Stochastic Evolution PDEs
Lectures 1–2

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Outline

Stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u) + g(u)\dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0. \end{cases}$$

Stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f(u) + g(u)\dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases}$$

$\dot{W}$ is spatial and temporal noise

Outline

Stochastic Cahn-Hilliard equation (Cahn-Hilliard-Cook):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta v = \dot{W} & \text{in } \mathcal{D} \times [0, T] \\ v = -\Delta u + f(u) & \text{in } \mathcal{D} \times [0, T] \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D} \times [0, T] \\ u(0) = u_0 & \text{in } \mathcal{D} \end{cases}$$

Outline

Formulate as an abstract evolution problem in Hilbert space $\mathcal{H}$:

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W, \quad t > 0 \\ X(0) = X_0 \end{cases}$$

What does this mean? Strong formulation:

$$X(t) = X_0 + \int_0^t (-AX + F(X))\,\mathrm{d}s + \int_0^t G(X)\,\mathrm{d}W$$

Weak formulation:

$$\langle X(t), \eta \rangle = \langle X_0, \eta \rangle + \int_0^t \langle X(s), -A^*\eta \rangle + \langle F(X(s)), \eta \rangle\,\mathrm{d}s + \int_0^t \langle \eta, G(X(s))\,\mathrm{d}W(s) \rangle \quad \forall \eta \in D(A^*)$$

Outline

We will use the semigroup approach of Da Prato and Zabczyk [2] based on the mild formulation:

$$X(t) = \mathrm{e}^{-tA}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}G(X(s))\,\mathrm{d}W(s)$$

Here $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ is the semigroup of bounded linear operators generated by $-A$.
$\{W(t)\}_{t\geq 0}$ is a Q-Wiener process in another Hilbert space $\mathcal{U}$ and $\int_0^t \cdots \mathrm{d}W$ is a stochastic integral.
We often study the linear case, where $F(X) = f$, $G(X) = B$ are independent of X:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

Here $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. Often $f = 0$ for brevity.
Additive noise: $B\,\mathrm{d}W$. Multiplicative noise: $G(X)\,\mathrm{d}W$.
We shall explain these things.

Notation

- $\mathcal{D} \subset \mathbf{R}^d$ spatial domain, bounded, convex, with polygonal boundary
- $\mathcal{H}$, $\mathcal{U}$ real, separable Hilbert spaces
- $H = L_2(\mathcal{D})$ Lebesgue space
- $\mathcal{L}(\mathcal{U}, \mathcal{H})$ bounded linear operators, $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$
- $\mathcal{L}_2(\mathcal{U}, \mathcal{H})$ Hilbert-Schmidt operators, $\mathrm{HS} = \mathcal{L}_2(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$

Semigroup

A family $\{E(t)\}_{t\geq 0} \subset \mathcal{L}(\mathcal{H})$ is a semigroup of bounded linear operators on $\mathcal{H}$, if

- $E(0) = I$, (identity operator)
- $E(t+s) = E(t)E(s)$, $t, s \geq 0$. (semigroup multiplication)

It is strongly continuous, or C_0, if

$$\lim_{t\to 0+} E(t)x = x \quad \forall x \in \mathcal{H}.$$

Then the generator of the semigroup is the linear operator G defined by

$$Gx = \lim_{t\to 0+} \frac{E(t)x - x}{t}, \quad D(G) = \{x \in \mathcal{H} : Gx \text{ exists}\}.$$

G is usually unbounded but densely defined and closed.

Semigroup

$u(t) = E(t)u_0$ solves the initial-value problem

$$u'(t) = Gu(t), \;\; t > 0; \quad u(0) = u_0,$$

if $u_0 \in D(G)$. Therefore, writing $E(t) = \mathrm{e}^{tG}$ is justified.

There are $M \geq 1$, $\omega \in \mathbf{R}$, such that

$$\|E(t)\|_{\mathcal{L}(\mathcal{H})} \leq M\mathrm{e}^{\omega t}, \quad t \geq 0.$$

Without loss of generality we assume $\omega = 0$ (a shift of the operator $G \mapsto G - \omega I$). Contraction semigroup if also $M = 1$.

If $E(t)$ is invertible, $E(t)^{-1} = E(-t)$, then $\{E(t)\}_{t \in \mathbf{R}}$ is a group.

The semigroup is analytic (holomorphic), if $E(t)$ extends to a complex analytic function $E(z)$ in a sector containing the positive real axis $\mathrm{Re}\, z > 0$. Then the derivative

$$E'(t)u_0 = \frac{\mathrm{d}}{\mathrm{d}t}E(t)u_0 = GE(t)u_0, \quad t > 0,$$

exists for all $u_0 \in \mathcal{H}$, not just for $u_0 \in D(G)$. Moreover,

$$\|E'(t)u_0\|_{\mathcal{H}} = \|GE(t)u_0\|_{\mathcal{H}} \leq Ct^{-1}\|u_0\|_{\mathcal{H}}, \quad t > 0. \tag{1}$$

The inequality (1) is characteristic for analytic semigroups.

Semigroup

On the other hand, we may start with a closed, densely defined, linear operator A and ask for conditions under which $G = -A$ generates a semigroup $E(t) = \mathrm{e}^{-tA}$, so that $u(t) = E(t)u_0$ solves

$$u'(t) + Au(t) = 0, \ t > 0; \quad u(0) = u_0.$$

Such theorems exist, which characterize the generators of strongly continuous (C_0) semigroups, analytic semigroups, and groups. For example, Hille-Yosida theorem, Lumer-Phillips theorem, Stone's theorem.

For analytic semigroups a characterization is given in terms of the resolvent bound

$$\|(z-A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{|z-\omega|}, \quad \text{for } \operatorname{Re} z < \omega, \tag{2}$$

with ω as above ($\omega = 0$ without loss of generality).

Semigroup

The non-homogeneous equation

$$u'(t) + Au(t) = f(t),\ t > 0; \quad u(0) = u_0.$$

is then solved by the variation of constants formula (Duhamel's principle):

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(s)\,\mathrm{d}s,$$

provided that f has some small amount of regularity. This is the basis for our semigroup approach to SPDE.

Proof.

Multiply $u'(s) + Au(s) = f(s)$ by the integrating factor $\Phi(s) = E(t-s) = \mathrm{e}^{-(t-s)A}$, $t > s$, and integrate. □

Laplacian

Let $\mathcal{D} \subset \mathbf{R}^d$ be a bounded, convex, polygonal domain. Then

- finite element meshes can be exactly fitted to $\partial\mathcal{D}$;
- we have elliptic regularity:

$$\|v\|_{H^2(\mathcal{D})} \leq C\|\Delta v\|_{L_2(\mathcal{D})} \quad \forall v \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}).$$

Here $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian. In this way we avoid some technical difficulties associated with the finite element method in smooth domains.

Let $H = L_2(\mathcal{D})$ and $\Lambda = -\Delta$ with $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Then Λ is unbounded in H and self-adjoint with compact inverse Λ^{-1}. The spectral theorem gives eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \ \lambda_j \to \infty, \ \lambda_j \sim j^{2/d} \text{ as } j \to \infty,$$

and a corresponding orthonormal (ON) basis of eigenvectors $\{\varphi_j\}_{j=1}^\infty$.

Laplacian

Parseval's identity is

$$v = \sum_{j=1}^{\infty} \hat{v}_j \varphi_j, \quad \hat{v}_j = \langle v, \varphi_j \rangle_H, \quad \|v\|_H^2 = \sum_{j=1}^{\infty} \hat{v}_j^2, \quad v \in H.$$

Fractional powers:

$$\Lambda^\alpha v = \sum_{j=1}^{\infty} \lambda_j^\alpha \hat{v}_j \varphi_j, \quad \alpha \in \mathbf{R},$$

$$\|v\|_{\dot{H}^\alpha}^2 = \|\Lambda^{\alpha/2}\|_H^2 = \sum_{j=1}^{\infty} \lambda_j^\alpha \hat{v}_j^2, \quad \alpha \in \mathbf{R},$$

$$\dot{H}^\alpha = \{v \in H : \|v\|_{\dot{H}^\alpha} < \infty\} = D(\Lambda^{\alpha/2}), \quad \alpha \geq 0,$$

$$\dot{H}^{-\alpha} = \text{ closure of } H \text{ in the } \dot{H}^{-\alpha}\text{-norm}, \quad \alpha > 0,$$

Then $\dot{H}^{-\alpha}$ can be identified with the dual space $(\dot{H}^\alpha)^*$.

Laplacian

The integer order spaces can be identified with standard Sobolev spaces.

Theorem

(i) $\dot{H}^1 = H_0^1(\mathcal{D})$ *with* $\|v\|_{\dot{H}^1} \simeq \|v\|_{H^1(\mathcal{D})}$ $\forall v \in \dot{H}^1$.
(ii) $\dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ *with* $\|v\|_{\dot{H}^2} \simeq \|v\|_{H^2(\mathcal{D})}$ $\forall v \in \dot{H}^2$.

Proof.

A proof of this can be found in Thomée [5, Ch. 3]. The proof of (i) is based on the Poincaré inequality and the trace inequality. The proof of (ii) uses also the elliptic regularity. In general, we have only

$$\dot{H}^2 \supset H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}),$$

because, in a nonconvex polygonal domain for example, $\dot{H}^2 = D(\Lambda)$ may contain functions with corner singularities which are not in $H^2(\mathcal{D})$. □

Laplacian

We define the heat semigroup:

$$E(t)v = \mathrm{e}^{-t\Lambda}v = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t}\hat{v}_j\varphi_j.$$

It is analytic in the right half plane $\operatorname{Re} z > 0$. Important bounds:

$$\|E(t)v\|_H \le \|v\|_H, \quad t \ge 0, \tag{3}$$

$$\|\Lambda^\alpha E(t)v\|_H \le C_\alpha t^{-\alpha}\|v\|_H, \quad t > 0,\ \alpha \ge 0, \tag{4}$$

$$\int_0^t \|\Lambda^{1/2}E(s)v\|_H^2 \,\mathrm{d}s \le \tfrac{1}{2}\|v\|_H^2, \quad t \ge 0. \tag{5}$$

Recall from (1) that (4) is characteristic for analytic semigroups. It means that the operator $E(t)$ has a smoothing effect. The smoothing effect in (5) is true for the heat semigroup, but not for analytic semigroups in general.

Laplacian

Proof.

