Exploring projective norm graphs

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Let q be a prime power and $t \geq 2$ an integer. The projective norm graph $\mathrm{NG}(q,t)$ has vertex set $\mathbb{F}_{q^{t-1}} \times \mathbb{F}_q^*$ and two vertices (A,a) and (B,b) are adjacent if and only if $\mathrm{N}(A+B) = ab$, where N stands for the norm function from $\mathbb{F}_{q^{t-1}}$ to $\mathbb{F}_q$. They were introduced by Alon, Rónyai and Szabó as a tight construction for the Turán problem for complete bipartite graphs $K_{t,s}$ when $s > (t-1)!$. Since their first appearance they have served as useful constructions for other combinatorial problems as well, and many of their interesting properties have been proven.

Here we explore them further and, among others, determine their automorphis groups and study the size of the common neighbourhoods of pairs, triples and quadruples of vertices. The neighbourhood problem is equivalent to determining the number of solutions of some special systems of norm equations.

As a first application we prove that for $t = 4$ the result of Alon, Rónyai and Szabó cannot be strengthened, in the sense that $\mathrm{NG}(q,4)$ already contains many copies of $K_{4,6}$. As another application we count for general t the number of copies of any fixed 3-degenerate graph in $\mathrm{NG}(q,t)$ and find that projective norm graphs are quasirandom with respect to this parameter.