The evolution of random graphs on surfaces of non-constant genus

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By the *genus* of a graph G, we denote the smallest integer g for which G can be drawn on the orientable surface of genus g (that is, a sphere to which g handles have been attached) without crossing edges. Given non-negative integers g, m and $n > 0$, denote by $S_g(n, m)$ the graph chosen uniformly at random from the set of all graphs with vertex set $\{1, \ldots, n\}$ that have precisely m edges and genus at most g.

For both g and m being functions of n, we investigate the evolution of $S_g(n, m)$ as m increases. In particular, we show that for $g \ll n$, the size L_1 of the largest component of $S_g(n, m)$ undergoes *two* phase transitions. The first phase transition happens at around $m = \frac{n}{2}$, when the giant component emerges; the second phase transition takes place at around $m = n$, when the giant component covers almost all vertices. The exact behaviour of L_1 in these phase transitions depends on the order of $g = g(n)$. In the first phase transition, L_1 exhibits a smooth change between behaviours similar to random planar graphs (when g is "small") and to Erdős-Rényi random graphs (when g is "large"). In the second phase transition, the width of the critical window changes with the genus, with larger genus resulting in a wider critical window.