Second order regularity of transition layers in Allen-Cahn equation

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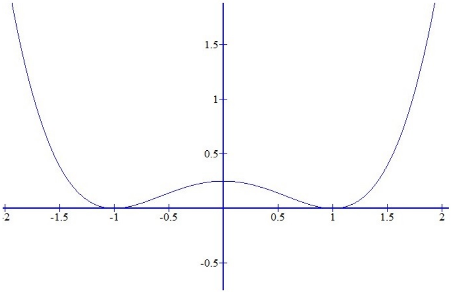
Allen-Cahn equation

$$\varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon}\left(u_\varepsilon^3 - u_\varepsilon\right) \qquad (AC)$$

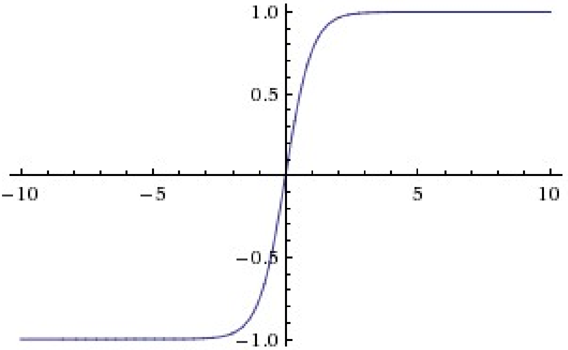
$$E_\varepsilon(u_\varepsilon) = \int \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon}\left(1 - u_\varepsilon^2\right)^2.$$



One dimensional solution

$$-g'' = g - g^3.$$

$$g(x) = \tanh \frac{x}{\sqrt{2}}.$$



Part I. Finite Morse index solutions

Uniform $C^{2,\alpha}$ regularity

Question: *Let u_ε be a sequence of solutions to (AC). Assume the level sets $\{u_\varepsilon = 0\}$ are uniformly $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Can we get a uniform $C^{2,\alpha}$ regularity?*

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This was used in our study on the structure of finite Morse index solutions.

Theorem (W.-Wei '17)

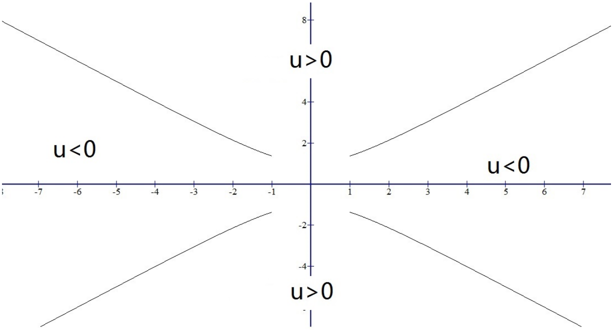
A finite Morse index solution of the Allen-Cahn equation

$$-\Delta u = u - u^3$$

in $\mathbb{R}^2$ has finitely many ends.

Solutions with finite ends

u looks like the 1d solution along each end. $\Longleftarrow$ Refined asymptotics, exponential convergence (Gui '08, Del Pino-Kowalczyk-Pacard '13).



A finiteness result for nodal domains

Let u_e be the directional derivative in e-direction, which satisfies the linearized equation

$$\Delta u_e = W''(u)u_e.$$

Lemma

If the Morse index is N, the number of connected components of $\{u_e \neq 0\}$ is at most $2N$.

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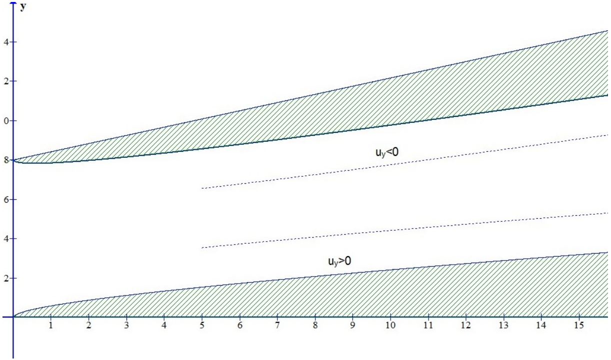
- Liouville theorem for the degenerate equation

$$\operatorname{div}\left(\varphi^2 \nabla \frac{u_e}{\varphi}\right) = 0.$$

- Similar to Courant's nodal domain theorem: entire space? $n = 2 \Longrightarrow$ log cut-off functions, Ambrosio-Cabré '03...