We use Parseval and $x^\alpha \mathrm{e}^{-x} \le C_\alpha$ for $x \ge 0$:

$$\|\Lambda^\alpha E(t)v\|_H^2 = \sum_{j=1}^{\infty} \left(\lambda_j^\alpha \mathrm{e}^{-\lambda_j t} \hat{v}_j\right)^2 = t^{-2\alpha} \sum_{j=1}^{\infty} (\lambda_j t)^{2\alpha} \mathrm{e}^{-2\lambda_j t} \hat{v}_j^2$$

$$\le C_\alpha^2 t^{-2\alpha} \sum_{j=1}^{\infty} \hat{v}_j^2 = C_\alpha^2 t^{-2\alpha} \|v\|_H^2.$$

This proves (3) and (4). Similarly,

$$\int_0^t \|\Lambda^{1/2} E(s)v\|_H^2 \, \mathrm{d}s = \int_0^t \sum_{j=1}^{\infty} \lambda_j \mathrm{e}^{-2\lambda_j s} \hat{v}_j^2 \, \mathrm{d}s$$

$$= \sum_{j=1}^{\infty} \int_0^t \lambda_j \mathrm{e}^{-2\lambda_j s} \, \mathrm{d}s \, \hat{v}_j^2 \le \tfrac{1}{2} \|v\|_H^2.$$

□

Remark. The above development based on the spectral representation of fractional powers and the heat semigroup carries over verbatim to more general self-adjoint elliptic operators:

$$\Lambda v = -\nabla \cdot (a(x)\nabla v) + c(x)v \quad \text{with } 0 < a_0 \le a(x) \le a_1,\ c(x) \ge 0,$$

for then we still have an ON basis of eigenvectors. For non-self-adjoint elliptic operators, the fractional powers and the semigroup may be constructed by means of an operator calculus based complex contour integration using the resolvent, see (2). The bounds (3) and (4) are part of the general theory and (5) can be proved by an energy argument if the operator satisfies the conditions of the Lax-Milgram lemma, for example,

$$\Lambda v = -\nabla \cdot (a(x)\nabla v) + b(x) \cdot \nabla v + c(x)v \quad \text{with } c(x) - \tfrac{1}{2}\nabla \cdot b(x) \ge 0,$$

so that

$$\langle \Lambda v, v \rangle_H \ge c\|v\|_{\dot{H}^1}^2.$$

See the following exercises.

Laplacian

Exercise 1. Prove (5) by the energy method: multiply

$$u'(t) + \Lambda u(t) = 0 \tag{6}$$

by $u(t)$ and integrate.

Exercise 2. Prove the special case $\alpha = \frac{1}{2}$ of (4) by the energy method: multiply (6) by $tu'(t)$ and integrate.

Stochastic ODE

$$\begin{cases} \mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma(X(t), t)\,\mathrm{d}B(t), \quad t \in [0, T] \\ X(0) = X_0. \end{cases}$$

This means

$$X(t) = X_0 + \int_0^t \mu(X(s), s)\,\mathrm{d}s + \int_0^t \sigma(X(s), s)\,\mathrm{d}B(s), \quad t \in [0, T].$$

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Could be a system:

$$\mathrm{d}X_i = \mu_i(X_1,\ldots,X_n,t)\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(X_1,\ldots,X_n,t)\,\mathrm{d}B_j(t), \quad i = 1,\ldots,n,$$

$$X = (X_1,\ldots,X_n)^T \in \mathbf{R}^n, \quad \mu : \mathbf{R}^n \times [0,T] \to \mathbf{R}^n, \quad \sigma : \mathbf{R}^n \times [0,T] \to \mathbf{R}^{n\times m},$$

and $B = (B_1,\ldots,B_m)^T$ an m-dimensional Brownian motion, consisting of m independent Brownian motions B_j.

Covariance

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dB(t)$$

If σ is constant:

$$\begin{aligned}
\mathbf{E}\big[(\sigma\Delta B) \otimes (\sigma\Delta B)\big] &= \mathbf{E}\big[(\sigma\Delta B)(\sigma\Delta B)^T\big] \\
&= \mathbf{E}\big[\sigma\Delta B\Delta B^T\sigma^T\big] \\
&= \sigma\mathbf{E}\big[\Delta B\Delta B^T\big]\sigma^T \\
&= \sigma\,(\Delta t\, I)\,\sigma^T = \Delta t\,\sigma\sigma^T = \Delta t\, Q
\end{aligned}$$

Covariance matrix: $Q = \sigma\sigma^T \qquad (n \times m) \times (m \times n) = n \times n$

It is symmetric positive semidefinite.

So σB is a vector-valued Wiener process with covariance matrix Q.

Covariance

$$\mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma(X(t), t)\,\mathrm{d}B(t)$$

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Conversely, given Q we may take $\sigma = Q^{1/2}$ and use $Q^{1/2}B$.

We want to do this in Hilbert space.

Q-Wiener process

Let $\mathcal{U}$ be a separable real Hilbert space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. A random variable is a measurable mapping $f : \Omega \to \mathcal{U}$, i.e.,

$$f^{-1}(A) \in \mathcal{F} \quad \forall A \in \mathcal{B}(\mathcal{U}) \text{ (= the Borel sigma algebra in } \mathcal{U}).$$

We define Lebesgue-Bochner spaces $L_p(\Omega, \mathcal{U})$:

$$\|f\|_{L_p(\Omega,\mathcal{U})} = \Big(\int_\Omega \|f(\omega)\|_{\mathcal{U}}^p \, \mathrm{d}\mathbf{P}(\omega) \Big)^{1/p} = (\mathbf{E}[\|f\|_{\mathcal{U}}^p])^{1/p},$$

and the expected value

$$\mathbf{E}[f] = \int_\Omega f \, \mathrm{d}\mathbf{P}, \quad f \in L_1(\Omega, \mathcal{U}).$$

Q-Wiener process

We start with a covariance operator $Q \in \mathcal{L}(\mathcal{U})$, self-adjoint, positive semidefinite. We assume that it has an eigenbasis:

$$Qe_j = \gamma_j e_j, \quad \gamma_j \geq 0, \quad \{e_j\}_{j=1}^{\infty} \text{ ON basis in } \mathcal{U}.$$

Let $\beta_j(t)$ be independent identically distributed, real-valued, Brownian motions. Define

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

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$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

Two important cases:

- $\mathrm{Tr}(Q) < \infty$. $W(t)$ converges in $L_2(\Omega, \mathcal{U})$:
$$\mathbf{E}\Big\|\sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j\Big\|_{\mathcal{U}}^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$

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  $$\mathbf{E}\Big\| \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j \Big\|_{\mathcal{U}}^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\, \mathrm{Tr}(Q) < \infty$$
- $Q = I$, "white noise". $W(t)$ is not $\mathcal{U}$-valued, since $\mathrm{Tr}(I) = \infty$, but converges in a weaker sense.

Q-Wiener process

If $\mathrm{Tr}(Q) < \infty$:

- $W(0) = 0$.
- continuous paths $t \mapsto W(t)$ in $\mathcal{U}$.
- independent increments: $W(t) - W(s)$ is independent of $W(r)$ for $0 \leq r \leq s \leq t$.
- Gaussian law: $\mathbf{P} \circ (W(t) - W(s))^{-1} \sim \mathcal{N}(0, (t-s)Q), \quad s \leq t$

Q-Wiener process

Proof.

(Covariance.) Let $\Delta W = W(t) - W(s)$. Then

$$
\begin{aligned}
\Big\langle \mathbf{E}\big[\Delta W \otimes \Delta W\big] u, v\Big\rangle_{\mathcal{U}} &= \mathbf{E}\big[\langle \Delta W, u\rangle_{\mathcal{U}} \langle \Delta W, v\rangle_{\mathcal{U}}\big] \\
&= \mathbf{E}\Big[\Big\langle \sum_{j=1}^{\infty} \gamma_j^{1/2} \Delta\beta_j e_j, u\Big\rangle_{\mathcal{U}} \Big\langle \sum_{k=1}^{\infty} \gamma_k^{1/2} \Delta\beta_k e_k, v\Big\rangle_{\mathcal{U}}\Big] \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_j^{1/2}\gamma_k^{1/2} \mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] \langle e_j, u\rangle_{\mathcal{U}} \langle e_k, v\rangle_{\mathcal{U}} \\
&= (t-s)\sum_{j=1}^{\infty} \gamma_j \langle e_j, u\rangle_{\mathcal{U}} \langle e_j, v\rangle_{\mathcal{U}} = (t-s)\langle Qu, v\rangle_{\mathcal{U}},
\end{aligned}
$$

because

$$
\mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] = \begin{cases} \mathbf{E}\big[\Delta\beta_j^2\big] = (t-s), & j = k, \\ \mathbf{E}\big[\Delta\beta_j\big]\,\mathbf{E}\big[\Delta\beta_k\big] = 0, & j \neq k. \end{cases}
$$

□

Q-Wiener process

Let $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and calculate the norm

$$
\begin{aligned}
\|B(W(t) - W(s))\|^2_{L_2(\Omega,\mathcal{H})} &= \mathbf{E}\big[\|B\Delta W\|^2_{\mathcal{H}}\big] \\
&= \mathbf{E}\Big[\Big\langle \sum_{j=1}^{\infty} \gamma_j^{1/2} \Delta\beta_j Be_j, \sum_{k=1}^{\infty} \gamma_k^{1/2} \Delta\beta_k Be_k \Big\rangle_{\mathcal{U}}\Big] \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \gamma_j^{1/2} \gamma_k^{1/2} \mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] \langle Be_j, Be_k \rangle_{\mathcal{U}} = (t-s) \sum_{j=1}^{\infty} \gamma_j \|Be_j\|^2_{\mathcal{H}} \\
&= (t-s) \sum_{j=1}^{\infty} \|B\gamma_j^{1/2} e_j\|^2_{\mathcal{H}} = (t-s) \sum_{j=1}^{\infty} \|BQ^{1/2} e_j\|^2_{\mathcal{H}} \\
&= (t-s) \|BQ^{1/2}\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = (t-s) \|B\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}.
\end{aligned}
$$

Here we used the Hilbert-Schmidt norm of a linear operator $T \colon \mathcal{U} \to \mathcal{H}$:

$$
\|T\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \|T\phi_j\|^2_{\mathcal{H}}, \quad \text{arbitrary ON-basis } \{\phi_j\}_{j=1}^{\infty} \text{ in } \mathcal{U}.
$$

Also, it is useful to introduce $\|B\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})} = \|BQ^{1/2}\|_{\mathcal{L}_2(\mathcal{U},\mathcal{H})}$.

Wiener integral

We want to define $\int_0^T \Phi(t)\,\mathrm{d}W(t)$, where $\Phi \in L_2([0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))$ is a deterministic integrand. The construction goes in three steps.

