The weak-A_∞ condition for harmonic measure: geometric characterization of the L^p solvability of the Dirichlet problem

Xavier Tolsa
(joint work with J. Azzam and M. Mourgoglou)

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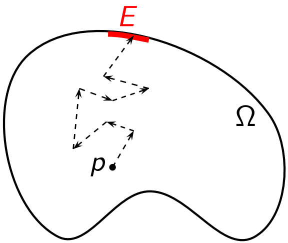
Probabilistic interpretation [Kakutani]:
When Ω is bounded, $\omega^p(E)$ is the probability that a particle with a Brownian movement leaving from $p \in \Omega$ escapes from Ω through E.



Rectifiability

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E is uniformly n-rectifiable if it is n-AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

Metric properties of harmonic measure

- In the plane if Ω is simply connected and $\mathcal{H}^1(\partial\Omega) < \infty$, then $\mathcal{H}^1 \approx \omega^p$. (F.& M. Riesz)
- Many results in $\mathbb{C}$ using complex analysis (Carleson, Makarov, Jones, Bishop, Wolff,...).
- The analogue of Riesz theorem fails in higher dimensions (counterexamples by Wu and Ziemer).
- In higher dimensions, need real analysis techniques.
- A basic result of Dahlberg: If Ω is a Lipschitz domain, then $\omega \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.

Uniform, semiuniform, and NTA domains

Let $\Omega \subset \mathbb{R}^{n+1}$ be open.

- For $x, y \in \overline{\Omega}$, a curve $\gamma \subset \overline{\Omega}$ from x to y is a C-cigar curve with bounded turning if
  - $\min(\mathcal{H}^1(\gamma(x,z)), \mathcal{H}^1(\gamma(y,z))) \leq C\,\mathrm{dist}(z, \Omega^c)$ for all $z \in \gamma$, and
  - $\mathcal{H}^1(\gamma) \leq C\,|x-y|$.

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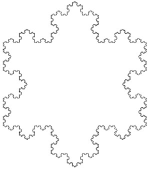
A non trivial NTA domain:



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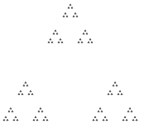
Example: The complement of this Cantor set is uniform but not NTA:



Harmonic measure in different types of domains

Definition: We say that $\omega \in A_\infty$ if, for any ball B centered in $\partial\Omega$ and $p \in \Omega \setminus 2B$, $\omega^p \in A_\infty(\mathcal{H}^n|_{\partial\Omega\cap B})$ uniformly.

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Let $\Omega \subset \mathbb{R}^{n+1}$, with $\partial\Omega$ n-AD-regular. TFAE:

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- Proof building on a previous result on uniform domains by Hofmann, Martell and Uriarte-Tuero.
- A previous partial result by Aikawa and Hirata.

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Let $\Omega \subset \mathbb{R}^{n+1}$, with $\partial\Omega$ n-AD-regular, satisfying the interior corkscrew condition. TFAE:

(a) *For some $p > 1$, the Dirichlet problem is L^p-solvable, i.e.*

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(b) $\omega \in \mathrm{weak}{-}A_\infty.$

Remarks

- Ω satisfies the interior corkscrew condition if for every ball B centered at $\partial\Omega$ with $r(B) \leq \mathrm{diam}(\Omega)$ there is another ball $B' \subset B \cap \Omega$ with $r(B') \approx r(B)$.

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- Problem: Find a geometric characterization of the weak$-A_\infty$ condition.

Geometric characterization of the weak$-A_\infty$ condition I

- $\omega \in \mathrm{weak}{-}A_\infty$ + interior corkscrew condition $\Longrightarrow$ $\partial\Omega$ is uniformly n-rectifiable [Hofmann, Martell], [Mourgoglou-T.].

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  This can be proven by showing that $\mathrm{weak}{-}A_\infty$ + interior corkscrew condition imply that the Riesz transform

$$\mathcal{R}f(x) = \int \frac{x-y}{|x-y|^{n+1}}\, f(y)\, d\mathcal{H}^n|_{\partial\Omega}(y)$$

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- The uniform n-rectifiability of $\partial\Omega$ can be characterized in terms of a corona type decomposition for harmonic measure (Garnett-Mourgoglou-T.).

Geometric characterization of the weak$-A_\infty$ condition II

- Given $x \in \Omega$, $y \in \partial\Omega$, a λ-carrot curve from x to y is a curve $\gamma \subset \Omega \cup \{y\}$ with end-points x and y such that $\text{dist}(z, \partial\Omega) \geq \lambda\, \mathcal{H}^1(\gamma(y,z))$ for all $z \in \gamma$, where $\gamma(y,z)$ is the arc in γ between y and z.

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- We denote $\delta_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$.

- We say that Ω satisfies the weak local John condition if there are $\lambda, \theta \in (0,1)$ such that for every $x \in \Omega$ there is a Borel set $F \subset B(x, 2\delta_\Omega(x)) \cap \partial\Omega$ with $\mathcal{H}^n(F) \geq \theta\, \mathcal{H}^n(B(x, 2\delta_\Omega(x)) \cap \partial\Omega)$ such that every $y \in F$ can be joined to x by a λ-carrot curve.

