# Nonconvergence in the first order logic of permutations

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Recently Albert, Bouvel and Féray introduced the *theory of two total orders* (TOTO) which allows one to express properties of permutations in first order logic. We are allowed to use the quantifiers $\forall, \exists$, variables $x, y, z, \ldots$, the logical connectives $\wedge, \vee, \neg$, etc., brackets and the relation symbols $=, <_1, <_2$. Thinking of a permutation $\pi$ as a map from $[n]$ two itself, if $x, y$ represent two elements of $[n]$ then $x <_1 y$ just means that $x < y$ while $x <_2 y$ means that $\pi(x) < \pi(y)$. For instance, the occurrence of the pattern 231 can be expressed as

$$\exists x, y, z : (x <_1 y) \wedge (y <_1 z) \wedge (z <_2 x) \wedge (x <_2 y).$$

We consider the probability that a given property expressible in TOTO holds for a random permutation. That is, the permutation $\pi_n$ chosen uniformly at random from all $n!$ permutations of $[n] := \{1, \ldots, n\}$. Answering a question of Albert, Bouvel and Féray in the negative, we show that there exists a property $\varphi$ expressible in TOTO, such that

$$\lim_{n\to\infty} \mathbb{P}(\pi_n \text{ satisfies } \varphi) \text{ does not exist.}$$

That is, we construct a $\varphi \in$ TOTO such that probability that $\pi_n$ satisfies it oscillates between zero and one. The construction builds on the seminal work of Shelah and Spencer on first order properties for the Erdős-Rényi random graph.

(Based on joint work with Fiona Skerman)