda(x_0) = 3/2$

  Signorini problem: $u(x) = \mathrm{Re}\left[(x_n + i\,|x_{n+1}|)^{\frac{3}{2}}\right]$

  - Athanasopoulos, Caffarelli & Salsa '08
  - Garofalo, Petrosyan & Smit Vega Garcia '16, Focardi & S. '16 (epiperimetric inequality)

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- Def. of singular points: $x_0 \in \mathrm{Sing}(u)$ if $\lambda(x_0) = 2m$ with $m \in \mathbb{N} \setminus \{0\}$

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- Def. of other points: $x_0 \in \mathrm{Other}(u)$ if $\lambda(x_0) \neq \frac{3}{2}, 2m$, with $m \in \mathbb{N} \setminus \{0\}$.

  Signorini problem: $u(x) = \mathrm{Re}\left[(x_n + i\,|x_{n+1}|)^{2m+\frac{1}{2}}\right]$

  - Focardi & S. '18

Rescalings & blowups

- Rescalings: $\forall x_0 \in \Gamma(u)$ and $\forall r \in (0, 1 - |x_0|)$, consider the rescalings

$$u_{x_0,r}(y) := \frac{r^{n/2} u(x_0 + ry)}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^n}.$$

By the monotonicity of the frequency, the functions $u_{x_0,r}$ converge (up to subsequences) in $C^{1,\alpha}_{loc}(B_1^+ \cup B_1')$, $\alpha < 1/2$, to some function $u_{x_0} \in C^{1,1/2}_{loc}(B_1^+ \cup B_1')$ as $r \downarrow 0$.

- Blowups: the limiting functions u_{x_0} are

1. solution to the Signorini problem;
2. $\lambda(x_0)$-homogeneous, as $I_{u_{x_0}}(\underline{0}, \rho) = \lambda(x_0) \ \forall \rho \in (0, 1)$.

Structure of the free boundary

Theorem (Focardi & S. '18)

Let u be a solution to the thin obstacle problem for the nonparametric area functional. Then,

(i) $\Gamma(u)$ *has locally finite Minkowski's content of dimension* $(n-1)$*:* $\forall K \subset\subset B_1'$, $\exists\ C(K) > 0$ *s.t.*

$$\mathcal{L}^{n+1}\big(\mathcal{T}_r(\Gamma(u) \cap K)\big) \leq C(K)\, r^2 \quad \forall\ r \in (0,1),$$

where $\mathcal{T}_r(E) := \{x \in \mathbb{R}^{n+1} : \mathrm{dist}(x, E) < r\}$;

(ii) $\Gamma(u)$ *is countably* $\mathcal{H}^{n-1}$*-rectifiable, i.e.* $(n-1)$*-dimensional submanifolds* $\exists \{M_i\}_{i \in \mathbb{N}}$ *of class* C^1 *s.t.*

$$\mathcal{H}^{n-1}\Big(\Gamma(u) \setminus \bigcup_{i \in \mathbb{N}} M_i\Big) = 0;$$

(iii) $\exists\, \Sigma(u) \subseteq \mathrm{Other}(u)$, *with* $\dim_{\mathcal{H}} \Sigma(u) \leq n-2$, *s.t.* $\forall\ x_0 \in \mathrm{Other}(u) \setminus \Sigma(u)$

$$\lambda(x_0) \in \Big\{2m - {}^1\!/_2\Big\}_{m \in \mathbb{N} \setminus \{0,1\}} \cup \Big\{2m+1\Big\}_{m \in \mathbb{N}\setminus\{0\}}$$

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- The computation performed in the proof of the theorem extends the regularity result of Simon & Wickramasekera '16 for two-valued minimal graphs.

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- It extends the analogous result for the Signorini problem proven in Focardi & S. ('18), which is the first instance of a global structure result for the free boundary for lower dimensional obstacle problems.
- Similar results are proven for multiple-valued Dirichlet minimizing functions in De Lellis, Marchese, S. & Valtorta '18 (see also Krummel & Wickramasekera '18 for related results).
- The computation performed in the proof of the theorem extends the regularity result of Simon & Wickramasekera '16 for two-valued minimal graphs.
- Regarding point (iii): the classification of the frequency at free boundary points is an open question; moreover, there are no examples of free boundary points with frequency $2m + 1$.

Sketch of proof: Mean flatness and rectifiability criterium

- MEAN-FLATNESS:
  μ Radon measure with $\mathrm{supp}\,(\mu) \subset \Gamma(u)$; $x_0 \in \Gamma(u)$, $r > 0$,

$$\beta_\mu(x_0, r) := \inf_{\mathcal{L}} \left(r^{-n-1} \int_{B_r(x)} \mathrm{dist}(y, \mathcal{L})^2 \, d\mu(y) \right)^{\frac{1}{2}},$$

  where the infimum is taken among all affine $(n-1)$-dimensional planes $\mathcal{L} \subset \mathbb{R}^{n+1}$.