Transferring finiteness information

If each end of $\{u = 0\}$ has an asymptotic direction at infinity, finiteness of nodal domains of u_e can be transformed into finiteness of ends.



Curvature decay

Theorem

Let u be a finite Morse index solution of the Allen-Cahn equation in $\mathbb{R}^2$. For all x large,

$$|A(x)|^2 := \frac{|\nabla^2 u(x)|^2 - |\nabla|\nabla u(x)||^2}{|\nabla u(x)|^2} \leq \frac{C}{|x|^2}.$$

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Theorem (Schoen '83)

Given a three dimensional manifold M (with some curvature bounds). Let Σ be a stable immersed minimal surface in a ball $B_R(p) \subset M$ with $\partial\Sigma \subset \partial B_R(p)$. Then

$$\sup_{B_{R/2}(p)\cap\Sigma} |A_\Sigma|^2 \leq \frac{C}{R^2}.$$

Sternberg-Zumbrun inequality

$$\text{Stability} \Leftrightarrow \int |\nabla\varphi|^2 |\nabla u|^2 \geq \int \varphi^2 \left[|\nabla^2 u|^2 - |\nabla|\nabla u||^2 \right].$$

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- If $|\nabla u| \neq 0$,

$$\frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2} = |A|^2 + |\nabla_T \log|\nabla u||^2,$$

  where A is the second fundamental form of level sets $\{u = const.\}$ and ∇_T is the tangential derivatives along these level sets.

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  where A is the second fundamental form of level sets $\{u = const.\}$ and ∇_T is the tangential derivatives along these level sets.
- Simons inequality for this curvature term? Not found yet. Seems to be a common difficulty in semilinear problems.

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- Find y_k satisfying

$$|A(y_k)| \geq |A(x_k)|, \qquad |A(y_k)||y_k| \geq k,$$

$$|A(x)| \leq 2|A(y_k)|, \quad \forall x \in B_{k|A(y_k)|^{-1}}(y_k).$$

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- Let $\varepsilon_k := |A(y_k)|$ and define $u_k(x) := u(y_k + \varepsilon_k^{-1} x)$.
$|y_k| \to +\infty$ and $\varepsilon_k \to 0 \Longleftarrow$ Locally close to $1D$ solution, by stable De Giorgi for $n = 2$.

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  $|y_k| \to +\infty$ and $\varepsilon_k \to 0 \Longleftarrow$ Locally close to $1D$ solution, by stable De Giorgi for $n = 2$.
- In $B_k(0)$, u_k is a stable solution of (AC) with parameter ε_k.
- The curvature of $\{u_{\varepsilon_k} = 0\}$ is uniformly bounded, and it equals 1 at the origin.

Second order regularity

Theorem (W.-Wei '18)

Let u_ε be a sequence of stable solutions to (AC) such that $\{u_\varepsilon = 0\}$ are uniformly $C^{1,\beta}$ for some $\beta \in (0,1)$. If $n \leq 10$, then $\{u_\varepsilon = 0\}$ are uniformly bounded in $C^{2,\alpha}$ for any $\alpha \in (0,1)$. Moreover, the mean curvature is of the order $O(\varepsilon^\alpha)$.

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Chodosh-Mantoulidis '18 has obtained the same result in dimension 3, which was used in their study of min-max minimal surfaces in three manifolds (Multiplicity one conjecture of Marques-Neves, existence of infinitely many minimal surfaces in generic case).