1. Simple functions.

$$0 = t_0 < \cdots < t_j < \cdots < t_N = T,\ \Phi = \sum_{j=0}^{N-1} \Phi_j \mathbf{1}_{[t_j,t_{j+1})},\ \Phi_j \in \mathcal{L}_2^0(\mathcal{U},\mathcal{H}).$$

Define

$$\int_0^T \Phi(t)\,\mathrm{d}W(t) = \sum_{j=0}^{N-1} \Phi_j\big(W(t_{j+1}) - W(t_j)\big).$$

Wiener integral

2. Itô isometry for simple functions. Using the independence of increments and the previous norm calculation:

$$\Big\|\int_0^T \Phi(t)\,\mathrm{d}W(t)\Big\|^2_{L_2(\Omega,\mathcal{H})} = \mathbf{E}\Big[\Big\|\sum_{j=0}^{N-1}\Phi_j\big(W(t_{j+1})-W(t_j)\big)\Big\|^2_{\mathcal{H}}\Big]$$

$$= \sum_{j=0}^{N-1}\mathbf{E}\Big[\Big\|\Phi_j\big(W(t_{j+1})-W(t_j)\big)\Big\|^2_{\mathcal{H}}\Big]$$

$$= \sum_{j=0}^{N-1}\|\Phi_j\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}(t_{j+1}-t_j) = \int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}\,\mathrm{d}t.$$

So we have an isometry for simple functions:

$$\Phi \mapsto \int_0^T \Phi\,\mathrm{d}W,$$
$$L_2([0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H})) \to L_2(\Omega,\mathcal{H}).$$

3. Extend to all of $L_2([0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))$ by density.

Itô integral

For a random integrand the Itô integral $\int_0^T \Phi \, \mathrm{d}W$ can be defined together with the isometry

$$\mathbf{E}\Big[\Big\| \int_0^T \Phi(t) \, \mathrm{d}W(t) \Big\|_{\mathcal{H}}^2\Big] = \int_0^T \mathbf{E}\big[\|\Phi(t)\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}\big] \, \mathrm{d}t.$$

Here the process $\Phi \colon [0, T] \to \mathcal{L}_2^0(\mathcal{U}, \mathcal{H})$ must be predictable and adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by W and

$$\int_0^T \mathbf{E}\big[\|\Phi(t)\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}\big] \, \mathrm{d}t < \infty.$$

No details here...

Stochastic evolution equation

It is now possible to study the mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

in a situation where E is a general C_0-semigroup. But I do not find this so useful, at least when you have numerical analysis of the equation in mind. I therefore specialize to the heat and wave equations (and, later, Cahn-Hilliard).

Linear stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t > 0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t > 0 \\ u(\xi,0) = u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = L_2(\mathcal{D})$, $\|\cdot\|$, $(\cdot,\cdot)$, $\mathcal{D} \subset \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H^1_0(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{H}$
- $X(t)$, $\mathcal{H}$-valued stochastic process
- $E(t) = e^{-tA}$, analytic semigroup generated by $-A$

Mild solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

Regularity

$$\|v\|_\beta = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta (v, \phi_j)^2 \Big)^{1/2}, \quad \dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

Mean square norm: $\|v\|^2_{L_2(\Omega, \dot{H}^\beta)} = \mathbf{E}(\|v\|^2_\beta), \quad \beta \in \mathbf{R}$

Hilbert-Schmidt norm: $\|T\|_{\mathrm{HS}} = \|T\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}$

Theorem. If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then
$\|X(t)\|_{L_2(\Omega, \dot{H}^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega, \dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} \Big)$

Regularity

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Two cases:

- If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \sum_{j=1}^\infty \|Q^{1/2} e_j\|^2 = \sum_{j=1}^\infty \gamma_j = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ for $\beta < 1/2$.

  $\|\Lambda^{(\beta-1)/2}\|^2_{\mathrm{HS}} = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty$ iff $d = 1$, $\beta < 1/2$

Proof with $X_0 = 0$

(Isometry, arbitrary ON-basis $\{\phi_j\}$)

$$
\begin{aligned}
\|X(t)\|^2_{L_2(\Omega,\dot{H}^\beta)} &= \mathbf{E}\Big(\Big\| \int_0^t \Lambda^{\beta/2} E(t-s)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \int_0^t \|\Lambda^{\beta/2} E(s) Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \int_0^t \|\Lambda^{1/2} E(s) \Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \sum_{k=1}^{\infty} \int_0^t \|\Lambda^{1/2} E(s) \Lambda^{(\beta-1)/2} Q^{1/2}\phi_k\|^2\,\mathrm{d}s \\
&\le C \sum_{k=1}^{\infty} \|\Lambda^{(\beta-1)/2} Q^{1/2}\phi_k\|^2 \\
&= C \|\Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}
$$

$$\boxed{\int_0^t \|\Lambda^{1/2} E(s) v\|^2\,\mathrm{d}s \le \tfrac{1}{2}\|v\|^2}$$

Regularity

We used smoothing of order 1 in the form (5):

$$\int_0^t \|\Lambda^{1/2} E(s) v\|^2 \, \mathrm{d}s \le \tfrac{1}{2} \|v\|^2,$$

which holds for the heat semigroup.
For an analytic semigroup in general we have only (4):

$$\|\Lambda^{\alpha} E(t) v\| \le C_{\alpha} t^{-\alpha} \|v\|,$$

which leads to

$$\sum_{k=1}^{\infty} \int_0^t \|\Lambda^{\frac{1-\epsilon}{2}} E(s) \Lambda^{\frac{\beta+\epsilon-1}{2}} Q^{1/2} \phi_k\|^2 \, \mathrm{d}s$$
$$\le C \int_0^t s^{-1+\epsilon} \, \mathrm{d}s \sum_{k=1}^{\infty} \|\Lambda^{\frac{\beta+\epsilon-1}{2}} Q^{1/2} \phi_k\|^2 \le C \|\Lambda^{\frac{\beta+\epsilon-1}{2}} Q^{1/2}\|_{\mathrm{HS}}^2.$$

An ϵ-loss!

Temporal regularity for the stochastic heat equation

Take $X_0 = 0$ so that (stochastic convolution)

$$X(t) = W_\Lambda(t) = \int_0^t e^{-(t-s)\Lambda} \, dW(s) = \int_0^t E(t-s) \, dW(s).$$

Theorem

If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{HS} < \infty$ *for some* $\beta \in [0, 1]$, *then*

$$\|X(t) - X(s)\|_{L_2(\Omega,H)} \leq C|t-s|^{\frac{\beta}{2}} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{HS}$$

Proof. Take $t > s$ and compute

$$\begin{aligned} X(t) - X(s) &= \int_0^t E(t-r) \, dW(r) - \int_0^s E(s-r) \, dW(r) \\ &= \int_0^s \big(E(t-r) - E(s-r)\big) \, dW(r) \qquad (7) \\ &\quad + \int_s^t E(t-r) \, dW(r) \end{aligned}$$

Temporal regularity

The terms are independent random variables with zero mean and therefore, using also Itô's isometry,

$$\begin{aligned}
\mathbf{E}(\|X(t)-X(s)\|^2) &= \mathbf{E}\Big(\Big\|\int_0^s \big(E(t-s)-I\big)E(s-r)\,\mathrm{d}W(r)\Big\|^2\Big) \\
&\quad + \mathbf{E}\Big(\Big\|\int_s^t E(t-r)\,\mathrm{d}W(r)\Big\|^2\Big) \\
&= \int_0^s \|(E(t-s)-I)E(r)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}r \\
&\quad + \int_0^{t-s} \|E(r)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}r
\end{aligned}$$

For the first term we use

$$\begin{aligned}
&\|(E(t)-I)v\| \le Ct^{\alpha}\|\Lambda^{\alpha}v\|, \quad t>0,\ \alpha\in[0,1], \\
&\|(E(t)-I)\Lambda^{-\alpha}v\| \le Ct^{\alpha}\|v\|, \\
&\|(E(t)-I)\Lambda^{-\alpha}\|_{\mathcal{L}(H)} \le Ct^{\alpha}.
\end{aligned}$$

Temporal regularity

With $\alpha = \beta/2$, $\beta \in [0, 2]$:

$$
\begin{aligned}
&\int_0^s \|(E(t-s) - I)E(r)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&\leq \int_0^s \|(E(t-s) - I)\Lambda^{-\frac{\beta}{2}}\Lambda^{\frac{\beta}{2}} E(r)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&\leq \|(E(t-s) - I)\Lambda^{-\frac{\beta}{2}}\|_{\mathcal{L}(H)}^2 \int_0^s \|\Lambda^{\frac{\beta}{2}} E(r)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&\leq C(t-s)^{\beta}\|\Lambda^{\frac{\beta-1}{2}} Q^{1/2}\|_{\mathrm{HS}}^2.
\end{aligned}
$$

For the second term we use a refined version of (5), see [3, Lemma 3.9]:

$$
\int_0^t \|\Lambda^{\frac{\alpha}{2}} E(s)v\|^2 \,\mathrm{d}s \leq Ct^{1-\alpha}\|v\|^2, \quad \alpha \in [0, 1],
$$

with $\alpha = 1 - \beta$, $\beta \in [0, 1]$:

$$
\begin{aligned}
&\int_0^{t-s} \|E(r)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&= \int_0^{t-s} \|\Lambda^{\frac{1-\beta}{2}} E(r)\Lambda^{\frac{\beta-1}{2}} Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \leq C(t-s)^{\beta}\|\Lambda^{\frac{\beta-1}{2}} Q^{1/2}\|_{\mathrm{HS}}^2.
\end{aligned}
$$

The linear stochastic wave equation

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$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi, t) - \Delta u(\xi, t) = \dot{W}(\xi, t), & \xi \in \mathcal{D} \subset \mathbf{R}^d, \ t > 0 \\ u(\xi, t) = 0, & \xi \in \partial\mathcal{D}, \ t > 0 \\ u(\xi, 0) = u_0, \ \dfrac{\partial u}{\partial t}(\xi, 0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

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$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W,$$

$$X = \begin{bmatrix} u \\ u_t \end{bmatrix},\ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{U} = \dot{H}^0 = L_2(\mathcal{D})$$

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$$\mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}, \quad \mathcal{H} = \mathcal{H}^0 = \dot{H}^0 \times \dot{H}^{-1}, \quad D(A) = \mathcal{H}^1$$

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Here

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})\langle v, \varphi_j\rangle\varphi_j, \quad (\lambda_j, \varphi_j) \text{ are eigenpairs of } \Lambda$$

Regularity

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Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique weak solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s) B \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2} \sin\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega,\mathcal{H}^\beta)} \leq t \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}. \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}$$

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- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.