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Theorem (Hofmann, Martell)

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Theorem (Azzam, Mourgoglou, T.)

Let $\Omega \subset \mathbb{R}^{n+1}$ be open with n-AD-regular boundary. If $\omega \in \mathrm{weak}{-}A_\infty$, then Ω satisfies the weak local John condition.

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Since IBPCAD $\Rightarrow \omega \in \mathrm{weak}{-}A_\infty$ (Bennewitz, Lewis), we have

$$(a) \iff (b) \iff \text{IBPCAD}.$$

The main result II

Putting all together:

Theorem

Let $\Omega \subset \mathbb{R}^{n+1}$ be open with n-AD-regular boundary, satisfying the interior corkscrew condition. TFAE:

(a) $\omega \in \text{weak}-A_\infty$.

(b) *$\partial\Omega$ is uniformly n-rectifiable and Ω satisfies the weak local John condition.*

(c) *Ω has IBPCAD.*

Some ideas for the proof that (a) $\Rightarrow$ weak local John

- For $p \in \Omega$, we have to build carrot curves that connect a big proportion of the points from $B(p, 2\delta_\Omega(p)) \cap \partial\Omega$ to p.

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  Otherwise we could argue with different poles $p_1, p_2, \ldots$
- Let $\mu = \mathcal{H}^n|_{\partial\Omega}$. We consider the good set G of points $x \in \partial\Omega \cap B(p, 2\delta_\Omega(p))$ such that

$$\omega^p(B(x,r)) \approx \frac{1}{\delta_\Omega(p)^n}\,\mu(B(x,r)) \qquad \forall r < \delta_\Omega(p).$$

  By the weak-A_∞ property, $\mu(G) \approx \mu(B(p, 2\delta_\Omega(p))) \approx \delta_\Omega(p)^n$.
  We want to connect points in G to p.

The ACF formula

We use Alt-Caffarelli-Friedman (ACF) monotonicity formula to connect a corkscrew point $x \in \Omega$ to another point $x' \in \Omega$, with $\delta_\Omega(x') \approx 100\,\delta_\Omega(x)$.

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Theorem (ACF)

Let $B(x,R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ be nonnegative subharmonic functions. Suppose that $u_1(x) = u_2(x) = 0$ and $u_1 \cdot u_2 \equiv 0$. Set

$$J(x,r) = \left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y-x|^{n-1}}dy\right)\cdot\left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y-x|^{n-1}}dy\right).$$

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This formula is a basic tool in free boundary problems.
It can be used to "prove connectivity".

How to use the ACF formula

Let $x, x_2 \in \Omega$ with $\delta_\Omega(x_2) \approx 10\,\delta_\Omega(x)$ such that

$$g(p,x) > \lambda \approx \frac{\delta_\Omega(x)}{\delta_\Omega(p)^n}, \qquad g(p,x_2) > \lambda \approx \frac{\delta_\Omega(x_2)}{\delta_\Omega(p)^n}.$$

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Denote

$$u(y) = (g(p,y) - \tfrac{1}{2}\lambda)^+.$$

For a big ball B centered at $\partial\Omega$ with

$$\delta_\Omega(x) \approx \delta_\Omega(x_2) \approx r(B), \quad x, x' \in \tfrac{1}{100}B,$$

consider the open set $U = \{y \in B : u(y) > 0\}$.

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Otherwise we apply the ACF formula to $u_1 = u\,\chi_{U_1}$ and $u_2 = u\,\chi_{U_2}$.
For $r(B)$ big enough, u_1 behaves as an affine function and U_1 is close to a half ball and thus one finds x' easily in U_1 that can be connected to x by a non-tangential curve.

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Problem: When we iterate many times the preceding argument, the constants worsen and this collapses.

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(a) *The family of roots of $\mathcal{T} \in I$ fulfils the packing condition*

$$\sum_{\mathcal{T}\in I:\mathrm{Root}(\mathcal{T})\subset S} \mu(\mathrm{Root}(\mathcal{T})) \leq C\,\mu(S) \quad \textit{for all } S \in \mathcal{D}_\mu.$$

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(b) *In each $\mathcal{T} \in I$, E is "very well approximated" by an n-dimensional Lipschitz graph $\Gamma_{\mathcal{T}}$. That is, for all $Q \in \mathcal{T}$, $\mathrm{dist}(Q, \Gamma_{\mathcal{T}}) \leq \ell(Q)$.*

The very good set VG

Recall that G is the set of points $x \in \partial\Omega \cap B(p, 2\delta_\Omega(p))$ such that

$$\omega^p(B(x,r)) \approx \frac{1}{\delta_\Omega(p)^n}\,\mu(B(x,r)).$$

For some $M \gg 1$, let

$$VG = \Big\{x \in G : \sum_{\mathcal{T}\in I} \chi_{\mathrm{Root}(\mathcal{T})}(x) \leq M\Big\}.$$

We build carrot curves that connect most points from VG to p.
Difficulty: control the estimates when $M \to \infty$.

Thank you!