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Theorem

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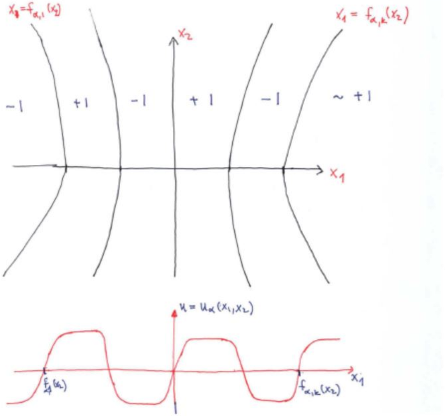
Caffarelli and Córdoba '06 and Savin '09: Lipschitz or flat $\Longrightarrow$ uniform $C^{1,\alpha}$.

Part II. Proof of $C^{2,\alpha}$ estimates

Clustering interfaces

There could be more and more connected components of $\{u_\varepsilon = 0\}$, which can collapse to the same limit as $\varepsilon \to 0$.



Toda system

Curvature bound on $\{u_\varepsilon = 0\} \Longrightarrow \{u_\varepsilon = 0\}$ locally represented by graphs $\cup_k \Gamma_{k,\varepsilon}$, where

$$\Gamma_{k,\varepsilon} = \{x_2 = f_{k,\varepsilon}(x_1)\}, \quad \cdots < f_{k-1,\varepsilon} < f_{k,\varepsilon} < f_{k+1,\varepsilon} < \cdots .$$

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Interaction between different interfaces has the form

$$\operatorname{div}\left(\frac{\nabla f_{k,\varepsilon}}{\sqrt{1+|\nabla f_{k,\varepsilon}|^2}}\right) = \frac{A}{\varepsilon}\left[e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon}-f_{k-1,\varepsilon}\right)} - e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k+1,\varepsilon}-f_{k,\varepsilon}\right)}\right] + h.o.t.$$

Infinite dimensional Lyapunov-Schmidt reduction of Del Pino, Kowalczyk and Wei.

Obstruction to $C^{2,\alpha}$ estimates of $f_{k,\varepsilon}$

$$\Delta f_{k,\varepsilon} = \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)} - \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k+1,\varepsilon} - f_{k,\varepsilon}\right)}.$$

$$f_{k+1,\varepsilon} - f_{k,\varepsilon} \geq \frac{\sqrt{2}\,(1+\alpha)}{2} \varepsilon |\log \varepsilon| - C\varepsilon \Longrightarrow f_{k,\varepsilon} \in C^{2,\alpha}.$$

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On the other hand, if

$$f_{k+1,\varepsilon} - f_{k,\varepsilon} \leq \frac{\sqrt{2}}{2}\varepsilon|\log \varepsilon| + C\varepsilon,$$

define the blow up sequence

$$\widetilde{f}_{k,\varepsilon}(x) := \frac{1}{\varepsilon} f_{k,\varepsilon}\left(\varepsilon^{\frac{1}{2}} x\right) - \frac{\sqrt{2}\alpha}{2}|\log \varepsilon|.$$

They converge to an entire solution of the Toda system

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}, \quad \text{in } \mathbb{R}^{n-1}.$$

Example I: Two end solutions of Agudelo-del Pino-Wei

- For $n \geq 10$, the Liouville equation ($=$ Two component Toda system)

$$\Delta f = e^{-\sqrt{2}f}$$

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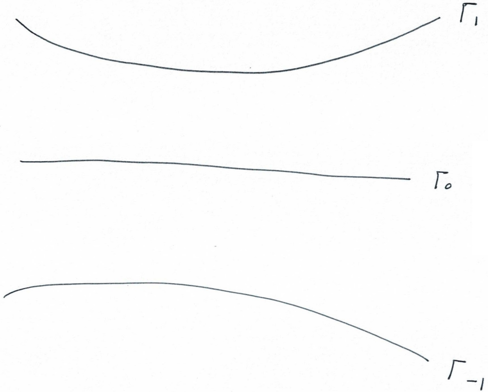
- For λ small ($\iff f_\lambda(0) \gg 1$), there exists a monotone (in λ) family of solutions u^λ to the unscaled (AC). $\Longrightarrow$ stable.
- Let $u_\varepsilon(x) := u^{\varepsilon^{1/2}}\left(\varepsilon^{-1}x\right)$. Its nodal set $\{u_\varepsilon = 0\}$ is given by the graph of
$$f_\varepsilon(x) \approx \varepsilon f\left(\varepsilon^{-\frac{1}{2}}x\right) + \frac{\sqrt{2}\alpha}{2}\varepsilon|\log\varepsilon|,$$
which satisfies
$$|\nabla^2 f_\varepsilon(0)| = |\nabla^2 f(0)|, \quad |\nabla^2 f_\varepsilon(x)| \to 0, \quad \forall x \neq 0.$$