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- If $\|Q^{1/2}\|_{\mathrm{HS}}^2=\mathrm{Tr}(Q)<\infty$, then $\beta=1$.
- If $Q=I$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}}<\infty$ iff $d=1$, $\beta<1/2$.

Proof for X_1

Isometry:

$$
\begin{aligned}
\|X_1(t)\|^2_{L_2(\Omega,\dot{H}^\beta)} &= \mathbf{E}\Big(\Big\| \int_0^t \Lambda^{\beta/2}\Lambda^{-1/2}\sin\big((t-s)\Lambda^{1/2}\big)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \int_0^t \|\Lambda^{(\beta-1)/2}\sin\big(s\Lambda^{1/2}\big)\,Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \int_0^t \|\sin\big(s\Lambda^{1/2}\big)\Lambda^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&\leq \int_0^t \|\sin\big(s\Lambda^{1/2}\big)\|^2_{\mathcal{L}(H)}\,\mathrm{d}s\,\|\Lambda^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}} \\
&\leq t\|\Lambda^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}
$$

An alternative condition

Note: we do not assume that Λ and Q commute, i.e., we do not assume that they have a common eigenbasis. However, then it may be difficult to verify the condition $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$.

The following theorem gives alternative conditions that imply this.

Theorem

Assume that $Q \in \mathcal{L}(H)$ is selfadjoint, positive semidefinite and that Λ is a densely defined, unbounded, selfadjoint, positive definite, linear operator on H with an orthonormal basis of eigenvectors. Then the following inequalities hold, for $s \in \mathbf{R}$, $\alpha > 0$,

$$\|\Lambda^{\frac{s}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \leq \|\Lambda^s Q\|_{\mathrm{Tr}} \leq \|\Lambda^{s+\alpha} Q\|_{\mathcal{L}(H)} \|\|\Lambda^{-\alpha}\|_{\mathrm{Tr}}, \tag{8}$$

$$\|\Lambda^{\frac{s}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \leq \|\Lambda^{s+\frac{1}{2}} Q \Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}}, \tag{9}$$

provided that the respective norms are finite. Furthermore, if Λ and Q have a common basis of eigenvectors, in particular, if $Q = I$, then

$$\|\Lambda^{\frac{s}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 = \|\Lambda^s Q\|_{\mathrm{Tr}} = \|\Lambda^{s+\frac{1}{2}} Q \Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}}. \tag{10}$$

This is Theorem 2.1 in [5].

An alternative condition

Here $\|T\|_{\mathrm{Tr}} = \|T\|_{\mathcal{L}_1(H)} = \sum_{j=1}^{\infty} \sigma_j$ is the trace norm defined in terms of the singular values σ_j of the trace class operator T, i.e., σ_j are the non-negative square roots of the eigenvalues of TT^*. We have $\|T\|_{\mathrm{Tr}} = \mathrm{Tr}(T)$ if T is self-adjoint positive semidefinite.

Therefore, using (8):

$$\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \leq \|\Lambda^{\beta-1+\alpha} Q\|_{\mathcal{L}(H)} \|\Lambda^{-\alpha}\|_{\mathrm{Tr}},$$

we select $\alpha > 0$ such that $\mathrm{Tr}(\Lambda^{-\alpha}) = \sum_{j=1}^{\infty} \lambda_j^{-\alpha} < \infty$, which is possible. Then it suffices to verify that $\|\Lambda^{\beta-1+\alpha} Q\|_{\mathcal{L}(H)} < \infty$.

Recall the linear stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t>0 \\ X(0) = X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = L_2(\mathcal{D})$, $\|\cdot\|$, $(\cdot,\cdot)$, $\mathcal{D} \subset \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{H}$
- $X(t)$, $\mathcal{H}$-valued stochastic process
- $E(t) = e^{-tA}$, analytic semigroup generated by $-A$

Mild solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

Regularity

$$\|v\|_\beta = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^{\beta} (v, \phi_j)^2 \Big)^{1/2}, \quad \dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

Mean square norm: $\|v\|^2_{L_2(\Omega, \dot{H}^\beta)} = \mathbf{E}(\|v\|^2_\beta), \quad \beta \in \mathbf{R}$

Hilbert-Schmidt norm: $\|T\|_{\mathrm{HS}} = \|T\|_{\mathcal{L}_2(\mathcal{H},\mathcal{H})}$

Theorem. If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then
$\|X(t)\|_{L_2(\Omega, \dot{H}^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega, \dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} \Big)$

Two cases:

- If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \sum_{j=1}^{\infty} \|Q^{1/2} e_j\|^2 = \sum_{j=1}^{\infty} \gamma_j = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ for $\beta < 1/2$.

  $\|\Lambda^{(\beta-1)/2}\|^2_{\mathrm{HS}} = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty$ iff $d = 1$, $\beta < 1/2$

The finite element method

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}$, $\quad S_h \subset H_0^1(\mathcal{D}) = \dot{H}^1$
- S_h continuous piecewise linear functions
- $X_h(t) \in S_h;\ (\mathrm{d}X_h, \chi) + (\nabla X_h, \nabla \chi)\,\mathrm{d}t = (\mathrm{d}W, \chi)\ \forall \chi \in S_h,\ \ t > 0$
- $\Lambda_h \colon S_h \to S_h,$ discrete Laplacian, $(\Lambda_h \psi, \chi) = (\nabla \psi, \nabla \chi)\ \forall \psi, \chi \in S_h$
- $A_h = \Lambda_h$
- $P_h \colon L_2 \to S_h,$ orthogonal projection, $\quad (P_h f, \chi) = (f, \chi)\ \forall \chi \in S_h$

$$\begin{cases} X_h(t) \in S_h, \quad X_h(0) = P_h X_0 \\ \mathrm{d}X_h + A_h X_h\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t > 0 \end{cases}$$

$P_h W(t)$ is a Q_h-Wiener process with $Q_h = P_h Q P_h$.

Mild solution, with $E_h(t) v_h = e^{-tA_h} v_h = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j}$:

$$X_h(t) = E_h(t) P_h X_0 + \int_0^t E_h(t-s) P_h\,\mathrm{d}W(s)$$

Error analysis for elliptic problems

$$u \in H_0^1; \quad \langle \nabla u, \nabla \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in H_0^1$$

$$u_h \in S_h; \quad \langle \nabla u_h, \nabla \chi \rangle = \langle f, \chi \rangle \quad \forall \chi \in S_h$$

Then $u_h = R_h u$, where R_h is the Ritz projector:

$R_h \colon H_0^1 \to S_h$
$\langle \nabla R_h v, \nabla \chi \rangle = \langle \nabla v, \nabla \chi \rangle \quad \forall \chi \in S_h$

Error estimate (elliptic regularity, Aubin-Nitsche duality argument):

$$\|R_h v - v\| \leq C h^2 \|v\|_{H^2} \quad \forall v \in H^2 \cap H_0^1$$

But $\dot{H}^2 = H^2 \cap H_0^1$ with equivalent norms, so that

$$\|R_h v - v\| \leq C h^2 \|v\|_{\dot{H}^2} \quad \forall v \in \dot{H}^2$$

Also:

$$\|P_h v - v\| \leq C h^2 \|v\|_{\dot{H}^2} \quad \forall v \in \dot{H}^2$$

Approximation of the semigroup

$$\begin{cases} u_t + \Lambda u = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} u_{h,t} + \Lambda_h u_h = 0, & t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad u_h(t) = E_h(t)P_h v$$

Denote

$$F_h(t)v = E_h(t)P_h v - E(t)v, \quad \|v\|_\beta = \|\Lambda^{\beta/2} v\|.$$

We have, for $0 \le \beta \le 2$,

- $\|F_h(t)v\| \le Ch^\beta \|v\|_\beta, \quad t \ge 0$
- $\Big(\int_0^t \|F_h(s)v\|^2 \, \mathrm{d}s\Big)^{1/2} \le Ch^\beta \|v\|_{\beta-1}, \quad t \ge 0$
- $\|F_h(t)v\| \le Ch^\beta t^{-\beta/2} \|v\|, \quad t > 0$

First prove for $\beta = 2$ and $\beta = 0$, then interpolate. This can be found in Thomée [5, Chapters 1, 3].

Proof the second error estimate

We prove $\left(\int_0^t \|F_h(s)v\|^2 \, \mathrm{d}s \right)^{1/2} \le Ch^2 \|v\|_1 \quad (\beta = 2).$

Proof. Recall $F_h(t)v = E_h(t)P_h v - E(t)v = u_h(t) - u(t)$, where

$$\langle u_t, \phi \rangle + \langle \nabla u, \nabla \phi \rangle = 0 \quad \forall \phi \in H_0^1(\mathcal{D})$$
$$\langle u_{h,t}, \phi_h \rangle + \langle \nabla u_h, \nabla \phi_h \rangle = 0 \quad \forall \phi_h \in S_h$$

Take $\phi = \phi_h$ and subtract (with $e = u_h - u$):

$$\langle e_t, \phi_h \rangle + \langle \nabla e, \nabla \phi_h \rangle = 0 \quad \forall \phi_h \in S_h$$

Write $e = (u_h - P_h u) + (P_h u - u) = \theta + \rho$. Then $\theta(0) = 0$ and

$$\langle \theta_t, \phi_h \rangle + \langle \nabla \theta, \nabla \phi_h \rangle = -\langle \rho_t, \phi_h \rangle - \langle \nabla \rho, \nabla \phi_h \rangle = -\langle \nabla \rho, \nabla \phi_h \rangle \quad \forall \phi_h \in S_h$$

Proof

We have

$$\langle \theta_t, \phi_h \rangle + \langle \nabla\theta, \nabla\phi_h \rangle = -\langle \nabla\rho, \nabla\phi_h \rangle \quad \forall \phi_h \in S_h; \qquad \theta(0) = 0.$$

Take $\phi_h = \Lambda_h^{-1}\theta$:

$$\begin{aligned}
&\langle \theta_t, \Lambda_h^{-1}\theta \rangle + \langle \nabla\theta, \nabla\Lambda_h^{-1}\theta \rangle = -\langle \nabla\rho, \nabla\Lambda_h^{-1}\theta \rangle \\
&\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_h^{-1/2}\theta\|^2 + \|\theta\|^2 = -\langle \rho, \theta \rangle \le \|\rho\|\,\|\theta\| \le \tfrac{1}{2}\|\rho\|^2 + \tfrac{1}{2}\|\theta\|^2 \\
&\int_0^t \|\theta\|^2 \,\mathrm{d}s \le \int_0^t \|\rho\|^2 \,\mathrm{d}s
\end{aligned}$$