- Theorem (Azzam–Tolsa '15, Naber–Valtorta '17)
  Let μ be a finite Borel measure with $\theta^{n-1,\star}(x, \mu) < +\infty$ for μ-a.e. x. Then, μ is $(n-1)$-rectifiable if

$$\sum_{k=0}^{\infty} \beta_\mu^2(x, 2^{-k}) < +\infty \quad \textit{for } \mu\textit{-a.e. } x.$$

Sketch of proof:
(C) Control of the mean-flatness

Proposition (C)

Let u be a solution to the thin obstacle problem for nonparametric minimal surfaces and let μ be a finite Borel measure with $\mathrm{supp}\,(\mu) \subseteq \Gamma(u)$. Then

$$\beta_\mu^2(p,r) \leq \frac{C}{r^{n-1}} \left(\int_{B_r(p)} \Big(I_u(x, 2\,r) - I_u(x, {}^r\!/_2) \Big)\, d\mu(x) + r^2 \mu(B_r(p)) \right)$$

$\forall\ p \in \Gamma(u) \cap B_1$ *and* $\forall\ r \in (0,1)$.

Sketch of proof: (C) $\Longrightarrow$ rectifiability

- Assume $\mathcal{H}^{n-1}(\Gamma(u)\cap B_{1/2}) < +\infty$. Then, w.l.o.g.

$$\frac{\mathcal{H}^{n-1}(\Gamma(u)\cap B_r(x))}{r^{n-1}} \leq C \qquad (\star)$$

for μ-a.e. x and every $r \in (0,1)$.

- Set $\mu := \mathcal{H}^{n-1} \llcorner \Gamma(u)$;

$$\sum_{k=0}^{+\infty} \int_{B_1} \beta_\mu^2(y, 4^{-k})\, d\mu(y)$$

$$\leq \sum_{k=0}^{+\infty} \frac{C}{4^{-k(n-1)}} \int_{B_1} \int_{B_{4^{-k}}(y)} \left(I_u(x, 2^{-2k+1}) - I_u(x, 2^{-2k-1}) \right) d\mu(x)\, d\mu(y) + C$$

$$\leq \sum_{k=0}^{+\infty} \frac{C}{4^{-k(n-1)}} \int_{B_{3/2}} \mu(B_{4^{-k}}(x)) \left(I_u(x, 2^{-2k+1}) - I_u(x, 2^{-2k-1}) \right) d\mu(x) + C$$

$$\overset{(\star)}{\leq} C \int_{B_{3/2}} \sum_{k=0}^{+\infty} \left(I_u(x, 2^{-2k+1}) - I_u(x, 2^{-2k-1}) \right) d\mu(x) + C$$

$$\leq C \int_{B_{3/2}} I_u(x, 2) d\mu(x) + C < +\infty.$$

Sketch of proof:
Homogeneous solutions

$$u(x) = |x|^\lambda \, u\left(x/|x|\right)$$

- Spine of u: maximal space of translation invariance

$$S(u) := \left\{ y \in \mathbb{R}^n \times \{0\} \ : \ u(x+y) = u(x) \quad \forall\, x \in \mathbb{R}^{n+1} \right\}.$$

- Lemma 1. Let u be a homogeneous solution. The following are equivalent:

  (i) $x_0 \in S(u)$,
  (ii) $I_u(x_0, r) = I_u(x_0, 0^+)$ for all $r > 0$.

- Lemma 2. $\dim(S(u)) \leq n - 1$.

Sketch of proof:
Estimate of the spatial derivative of the frequency

Proposition (C)

Let u be a solution to the thin obstacle problem for nonparametric minimal surfaces and let μ be a finite Borel measure with $\operatorname{supp}(\mu) \subseteq \Gamma(u)$. Then

$$\beta_{\mu}^{2}(p,r) \leq \frac{C}{r^{n-1}} \left(\int_{B_r(p)} \Big(I_u(x,2\,r) - I_u(x,{}^{r}\!/\!_{2}) \Big)\, d\mu(x) + r^2 \mu(B_r(p)) \right)$$

$\forall\, p \in \Gamma(u) \cap B_1$ *and* $\forall\, r \in (0,1)$.

- Heuristics: if $I_u(x,2\,r) - I_u(x,{}^{r}\!/\!_{2}) = 0$ for all $x \in \Gamma(u)$, then u is homogeneous at every point of $\Gamma(u)$ and by Lemma 1 and 2

$$\Gamma(u) \subset S(u), \quad \dim(\Gamma(u)) \leq n-1 \quad \Longrightarrow \quad \beta_{\mu}^{2}(p,r) = 0.$$

- Proof: Quantitive version of Lemma 1 based on new variational identities for the derivative of $x \mapsto I_u(x,r)$. □

Thanks for your attention