Ramsey properties of randomly perturbed dense sets of integers and the asymmetric random Rado theorem

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For n sufficiently large, given an n-vertex graph G with positive edge-density, the distribution $G \cup \mathbb{G}(n,p)$ is viewed as a random perturbation of G. The limiting behaviour of *symmetric* and *asymmetric* Ramsey properties of $G \cup \mathbb{G}(n,p)$ have been studied by Krivelevich, Sudakov, and Tetali (2006), and more recently by Das and Treglown (2018) as well as Powierski (2018). The Kohayakawa-Kreuter conjecture (1997) regarding asymmetric Ramsey properties of $\mathbb{G}(n,p)$, whose 1-statement has recently been established by Mousset, Nenadov, and Samotij (2018), arises quite naturally in the study of the threshold for the symmetric Ramsey property $G \cup \mathbb{G}(n,p) \to (H)_2$ for a prescribed graph H.

Much less is known regarding the Ramsey properties of randomly perturbed dense sets of integers captured by the distribution $A \cup [n]_p$ where here $A \subseteq [n]$ has positive density. We prove that $p = n^{-2/3}$ is the threshold for the Schur property in this model. Nothing (as far as we know) is known regarding the more general *Rado* property $A \cup [n]_p \to (M)_2$ where here M is a fixed *Rado matrix* (i.e., irredundant and partition-regular) that does not correspond to the Schur equation. If the graph case is of any consequence, a counterpart to the Kohayakawa-Kreuter conjecture fitting the integers is missing.

Let A and B be two Rado matrices, where A is an $\ell_A \times k_A$-matrix and B is an $\ell_B \times k_B$-matrix. Set

$$m(A) := \max_{\substack{W \dot\cup \overline{W} = [k_A] \\ |W| \geq 2}} \frac{|W| - 1}{|W| - 1 + \boldsymbol{rk}\,(A_{\overline{W}}) - \boldsymbol{rk}\,A}.$$

We put forth the parameter

$$m(A,B) := \max_{\substack{W \subseteq [k_A] \\ |W| \geq 2}} \frac{|W|}{|W| - \boldsymbol{rk}\,A + \boldsymbol{rk}\,(A_{\overline{W}}) - 1 + 1/m(B)}$$

and prove that whenever $A_1, \ldots, A_r$ are r Rado matrices satisfying $m(A_1) \geq m(A_2) \geq \cdots \geq m(A_r)$, then there exists a constant $C > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}\left[[n]_p \to (A_1, \ldots, A_r)\right] = 1.$$

whenever $p \geq Cn^{-1/m(A_1,A_2)}$; we conjecture that $n^{-1/m(A_1,A_2)}$ is in fact the threshold. In the special case of arithmetic progressions Zohar (2019) verified this conjecture.

For symmetric random Ramsey-type results, Friedgut, Rödl, and Schacht (2010) have established a *meta theorem* through which 1-statements for various *symmetric* random Ramsey properties in (hyper)graphs and the integers can be attained at the correct threshold. Building upon the arguments of Mousset, Nenadov and Samotij (2018) we establish a corresponding meta-theorem for the asymmetric setting and through which are able for instance to recover the 1-statement of, say, the Kohayakawa-Kreuter conjecture and in particular the aforementioned 1-statement for the asymmetric random Rado property.