Finally, by $e = \theta + \rho$ and smoothing of order 1, see (5),

$$\int_0^t \|e\|^2 \,\mathrm{d}s \le 2\int_0^t \|\rho\|^2 \,\mathrm{d}s \le Ch^4 \int_0^t \|u\|_2^2 \,\mathrm{d}s \le Ch^4 \|v\|_1^2.$$

Proof

For $\beta = 0$, smoothing of order 1, see (5), holds analogously for E_h:

$$\int_0^t \|\Lambda_h^{1/2} E_h(s) v_h\|^2 \, \mathrm{d}s \le \tfrac{1}{2} \|v_h\|^2.$$

Then

$$\begin{aligned}
\int_0^t \|F_h(s)v\|^2 \, \mathrm{d}s &\le 2\int_0^t \|E_h(s)P_h v\|^2 \, \mathrm{d}s + 2\int_0^t \|E(s)v\|^2 \, \mathrm{d}s \\
&= 2\int_0^t \|\Lambda_h^{1/2} E_h(s) \Lambda_h^{-1/2} P_h v\|^2 \, \mathrm{d}s \\
&\quad + 2\int_0^t \|\Lambda^{1/2} E(s) \Lambda^{-1/2} v\|^2 \, \mathrm{d}s \\
&\le \|\Lambda_h^{-1/2} P_h v\|^2 + \|\Lambda^{-1/2} v\|^2 \le 2\|v\|_{-1}^2
\end{aligned}$$

Proof of $\|\Lambda_h^{-1/2} P_h v\| \le \|\Lambda^{-1/2} v\| = \|v\|_{-1}$ on the next page.

Proof

Proof of $\|\Lambda_h^{-1/2} P_h v\| \leq \|\Lambda^{-1/2} v\| = \|v\|_{-1}$.

$$\begin{aligned}
\|\Lambda_h^{-\frac{1}{2}} P_h v\| &= \sup_{v_h \in S_h} \frac{|(\Lambda_h^{-\frac{1}{2}} P_h v, v_h)|}{\|v_h\|} = \sup_{v_h \in S_h} \frac{|(v, \Lambda_h^{-\frac{1}{2}} v_h)|}{\|v_h\|} \\
&= \sup_{w_h \in S_h} \frac{|(v, w_h)|}{\|\Lambda_h^{\frac{1}{2}} w_h\|} \leq \sup_{w_h \in S_h} \frac{|(v, w_h)|}{\|w_h\|_1} \\
&\leq \sup_{w \in \dot{H}^1} \frac{|(v, w)|}{\|w\|_1} = \sup_{h \in H} \frac{|(v, \Lambda^{-\frac{1}{2}} h)|}{\|h\|} \\
&= \sup_{h \in H} \frac{|(\Lambda^{-\frac{1}{2}} v, h)|}{\|h\|} = \|\Lambda^{-\frac{1}{2}} v\| = \|v\|_{-1}.
\end{aligned}$$

Strong convergence

Theorem

If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,2]$, *then*

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \leq Ch^\beta \Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big).$$

Optimal result: the order of regularity equals the order of convergence.

Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then the convergence rate is $O(h)$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then the rate is almost $O(h^{1/2})$.

No result for $Q = I$, $d \geq 2$.

Strong convergence: proof

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s)$$

$$X_h(t) = E_h(t)P_hX_0 + \int_0^t E_h(t-s)\,P_h\,\mathrm{d}W(s)$$

$$F_h(t) = E_h(t)P_h - E(t)$$

$$X_h(t) - X(t) = F_h(t)X_0 + \int_0^t F_h(t-s)\,\mathrm{d}W(s) = e_1(t) + e_2(t)$$

$$\|F_h(t)X_0\| \le Ch^\beta \|X_0\|_\beta$$

$$\implies \quad \|e_1(t)\|_{L_2(\Omega,H)} \le Ch^\beta \|X_0\|_{L_2(\Omega,\dot{H}^\beta)}$$

Strong convergence: proof

$$\begin{cases} \mathbf{E}\Big\| \int_0^t B(s)\,\mathrm{d}W(s) \Big\|^2 = \mathbf{E} \int_0^t \|B(s)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}s \text{ (isometry)} \\ \Big(\int_0^t \|F_h(s)v\|^2 \,\mathrm{d}s \Big)^{1/2} \le Ch^\beta \|v\|_{\beta-1}, \text{ with } v = Q^{1/2}\varphi_l \end{cases}$$

$\Longrightarrow$

$$\begin{aligned} \|e_2(t)\|_{L_2(\Omega,H)}^2 &= \mathbf{E}\Big\| \int_0^t F_h(t-s)\,\mathrm{d}W(s) \Big\|^2 = \int_0^t \|F_h(t-s)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}s \\ &= \sum_{l=1}^\infty \int_0^t \|F_h(t-s)Q^{1/2}\varphi_l\|^2 \,\mathrm{d}s \le C \sum_{l=1}^\infty h^{2\beta} \|Q^{1/2}\varphi_l\|_{\beta-1}^2 \\ &= Ch^{2\beta} \sum_{l=1}^\infty \|\Lambda^{(\beta-1)/2} Q^{1/2}\varphi_l\|^2 = Ch^{2\beta} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}^2 \end{aligned}$$

Another type of estimate

Take $X_0 = 0$ so that $X(t) = W_\Lambda(t) = \int_0^t E(t-s)\,\mathrm{d}W(s)$.

We have shown

$$\Big(\sup_{t\in[0,T]} \mathbf{E}\Big[\|W_\Lambda(t)\|_\beta^2\Big]\Big)^{1/2} \le C\|\Lambda^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathrm{HS}}$$

$$\Big(\sup_{t\in[0,T]} \mathbf{E}\Big[\|W_{\Lambda_h}(t) - W_\Lambda(t)\|^2\Big]\Big)^{1/2} \le Ch^\beta\|\Lambda^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathrm{HS}}$$

Theorem

Let $\epsilon \in (0,1]$ and $p > 2/\epsilon$. Then

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]} \|W_\Lambda(t)\|_\beta^p\Big]\Big)^{1/p} \le C_\epsilon\|\Lambda^{\frac{\beta-1}{2}+\epsilon}Q^{1/2}\|_{\mathrm{HS}}$$

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]} \|W_{\Lambda_h}(t) - W_\Lambda(t)\|^p\Big]\Big)^{1/p} \le C_\epsilon h^\beta\|\Lambda^{\frac{\beta-1}{2}+\epsilon}Q^{1/2}\|_{\mathrm{HS}}$$

Another type of estimate

Proof. The proof is an adaptation of the 'factorization method' in the proof of [2, Theorem 5.9, Remark 5.11]:

$$
\begin{aligned}
W_\Lambda(t) &= \int_0^t E(t-s)\,\mathrm{d}W(s) \\
&= c_\alpha \int_0^t E(t-s) \int_\sigma^t (t-s)^{-1+\alpha}(s-\sigma)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}W(s) \\
&= c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma)\,\mathrm{d}W(\sigma)\,\mathrm{d}s \\
&= c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s)\, Y(s)\,\mathrm{d}s \\
Y(s) &= \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma)\,\mathrm{d}W(\sigma) \\
c_\alpha^{-1} &= \int_\sigma^t (t-s)^{-1+\alpha}(s-\sigma)^{-\alpha}\,\mathrm{d}s = \frac{\pi}{\sin(\pi\alpha)}
\end{aligned}
$$

Another type of estimate

Idea of the proof:

$$Y(s) = \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma)\,\mathrm{d}W(\sigma)$$
$$W_\Lambda(t) = c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) Y(s)\,\mathrm{d}s$$

Hölder:

$$\|\Lambda^{\frac{\beta}{2}} W_\Lambda(t)\|^p \le c_\alpha \Big(\int_0^T \big(s^{-1+\alpha}\|E(s)\|\big)^{\frac{p}{p-1}}\,\mathrm{d}s\Big)^{p-1} \int_0^T \|\Lambda^{\frac{\beta}{2}} Y(s)\|^p\,\mathrm{d}s$$

and, hence,

$$\mathbf{E}\Big[\sup_{t\in[0,T]} \|\Lambda^{\frac{\beta}{2}} W_\Lambda(t)\|^p\Big] \le c_\alpha \Big(\int_0^T \big(s^{-1+\alpha}\|E(s)\|\big)^{\frac{p}{p-1}}\,\mathrm{d}s\Big)^{p-1} \times \int_0^T \mathbf{E}\big[\|\Lambda^{\frac{\beta}{2}} Y(s)\|^p\big]\,\mathrm{d}s$$

Time discretization

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

The implicit Euler method:
$k = \Delta t,\ t_n = nk,\ \Delta W^n = W(t_n) - W(t_{n-1})$

$$\begin{cases} X_h^n \in S_h, \quad X_h^0 = P_h X_0 \\ X_h^n - X_h^{n-1} + kA_h X_h^n = P_h \Delta W^n, \end{cases}$$

$$X_h^n = E_{kh} X_h^{n-1} + E_{kh} P_h \Delta W^n, \quad E_{kh} = (I + kA_h)^{-1}$$

$$X_h^n = E_{kh}^n P_h X_0 + \sum_{j=1}^{n} E_{kh}^{n-j+1} P_h \Delta W^j$$

$$X(t_n) = E(t_n) X_0 + \int_0^{t_n} E(t_n - s)\,\mathrm{d}W(s)$$

Approximation of the semigroup

Denote $F_n = E_{kh}^n P_h - E(t_n)$

We have the following estimates for $0 \leq \beta \leq 2$:

- $\|F_n v\| \leq C(k^{\beta/2} + h^{\beta})\|v\|_{\beta}$
- $\Big(k \sum_{j=1}^{n} \|F_j v\|^2\Big)^{1/2} \leq C(k^{\beta/2} + h^{\beta})\|v\|_{\beta-1}$

See Thomée [5].

Strong convergence

Theorem
If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,2]$*, then, with* $e^n = X_h^n - X(t_n)$,

$$\|e^n\|_{L_2(\Omega,H)} \leq C(k^{\beta/2} + h^{\beta})\Big(\|X_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big)$$

The reason why we can have k^1 (when $\beta = 2$) is that the Euler-Maruyama method is exact in the stochastic integral for additive noise.