Example II: Multiple end solutions in $\mathbb{R}^2$

Del Pino-Kowalczyk-Pacard-Wei '10 : Unstable solutions with $\{u = 0\}$ close to the graph of Toda solutions:

$$f_k''(x) = e^{-\sqrt{2}(f_k(x)-f_{k-1}(x))} - e^{-\sqrt{2}(f_{k+1}(x)-f_k(x))}, \quad x \in \mathbb{R}, \quad 1 \le k \le Q.$$



Reduction of the stability condition

- If u_ε is stable, $(f_{k,\varepsilon})$ satisfies a stability condition:

$$\sum_k \int |\nabla \eta_k|^2 \geq \frac{\sqrt{2}A}{\varepsilon^2} \sum_k \int (\eta_k - \eta_{k-1})^2 e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)} - h.o.t..$$

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$\Longrightarrow$ Existence of entire stable solutions of Toda system
Liouville theorem: No stable solution of Toda in $\mathbb{R}^n$ for $n \leq 9$.

An ε-regularity theorem

Theorem

For any n, there exists a universal constant η such that, if (f_k) is a stable solution to the Toda lattice

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)} \quad \text{in } B_1 \subset \mathbb{R}^n,$$

then

$$\int_{B_1} e^{-\sqrt{2}(f_k - f_{k-1})} \leq \eta(n) \quad \Longrightarrow \quad \sup_{B_{1/2}} e^{-\sqrt{2}(f_k - f_{k-1})} \leq \frac{1}{2}.$$

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Applying this ε-regularity to suitable rescalings of Toda system constructed from (AC), gives a decay estimate on $e^{-\frac{\sqrt{2}}{\varepsilon}(f_{k,\varepsilon} - f_{k-1,\varepsilon})}$ in shrinking balls, leading finally to

$$e^{-\frac{\sqrt{2}}{\varepsilon}(f_{k,\varepsilon} - f_{k-1,\varepsilon})} \lesssim \varepsilon^{1+\alpha}, \qquad \text{in the interior.}$$

Proof of ε-regularity theorem

Without the stability condition, for the Liouville equation

$$\Delta f = e^{-f},$$

this ε-regularity has been proved by Brezis-Merle '91 in 2 dimension and F. Da Lio '08 in 3 dimension. Higher dimensions are not known.

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Our proof relies essentially on the stability condition, which gives us an integral estimate (as in Farina '07)

$$\int_{B_r} e^{-2\sqrt{2}(f_k - f_{k-1})} \leq Cr^{-2} \int_{B_{2r}} e^{-\sqrt{2}(f_k - f_{k-1})}.$$

Part III. Derivation of the Toda system

Lyapunov-Schmidt reduction

Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, consider a nonlinear map $G \in C^1(\mathcal{X}, \mathcal{Y})$.
Let $E := \ker DF(0)$ and $F := \operatorname{coker} DF(0)$. Assume $\mathcal{X} = E \bigoplus E^\perp$ and $\mathcal{Y} = F \bigoplus F^\perp$ (e.g. when E and F are finite dimensional).
Then

$$G(x_1, x_2) = 0 \Longleftrightarrow \begin{cases} \prod_F \circ G(x_1, x_2) = 0, \\ \prod_{F^\perp} \circ G(x_1, x_2) = 0. \end{cases}$$

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- Solve the second equation by Implicit Function Theorem.
- Substituting the second equation into the first one $\Longrightarrow$ an equation defined on E.
- If E is finite dimensional: finite dimensional reduction.
- Even if E is infinite dimensional, sometimes the reduction problem is still a good one.

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We use this method in a reverse order.

- We already have a solution u_ε of the Allen-Cahn equation.

