The rank of sparse random matrices

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We determine the rank of a random matrix A over an arbitrary field F with prescribed numbers of non-zero entries in each row and column. As an application we obtain a formula for the rate of low-density parity check codes. This formula verifies a conjecture of Lelarge [1]. The proofs are based on coupling arguments.

Specifically, let $\chi \neq 0$ be a random variable that takes values in a field F. Moreover, let $d \geq 1, k \geq 3$ be integer-valued random variables such that $E[d^r]+E[k^r] < \infty$ for a real $r > 2$ and set $\bar{d} = E[d]$, $\bar{k} = E[k]$. Let $n > 0$ be an integer divisible by the greatest common divisor of the support of k and let $m \sim \mathrm{Po}(dn/k)$. Further, let $(d_i, k_i, \chi_{i,j})_{i,j\geq 1}$ be copies of d, k, χ, respectively, mutually independent and independent of m. Given $\sum_{i=1}^{n} d_i = \sum_{i=1}^{m} k_i$, draw a simple bipartite graph G comprising a set $\{a_1, \ldots, a_m\}$ of *check nodes* and a set $\{x_1, \ldots, x_n\}$ of *variable nodes* such that the degree of a_i equals k_i and the degree of x_j equals d_j for all i, j uniformly at random. Then let A be the $m \times n$-matrix with entries

$$A_{ij} = 1\{a_i x_j \in E(G)\} \cdot \chi_{i,j}.$$

Since d, k have finite means the matrix A is sparse, i.e., the expected number of non-zero entries is $O(n)$.

The following theorem provides a formula for the asymptotic rank of A. Let $D(x)$ and $K(x)$ denote the probability generating functions of d and k, respectively.

Theorem 1. *Let*

$$\Phi(\alpha) = D\left(1 - K'(\alpha)/\bar{k}\right) + \frac{\bar{d}}{\bar{k}}(K(\alpha) + (1-\alpha)K'(\alpha) - 1).$$

Then

$$\lim_{n\to\infty} \frac{\mathrm{rk}(A)}{n} = 1 - \max_{\alpha\in[0,1]} \Phi(\alpha) \qquad \textit{in probability.}$$

The upper bound on the rank already follows from [1].

REFERENCES

[1] M. Lelarge: Bypassing correlation decay for matchings with an application to XORSAT. Proc. IEEE Information Theory Workshop (2013) 1–5.