J. Printems [4] (only time-discretization)

Y. Yan [2, 3]

Recall the linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \|v\|_\beta = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta (v,\varphi_j)^2 \Big)^{1/2}, \quad \beta \in \mathbf{R}$$

$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W,$$

$$X = \begin{bmatrix} u \\ u_t \end{bmatrix},\ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{U} = \dot{H}^0 = L_2(\mathcal{D})$$

$$\mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}, \quad \mathcal{H} = \mathcal{H}^0 = \dot{H}^0 \times \dot{H}^{-1}, \quad D(A) = \mathcal{H}^1$$

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

- $\{X(t)\}_{t\geq 0}$, $\mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $\{W(t)\}_{t\geq 0}$, $\mathcal{U} = \dot{H}^0$-valued Q-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$
- $E(t) = e^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}$,
  C_0-semigroup on $\mathcal{H}$ (actually a group) but not analytic

Here

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})\langle v, \varphi_j\rangle \phi_j, \quad (\lambda_j, \phi_j) \text{ are eigenpairs of } \Lambda$$

Regularity

Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique weak solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s) B \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2} \sin\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega, \mathcal{H}^\beta)} \leq C(t) \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}, \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}.$$

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- $P_h \colon \dot{H}^0 \to S_h,$ orthogonal projection, $\quad (P_h f, \chi) = (f, \chi), \ \forall \chi \in S_h$
- $A_h = \begin{bmatrix} 0 & -I \\ \Lambda_h & 0 \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ P_h \end{bmatrix}$

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- $\Lambda_h \colon S_h \to S_h,$ discrete Laplacian, $\quad (\Lambda_h \psi, \chi) = (\nabla \psi, \nabla \chi),\ \forall \chi \in S_h$
- $P_h \colon \dot{H}^0 \to S_h,$ orthogonal projection, $\quad (P_h f, \chi) = (f, \chi),\ \forall \chi \in S_h$
- $A_h = \begin{bmatrix} 0 & -I \\ \Lambda_h & 0 \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ P_h \end{bmatrix}$
- $\begin{cases} \mathrm{d}X_h(t) + A_h X_h(t)\,\mathrm{d}t = B_h\,\mathrm{d}W(t), & t > 0 \\ X_h(0) = X_{0,h} \end{cases}$

Spatial discretization

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}$
- $S_h \subset \dot{H}^1 = H_0^1(\mathcal{D})$ continuous piecewise polynomials of degree ≤ 1
- $\Lambda_h \colon S_h \to S_h,$ discrete Laplacian, $\quad (\Lambda_h \psi, \chi) = (\nabla \psi, \nabla \chi), \ \forall \chi \in S_h$
- $P_h \colon \dot{H}^0 \to S_h,$ orthogonal projection, $\quad (P_h f, \chi) = (f, \chi), \ \forall \chi \in S_h$
- $A_h = \begin{bmatrix} 0 & -I \\ \Lambda_h & 0 \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ P_h \end{bmatrix}$
- $\begin{cases} \mathrm{d}X_h(t) + A_h X_h(t)\,\mathrm{d}t = B_h\,\mathrm{d}W(t), & t > 0 \\ X_h(0) = X_{0,h} \end{cases}$
- $E_h(t) = e^{-tA_h} = \begin{bmatrix} \cos(t\Lambda_h^{1/2}) & \Lambda_h^{-1/2}\sin(t\Lambda_h^{1/2}) \\ -\Lambda_h^{1/2}\sin(t\Lambda_h^{1/2}) & \cos(t\Lambda_h^{1/2}) \end{bmatrix}$

Spatial discretization

The weak solution is:

$$X_h(t) = \begin{bmatrix} X_{h,1}(t) \\ X_{h,2}(t) \end{bmatrix}$$

$$= \int_0^t E_h(t-s) B_h \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda_h^{-1/2} \sin\left((t-s)\Lambda_h^{1/2}\right) P_h \, \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda_h^{1/2}\right) P_h \, \mathrm{d}W(s) \end{bmatrix}$$

where, for example,

$$\cos(t\Lambda_h^{1/2})v = \sum_{j=1}^{N_h} \cos\left(t\sqrt{\lambda_{h,j}}\right) \langle v, \varphi_{h,j} \rangle \varphi_{h,j},$$

and $(\lambda_{h,j}, \varphi_{h,j})$ are eigenpairs of Λ_h.

Spatially semidiscrete: approximation of the semigroup

$$\begin{cases} v_{tt}(t) + \Lambda v(t) = 0, \ t > 0 \\ v(0) = 0, \ v_t(0) = f \end{cases} \qquad \Rightarrow v(t) = \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right) f$$

$$\begin{cases} v_{h,tt}(t) + \Lambda_h v_h(t) = 0, \ t > 0 \\ v_h(0) = 0, \ v_{h,t}(0) = P_h f \end{cases} \qquad \Rightarrow v_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h f$$

We have, for $K_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h - \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right)$ and $r = 2$,

$\|K_h(t)f\| \leq C(t)h^2 \|f\|_{\dot{H}^2}$ "initial regularity of order 3"

$\|K_h(t)f\| \leq 2\|f\|_{\dot{H}^{-1}}$ "initial regularity of order 0" (stability)

$$\|K_h(t)f\| \leq C(t)h^{\frac{2}{3}\beta} \|f\|_{\dot{H}^{\beta-1}}, \quad 0 \leq \beta \leq 3$$

$\beta - 1$ can not be replaced by $\beta - 1 - \epsilon$ for $\epsilon > 0$ (J. Rauch 1985)

Note: $\|v(t)\|_{\dot{H}^2} \leq \|f\|_{\dot{H}^1}$ "initial regularity of order 2"

Spatially semidiscrete: Strong convergence

Theorem. If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0, 3]$, then

$$\|X_{h,1}(t) - X_1(t)\|_{L_2(\Omega,\dot{H}^0)} \le C(t)\, h^{\frac{2}{3}\beta} \big\|\Lambda^{(\beta-1)/2}Q^{1/2}\big\|_{\mathrm{HS}}$$

Higher order FEM: $\quad O(h^{\frac{r}{r+1}\beta}), \quad \beta \in [0, r+1].$

Proof. $\{f_k\}$ an arbitrary ON basis in $\dot{H}^0$

$$\begin{aligned}
\|X_{h,1}(t) - X_1(t)\|^2_{L_2(\Omega,\dot{H}^0)} &= \mathbf{E}\big(\|X_{h,1}(t) - X_1(t)\|^2\big) \\
&= \mathbf{E}\Big(\Big\|\int_0^t K_h(t-s)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \int_0^t \big\|K_h(s)Q^{1/2}\big\|^2_{\mathrm{HS}}\,\mathrm{d}s = \int_0^t \sum_{k=1}^{\infty} \big\|K_h(s)Q^{1/2}f_k\big\|^2\,\mathrm{d}s \\
&\le C(t)h^{\frac{4}{3}\beta} \sum_{k=1}^{\infty} \big\|Q^{1/2}f_k\big\|^2_{\dot{H}^{\beta-1}} = C(t)h^{\frac{4}{3}\beta} \big\|\Lambda^{(\beta-1)/2}Q^{1/2}\big\|^2_{\mathrm{HS}}
\end{aligned}$$

This is from [2].
Time stepping is studied in [4].

Nonlinear problems

This kind of analysis carries over (with some limitations) to nonlinear problems

$$\mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W$$

if the operators F, G are globally Lipschitz in the appropriate senses.

Nonlinear problems

This kind of analysis carries over (with some limitations) to nonlinear problems

$$dX + AX\,dt = F(X)\,dt + G(X)\,dW$$

if the operators F, G are globally Lipschitz in the appropriate senses.

For example:

$$\begin{aligned} &\{E(t)\}_{t\geq 0} \text{ analytic}, \quad \operatorname{Tr}(Q) < \infty \\ &\|F(u) - F(v)\|_H \leq C\|u - v\|_H \\ &\|(G(u) - G(v))Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{U},H)} \leq C\|u - v\|_H \end{aligned}$$

Then we have spatial regularity: $L_p(\Omega, \dot{H}^\gamma)$, and temporal regularity: Hölder $\gamma/2$ in $L_p(\Omega, H)$ for $\gamma \in [0, 1)$, $p \geq 2$ [2].

Nonlinear problems

$$\mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W$$

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\end{aligned}$$

Nonlinear problems

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$$\|(G(u) - G(v))Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{U},H)} \leq C\|u - v\|_H$$

Jentzen and Röckner [3] introduced a linear growth bound:

$$\|A^{\frac{\beta-1}{2}}G(u)Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{U},H)} \leq C(1 + \|u\|_{\dot{H}^{\beta-1}})$$

Nonlinear problems

$$\mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W$$

For example:

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&\{E(t)\}_{t\geq 0} \text{ analytic}, \quad \mathrm{Tr}(Q) < \infty \\
&\|F(u) - F(v)\|_H \leq C\|u - v\|_H \\
&\|(G(u) - G(v))Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{U},H)} \leq C\|u - v\|_H
\end{aligned}$$

Jentzen and Röckner [3] introduced a linear growth bound:

$$\|A^{\frac{\beta-1}{2}} G(u)Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{U},H)} \leq C(1 + \|u\|_{\dot{H}^{\beta-1}})$$

Kruse and L [3] assumed that A is self-adjoint with compact inverse so that the "special smoothing of order 1" (5) holds.

Then, for $\beta \in [0, 2)$, $p \in [1, \infty)$,

$$\|X(t)\|_{L_p(\Omega,\dot{H}^\beta)} \leq C,$$

and Hölder in t with exponent $\min(\frac{1}{2}, \frac{\beta}{2})$ in $L_p(\Omega, H)$.

Nonlinear problems

I will speak about this in the next lecture.

Today: A problem with locally Lipschitz nonlinearity: Cahn-Hilliard-Cook

Cahn-Hilliard-Cook equation

$$\begin{cases} \mathrm{d}u - \Delta v \,\mathrm{d}t = \mathrm{d}W & \text{in } \mathcal{D} \times [0, T] \\ v = -\Delta u + f(u) & \text{in } \mathcal{D} \times [0, T] \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D} \times [0, T] \\ u(0) = u_0 & \text{in } \mathcal{D} \end{cases}$$

Here $f(u) = u^3 - u$. Eliminate v.

Set $X(t) = u(t) \in H = L_2(\mathcal{D})$.

H-valued stochastic process: $X(t)$.

Let $\Lambda = -\Delta$ be the Neumann Laplacian in H.

$W(t)$, a Q-Wiener process in H with respect to $\{\mathcal{F}_t\}$.

$$\begin{cases} \mathrm{d}X + \big(\Lambda^2 X + \Lambda f(X)\big)\,\mathrm{d}t = \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

The Cahn-Hilliard semigroup

$\Lambda = -\Delta$ is the Neumann Laplacian in H.