Infinite dimensional reduction method

- Finite dimensional reduction method has been used by many authors to construct solutions of nonlinear PDEs: gluing method.
- Construction of solutions with multiple ends or clustering interfaces from Toda systems by (Del Pino, Kowalczyk, Wei and their coauthors).

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- In order to get estimates on level sets, a good equation satisfied by these level sets is needed.
- Decouple the single equation (AC) into two: one is the equation for level sets (on the kernel space E), the other one is on $E^\perp$ which enjoys good a priori estimates.

The multiplicity one case: approximate solutions

Starting assumptions: close to a canonical configuration by assuming $C^{1,1}$ regularity.

Around $\Gamma_\varepsilon := \{u_\varepsilon = 0\}$, u_ε looks like

$$g_*(y,z) := g\left(\frac{z - h_\varepsilon(y)}{\varepsilon}\right),$$

where $z = \mathrm{dist}_{\Gamma_\varepsilon}$ is the signed distance to Γ_ε and y denotes a point on Γ_ε. $\Longrightarrow$ Introduction of Fermi coordinates w.r.t. Γ_ε.

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A small perturbation in normal direction is also needed: introduction of h_ε.

Fermi coordinates

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$$\Delta = \Delta_z - H\partial_z + \partial_{zz},$$

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$$\Delta d = \sum_i \frac{k_i}{1 - k_i d} = H + O(|A|^2),$$

where k_i are principal curvatures of Γ_ε.

Optimal approximation in Fermi coordinates

h_ε is a function defined on Γ_ε, which must be introduced so that the orthogonal condition holds:

$$\int_{-\infty}^{+\infty} \left(u_\varepsilon(y,z) - g_*(y,z)\right) g_*' dz = 0.$$

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For each $y \in \Gamma_\varepsilon$,

$$\min_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} \left| u_\varepsilon(y,z) - g\left(\frac{z-t}{\varepsilon}\right)\right|^2 dz$$

is attained at a unique point $h_\varepsilon(y)$.

Decoupling

The error between u_ε and g_*, $\phi_\varepsilon := u_\varepsilon - g_*$, satisfies

$$\varepsilon^2 \left(\Delta_0 \phi_\varepsilon + \frac{\partial^2 \phi_\varepsilon}{\partial z^2} \right) = W''(g_*)\phi_\varepsilon + \varepsilon \left[H_\varepsilon(y) + \Delta h_\varepsilon(y) \right] g_*' + h.o.t.$$

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- The parallel part gives the equation satisfied by $\{u_\varepsilon = 0\}$:

$$H_\varepsilon = O(\varepsilon).$$

  Then standard elliptic estimates on minimal surface equation gives the $C^{2,\alpha}$ estimate.

Decoupling: a model case

$$\begin{cases} \Delta_{\mathbb{R}^{n-1}}\phi + \partial_{zz}\phi = W''(g(z))\phi + a(y)g'(z) + E, \\ \int_{-\infty}^{+\infty} \phi(y,z)g'(y,z)dz = 0, \quad \forall y \in \mathbb{R}^{n-1}. \end{cases}$$

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$$a(y) \int_{-\infty}^{+\infty} g'(z)^2 dz = -\int_{-\infty}^{+\infty} E(y,z)g'(z)dz.$$

Clustering interfaces: approximate solutions

Starting assumptions: close to a canonical configuration.
Around $\Gamma_{k,\varepsilon}$, u_ε looks like

$$g_{k,\varepsilon} := g\left(\frac{\mathrm{dist}_{\Gamma_{k,\varepsilon}} - h_{k,\varepsilon}}{\varepsilon}\right),$$

where $\mathrm{dist}_{\Gamma_{k,\varepsilon}}$ is the signed distance to $\Gamma_{k,\varepsilon}$.