Eigenvalues: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty.$

Orthonormal eigenbasis: $\{\varphi_j\}_{j=0}^{\infty}, \quad \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$

$$E(t)v = \mathrm{e}^{-t\Lambda^2} v = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j + \langle v, \varphi_0 \rangle \varphi_0$$

$$\begin{cases} \dot{u} + \Lambda^2 u = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \Rightarrow u(t) = E(t)v$$

$$\begin{cases} \dot{u} + \Lambda^2 u = f, & t > 0 \\ u(0) = v \end{cases} \qquad \Rightarrow u(t) = E(t)v + \int_0^t E(t-s) f(s) \, \mathrm{d}s$$

Thus: $A = \Lambda^2$ here.

CHC: abstract formulation

$$\begin{cases} \mathrm{d}X + \Lambda^2 X \,\mathrm{d}t = -\Lambda f(X)\,\mathrm{d}t + \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

The mild solution is given by the equation:

$$\begin{aligned} X(t) &= \mathrm{e}^{-t\Lambda^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)\Lambda^2} \Lambda f(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)\Lambda^2}\,\mathrm{d}W(s) \\ &= Y(t) + W_\Lambda(t) \end{aligned}$$

Stochastic convolution:

$$W_\Lambda(t) = \int_0^t \mathrm{e}^{-(t-s)\Lambda^2}\,\mathrm{d}W(s)$$

Random evolution problem:

$$\begin{cases} \dot{Y} + \Lambda^2 Y + \Lambda f(Y + W_\Lambda) = 0, & t > 0 \\ Y(0) = X_0. \end{cases}$$

The finite element method

Spatial discretization

- family of triangulations of $\mathcal{D}$: $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- family of finite element spaces: $\{S_h\}_{0<h<1}$
- $S_h \subset H^1(\mathcal{D})$, continuous piecewise linear functions
- Galerkin method: $u_h(t), v_h(t) \in S_h$
  $$\begin{cases} \langle \mathrm{d}u_h, \chi\rangle + \langle \nabla v_h, \nabla\chi\rangle \,\mathrm{d}t = \langle \mathrm{d}W, \chi\rangle \quad \forall \chi \in S_h,\ t>0 \\ \langle v_h, \chi\rangle = \langle \nabla u_h, \nabla\chi\rangle + \langle f(u_h), \chi\rangle \quad \forall \chi \in S_h,\ t>0 \end{cases}$$
- $\Lambda_h\colon S_h \to S_h$, discrete Laplacian, $\langle \Lambda_h \psi, \chi\rangle = \langle \nabla\psi, \nabla\chi\rangle$, $\forall \chi \in S_h$
- $P_h\colon H \to S_h$, orthogonal projector, $\langle P_h f, \chi\rangle = \langle f, \chi\rangle$, $\forall \chi \in S_h$
- $$\begin{cases} \mathrm{d}X_h + \Lambda_h^2 X_h \,\mathrm{d}t + \Lambda_h P_h f(X_h)\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t>0 \\ X(0) = P_h X_0 \end{cases}$$
- eigenvalues: $0 = \lambda_{h,0} < \lambda_{h,1} \le \cdots \le \lambda_{h,j} \le \cdots \le \lambda_{h,N_h}$
- orthonormal eigenbasis: $\{\varphi_{h,j}\}_{j=0}^{N_h}$, $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$
- semigroup:
  $$E_h(t)v_h = \mathrm{e}^{-t\Lambda_h^2} v_h = \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v_h, \varphi_{h,j}\rangle \varphi_{h,j}$$

CHC: finite element approximation

$$\begin{cases} \mathrm{d}X_h + \Lambda_h^2 X_h \,\mathrm{d}t = -\Lambda_h P_h f(X_h)\,\mathrm{d}t + P_h\,\mathrm{d}W, \quad t > 0 \\ X(0) = P_h X_0 \end{cases}$$

The mild solution is given by the equation:

$$\begin{aligned} X_h(t) &= \mathrm{e}^{-t\Lambda_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)\Lambda_h^2} \Lambda_h P_h f(X_h(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)\Lambda_h^2} P_h\,\mathrm{d}W(s) \\ &= Y_h(t) + W_{\Lambda_h}(t) \end{aligned}$$

Stochastic convolution: $W_{\Lambda_h}(t) = \int_0^t \mathrm{e}^{-(t-s)\Lambda_h^2} P_h\,\mathrm{d}W(s)$

Random evolution problem:

$$\begin{cases} \dot{Y}_h + \Lambda_h^2 Y_h + \Lambda_h P_h f(Y_h + W_{\Lambda_h}) = 0, \quad t > 0 \\ Y_h(0) = P_h X_0 \end{cases}$$

Linear CHC: approximation of the semigroup

$$\begin{cases} \dot{u} + \Lambda^2 u = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} \dot{u}_h + \Lambda_h^2 u_h = 0, & t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad u_h(t) = E_h(t)P_h v$$

Error: $F_h(t)v = E_h(t)P_h v - E(t)v$, seminorm: $|v|_\beta = \|\Lambda^{\beta/2} v\|$

Theorem

- $\|F_h(t)Pv\| \le Ch^\beta |v|_\beta, \quad t \ge 0, \quad \beta \in [0,2]$
- $\Big(\int_0^t \|F_h(s)Pv\|^2 \, \mathrm{d}s \Big)^{1/2} \le Ch^\beta |\log(h)| \, |v|_{\beta-2}, \quad t \ge 0, \quad \beta \in [1,2]$

Note: our FEM is based on $(\Lambda_h)^2$ instead of $(\Lambda^2)_h$.

Linear CHC: regularity and strong convergence

Theorem

If $\|\Lambda^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [1,2]$, *then*

$$\|W_\Lambda(t)\|_{L_2(\Omega,\dot H^\beta)} \le C\,\|\Lambda^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}, \quad t \ge 0,$$

$$\|W_{\Lambda_h}(t) - W_\Lambda(t)\|_{L_2(\Omega,H)} \le Ch^\beta|\log(h)|\,\|\Lambda^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}, \quad t \ge 0.$$

Proof:

$$\begin{aligned}
&\|W_{\Lambda_h}(t) - W_\Lambda(t)\|^2_{L_2(\Omega,H)} \\
&\quad = \mathbf{E}\Big\|\int_0^t F_h(t-s)P\,\mathrm{d}W(s)\Big\|^2 = \int_0^t \|F_h(t-s)PQ^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&\quad = \sum_{j=1}^\infty \int_0^t \|F_h(t-s)PQ^{1/2}\phi_j\|^2\,\mathrm{d}s \le C\sum_{j=1}^\infty h^{2\beta}|\log(h)|^2|Q^{1/2}\phi_j|^2_{\beta-2} \\
&\quad = Ch^{2\beta}|\log(h)|^2\sum_{j=1}^\infty \|\Lambda^{(\beta-2)/2}Q^{1/2}\phi_j\|^2 \\
&\quad = Ch^{2\beta}|\log(h)|^2\|\Lambda^{(\beta-2)/2}Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}$$

Linear CHC

We also show

Theorem

Let $\epsilon \in (0,1]$ *and* $p > 2/\epsilon$. *Then*

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]} \|W_\Lambda(t)\|_3^p\Big]\Big)^{1/p} \le C_\epsilon \|\Lambda^{\frac{1}{2}+\epsilon} Q^{1/2}\|_{\mathrm{HS}} \quad (\beta = 3 + eps)$$

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]} \|W_{\Lambda_h}(t) - W_\Lambda(t)\|^p\Big]\Big)^{1/p} \le C_\epsilon h^2 \|\Lambda^{\epsilon} Q^{1/2}\|_{\mathrm{HS}} \quad (\beta = 2 + eps)$$

Proof. The factorization method.

Linear CHC: strong convergence

Larsson and Mesforush, IMAJNA (2011).
Euler timestepping is also studied here.

Kossioris and Zouraris, M2AN (2010) (1-D)

Cahn-Hilliard-Cook equation

$$\begin{cases} \mathrm{d}X + \Lambda^2 X\,\mathrm{d}t = -\Lambda f(X)\,\mathrm{d}t + \mathrm{d}W, \quad t > 0 \\ X(0) = X_0 \end{cases}$$

The mild solution is given by the equation:

$$\begin{aligned} X(t) &= \mathrm{e}^{-t\Lambda^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)\Lambda^2} \Lambda f(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)\Lambda^2}\,\mathrm{d}W(s) \\ &= Y(t) + W_\Lambda(t) \end{aligned}$$

The stochastic convolution is now known:

$$W_\Lambda(t) = \int_0^t \mathrm{e}^{-(t-s)\Lambda^2}\,\mathrm{d}W(s)$$

Remains to solve the random evolution problem:

$$\begin{cases} \dot{Y} + \Lambda^2 Y + \Lambda f(Y + W_\Lambda) = 0, \quad t > 0 \\ Y(0) = X_0. \end{cases}$$

Cahn-Hilliard-Cook equation

$$Y(t) = \mathrm{e}^{-t\Lambda^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)\Lambda^2} \Lambda f\big(Y(s) + W_\Lambda(s)\big)\,\mathrm{d}s$$

Controlling the non-linearity: $f(s) = s^3 - s$

- $\|\Lambda f(u)\| \leq C\big(1 + \|u\|_{H^1}^2\big)\|u\|_{H^3}$
- $|f(u) - f(v)|_{-1} \leq C\big(1 + \|u\|_{H^1}^2 + \|v\|_{H^1}^2\big)\|u - v\|$

Useful to bound $\|X(t)\|_{H^1}$ and $\|X_h(t)\|_{H^1}$.