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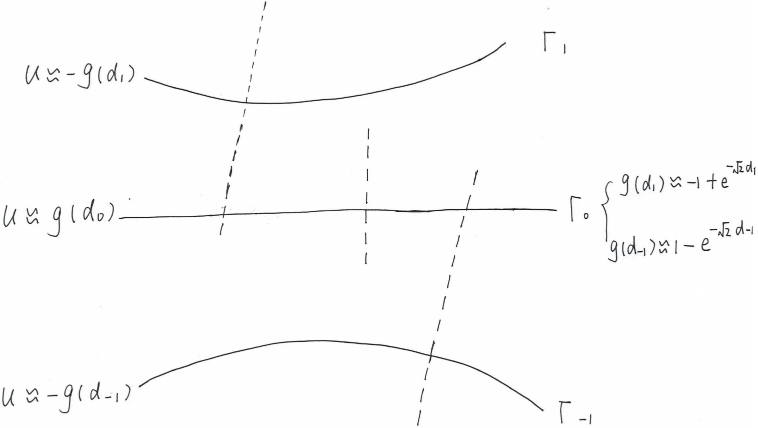
As before, we need to

- introduce Fermi coordinates w.r.t. $\Gamma_{k,\varepsilon}$;
- introduce a small perturbation $h_{k,\varepsilon}$ to obtain the orthogonal condition.

Interaction between transition layers

Approximate solution: near Γ_k,

$$g_* := g_k + \sum_{\ell<k}\left[g_\ell - (-1)^\ell\right] + \sum_{\ell>k}\left[g_\ell + (-1)^\ell\right].$$



Decoupling in clustering interfaces

Near Γ_k, the error between u_ε and g_*, $\phi_\varepsilon := u_\varepsilon - g_*$, satisfies

$$\varepsilon^2 \left(\Delta_0 \phi_\varepsilon + \frac{\partial^2 \phi_\varepsilon}{\partial z^2} \right) = W''(g_{k,\varepsilon})\phi_\varepsilon + \varepsilon \left[H_{k,\varepsilon} + \Delta h_{k,\varepsilon} \right] g'_{k,\varepsilon} + \mathcal{I}_{k,\varepsilon} + h.o.t.$$

where $\mathcal{I}_{k,\varepsilon}$ describes the interaction between $\Gamma_{k,\varepsilon}$ and other components.

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- The parallel part (w.r.t. $g'_{k,\varepsilon}$) gives the Toda system with remainder terms of higher order (quadratic in ϕ_ε, $e^{-\frac{\sqrt{2}}{\varepsilon}(f_{\alpha+1,\varepsilon} - f_{\alpha,\varepsilon})}$ etc.)

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- ϕ is mainly controlled by the interaction term:

$$\|\phi_\varepsilon\|_{C^{2,\alpha}} \lesssim \varepsilon^{1-\alpha} + \frac{1}{\varepsilon} \sup \left[e^{-\frac{f_{k+1,\varepsilon} - f_{k,\varepsilon}}{\varepsilon}} + e^{-\frac{f_{k,\varepsilon} - f_{k-1,\varepsilon}}{\varepsilon}} \right].$$

Reduction of the stability condition again

- In the stability condition for the Allen-Cahn equation,

$$\int \varepsilon |\nabla \psi|^2 \geq \int \frac{1}{\varepsilon} W''(u_\varepsilon)\psi^2,$$

choose (as in Agudelo-Del Pino-Wei '16)

$$\psi := \sum_k \eta_k g'_{k,\varepsilon},$$

where $\eta_k \in C_0^\infty(\Gamma_{k,\varepsilon})$.

- Good decomposition: $u_\varepsilon = g_* + \phi_\varepsilon$, i.e. good estimates on ϕ_ε $\Longrightarrow$ main order terms are

$$\sum_k \int |\nabla \eta_k|^2 \geq \frac{\sqrt{2}A}{\varepsilon^2} \sum_k \int (\eta_k - \eta_{k-1})^2 \, e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)}.$$

Concluding remarks

- Reduction method is basically a (partial) linearization procedure: when the solution u_ε is close to $\sum_i g_{i,\varepsilon}$, the ϕ_ε equation is almost a linearized one;

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- The reduced problem is still a nonlinear one in most cases, but its complexity is decreased;
- Long range interaction in these phase field models. This is helpful for the construction of Jacobi fields (Chodosh-Mantoulidis '18).

Thanks for your attention!