Cahn-Hilliard-Cook equation

Energy functional:

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u \in H^1, \quad F(s) = \tfrac{1}{4}s^4 - \tfrac{1}{2}s^2,$$

Deterministic case: $J(X(t)) \le J(X_0), \quad t \ge 0 \quad$ (Lyapunov functional)

Stochastic case:

Theorem

If $\|\Lambda^{1/2}Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$ ($\beta = 3$), then

$$\mathbf{E}\big[J(X(t))\big] \le C(t), \quad \mathbf{E}\big[J(X_h(t))\big] \le C(t), \quad t \ge 0.$$

Cahn-Hilliard-Cook equation

Proof for $J(X_h(t))$:

$$\begin{aligned} J(u_h) &= \frac{1}{2}\|\nabla u_h\|^2 + \int_{\mathcal{D}} F(u_h)\,\mathrm{d}x \\ J'(u_h) &= \Lambda_h u_h + P_h f(u_h) \\ \mathrm{d}X_h &= -\Lambda_h^2 X_h\,\mathrm{d}t - \Lambda_h P_h f(X_h)\,\mathrm{d}t + P_h\,\mathrm{d}W \\ &= -\Lambda_h J'(X_h)\,\mathrm{d}t + P_h\,\mathrm{d}W \end{aligned}$$

Itô's formula:

$$\begin{aligned} J(X_h(t)) &= J(X_h(0)) + \int_0^t \langle J'(X_h(s)), \mathrm{d}X_h(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s \\ &= J(P_h X_0) - \int_0^t \langle J'(X_h(s)), \Lambda_h J'(X_h(s))\rangle\,\mathrm{d}s \\ &\quad + \int_0^t \langle J'(X_h(s)), P_h\,\mathrm{d}W(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s. \end{aligned}$$

Cahn-Hilliard-Cook equation

$$
\begin{aligned}
&\mathbf{E}\big[J(X_h(t))\big] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2 \,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\big[J(P_h X_0)\big] + \underbrace{\mathbf{E}\Big[\int_0^t \langle J'(X_h(s)), P_h \,\mathrm{d}W(s)\rangle\Big]}_{=0} \\
&\qquad + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\big[J(P_h X_0)\big] + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}(\Lambda_h Q_h)\,\mathrm{d}s\Big] + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}([f'(X_h(s))\cdot]Q_h)\,\mathrm{d}s\Big]
\end{aligned}
$$

Cahn-Hilliard-Cook equation

If $\|\Lambda^{1/2}Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$ ($\beta = 3$), then

$$\mathbf{E}\big[J(X(t))\big] \le C(t), \quad \mathbf{E}\big[J(X_h(t))\big] \le C(t), \quad t \ge 0.$$

Hence: $\mathbf{E}\big[\|X(t)\|_{H^1}^2\big] \le C(t), \quad \mathbf{E}\big[\|X_h(t)\|_{H^1}^2\big] \le C(t), \quad t \ge 0.$

Generalization of Da Prato and Debussche (1996) [1]:

- do not assume common eigenbasis for Λ and Q.
- do not assume max-norm bound for the eigenbasis of Q: $\|e_j\|_{L_\infty(\mathcal{D})} \le C$.
- the growth $C(t)$ is quadratic instead of exponential.
- same bound for X_h.

We also show

$$\mathbf{E}\Big[\sup_{t\in[0,T]} \|X(t)\|_{H^1}^2\Big] \le K_T, \quad \mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2\Big] \le K_T.$$

Cahn-Hilliard-Cook equation

$$\mathbf{E}\big[\|X(t)\|_{H^1}^2\big] \le C(t), \quad t \ge 0.$$

$$\|\Lambda f(X)\| \le C\big(1 + \|X\|_{H^1}^2\big)\|X\|_{H^3}.$$

$$\mathbf{E}\Big[\|X\|_{H^1}^2\|X\|_{H^3}\Big] \le \mathbf{E}\Big[\|X\|_{H^1}^4\Big]^{\frac{1}{2}}\mathbf{E}\Big[\|Y + W_\Lambda\|_{H^3}^2\Big]^{\frac{1}{2}}.$$

Need higher moments!

Cahn-Hilliard-Cook equation

$$\mathbf{E}\big[\|X(t)\|_{H^1}^2\big] \le C(t), \quad t \ge 0.$$

$$\|\Lambda f(X)\| \le C\big(1 + \|X\|_{H^1}^2\big)\|X\|_{H^3}.$$

$$\mathbf{E}\Big[\|X\|_{H^1}^2\|X\|_{H^3}\Big] \le \mathbf{E}\Big[\|X\|_{H^1}^4\Big]^{\frac{1}{2}}\mathbf{E}\Big[\|Y + W_\Lambda\|_{H^3}^2\Big]^{\frac{1}{2}}.$$

Need higher moments!

Alternative: Apply Chebyshev's inequality to

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\big(\|X(t)\|_{H^1}^2 + \|X_h(t)\|_{H^1}^2\big)\Big] \le K_T.$$

For each $T > 0$, $h \in (0,1]$, and $\epsilon \in (0,1)$ there are K_T and $\Omega_\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_\epsilon) \ge 1 - \epsilon$ and such that

$$\sup_{t\in[0,T]}\big(\|X(t)\|_{H^1}^2 + \|X_h(t)\|_{H^1}^2\big) \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon.$$

Now we can control the nonlinearity pointwise on $\Omega_\epsilon \times [0, T]$.

Chebyshev's inequality

We have $\mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2\Big] \leq K_T$ and take $\epsilon > 0$.

Chebyshev:

$$\mathbf{P}\Big\{\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2 > \alpha\Big\} \leq \alpha^{-1}\mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2\Big] \leq \alpha^{-1}K_T$$

Chebyshev's inequality

We have $\mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2\Big] \leq K_T$ and take $\epsilon > 0$.

Chebyshev:

$$\mathbf{P}\Big\{\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2 > \alpha\Big\} \leq \alpha^{-1}\mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2\Big] \leq \alpha^{-1}K_T = \epsilon$$

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We have $\mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2\Big] \leq K_T$ and take $\epsilon > 0$.

Chebyshev:

$$\mathbf{P}\Big\{\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2 > \alpha\Big\} \leq \alpha^{-1}\mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2\Big] \leq \alpha^{-1}K_T = \epsilon$$

Choose $\alpha = \epsilon^{-1}K_T$

and $\Omega_\epsilon = \Big\{\sup_{t\in[0,T]} \|X_h(t)\|_{H^1}^2 \leq \alpha = \epsilon^{-1}K_T\Big\}$.

Then $\mathbf{P}(\Omega_\epsilon) \geq 1 - \epsilon$.

Nonlinear CHC

Now we can use deterministic analysis to prove bounds for $\|Y\|_3$ and $\|Y_h(t) - Y(t)\|$ pointwise on $\Omega_\epsilon \times [0, T]$.

In order to add W_Λ and W_{Λ_h}, we must apply Chebyshev to

$$\Big(\mathbf{E}\big[\sup_{t\in[0,T]} \|W_\Lambda(t)\|_3^p\big]\Big)^{1/p} \le C_\alpha \|\Lambda^{\frac{1}{2}+\alpha} Q^{1/2}\|_{\mathrm{HS}}$$

$$\Big(\mathbf{E}\big[\sup_{t\in[0,T]} \|W_{\Lambda_h}(t) - W_\Lambda(t)\|^p\big]\Big)^{1/p} \le C_\alpha h^2 \|\Lambda^{\alpha} Q^{1/2}\|_{\mathrm{HS}}$$

for $\alpha \in (0, 1]$ and $p > 2/\alpha$. Then we have pointwise bounds for

$$\|W_\Lambda(t)\|_3 \text{ and } \|W_{\Lambda_h}(t) - W_\Lambda(t)\| \text{ on } \Omega_\epsilon \times [0, T].$$

Non-homogeneous linear Cahn-Hilliard: error estimate

$$\begin{cases} u(t) \in H^1(\mathcal{D}), \quad u(0) = u_0 \\ \langle \dot{u}, \chi \rangle + \langle \nabla v, \nabla \chi \rangle = 0 \quad \forall \chi \in H^1(\mathcal{D}),\ t > 0 \\ \langle v, \chi \rangle = \langle \nabla u, \nabla \chi \rangle + \langle g, \chi \rangle \quad \forall \chi \in H^1(\mathcal{D}),\ t > 0 \end{cases}$$

$$\begin{cases} u_h(t) \in S_h, \quad u_h(0) = P_h u_0 \\ \langle \dot{u}_h, \chi \rangle + \langle \nabla v_h, \nabla \chi \rangle = 0 \quad \forall \chi \in S_h,\ t > 0 \\ \langle v_h, \chi \rangle = \langle \nabla u_h, \nabla \chi \rangle + \langle g, \chi \rangle \quad \forall \chi \in S_h,\ t > 0 \end{cases}$$

Time-derivative-free error estimate:

$$\begin{aligned} \|u_h(t) - u(t)\| &\leq Ch^2 \Big(|\log(h)| \sup_{s \in [0,t]} |u(s)|_2 + \Big(\int_0^t |v(s)|_2^2 \, \mathrm{d}s \Big)^{\frac{1}{2}} \Big) \\ &\leq Ch^2 |\log(h)| \Big(|u_0|_2^2 + \int_0^t |g(s)|_2^2 \, \mathrm{d}s \Big)^{\frac{1}{2}} \end{aligned}$$

To be used with $g(t) = f(X(t))$, so that $u(t) = Y(t)$.

Cahn-Hilliard-Cook equation: error analysis

$$X(t) = Y(t) + W_\Lambda(t)$$

$$X_h(t) = Y_h(t) + W_{\Lambda_h}(t)$$

$$Y(t) = \mathrm{e}^{-t\Lambda^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)\Lambda^2} \Lambda f(X(s))\,\mathrm{d}s = u(t)$$

$$Y_h(t) = \mathrm{e}^{-t\Lambda_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)\Lambda_h^2} \Lambda_h P_h f(X_h(s))\,\mathrm{d}s$$

$$Z_h(t) = \mathrm{e}^{-t\Lambda_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)\Lambda_h^2} \Lambda_h P_h f(X(s))\,\mathrm{d}s = u_h(t)$$

$$\begin{aligned} X_h(t) - X(t) &= (Y_h(t) + W_{\Lambda_h}(t)) - (Y(t) + W_\Lambda(t)) \pm Z_h(t) \\ &= \underbrace{(Y_h(t) - Z_h(t))}_{\text{Gronwall}} + \underbrace{(Z_h(t) - Y(t))}_{\text{deterministic error estimate}} + \underbrace{(W_{\Lambda_h}(t) - W_\Lambda(t))}_{\text{known}} \end{aligned}$$

Cahn-Hilliard-Cook equation

We show regularity and error estimates on $\Omega_\epsilon \times [0, T]$:

If $\|\Lambda^{(1+\gamma)/2} Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$, $\gamma > 0$ ($\beta = 3 + \gamma$), then

$$\|X(t)\|_{H^3} \leq C(\epsilon^{-1} K_T, T) \qquad \text{on } \Omega_\epsilon,\ t \in [0, T]$$

$$\|X_h(t) - X(t)\| \leq C(\epsilon^{-1} K_T, T) h^2 |\log(h)| \qquad \text{on } \Omega_\epsilon,\ t \in [0, T]$$

Strong convergence (without known rate):

$$\mathbf{E}\Big[\sup_{t\in[0,T]} \|X_h(t) - X(t)\|^2\Big] \to 0 \quad \text{as } h \to 0.$$

This is from [1].
Earlier work by Cardon-Weber [1], convergence for a finite difference method.

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