Stochastic Evolution PDEs
Lectures 3–4

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Outline

- Quick review of strong convergence analysis.
- Weak convergence analysis.

Semigroup approach

Semigroup approach

Linear SPDE with additive noise:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \;\; t > 0 \\ X(0) = X_0 \end{cases}$$

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- X_0 is an $\mathcal{F}_0$-measurable $\mathcal{H}$-valued random variable

Mild solution

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \;\; t > 0 \\ X(0) = X_0 \end{cases}$$

The unique solution is given by (mild solution)

$$X(t) = E(t)X_0 + \int_0^t E(t-s)B\,\mathrm{d}W(s)$$

Stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t>0 \\ X(0) = X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = L_2(\mathcal{D})$, $\|\cdot\|$, $(\cdot,\cdot)$, $\mathcal{D} \subset \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H^1_0(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{H}$
- $X(t)$, $\mathcal{H}$-valued stochastic process
- $E(t) = e^{-tA}$ analytic semigroup generated by $-A$

Mild solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

The finite element method

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}$, $\quad S_h \subset H_0^1(\mathcal{D}) = \dot{H}^1$
- S_h continuous piecewise poly degree $\le r-1$, $r \ge 2$
- $X_h(t) \in S_h$; $(\mathrm{d}X_h, \chi) + (\nabla X_h, \nabla \chi)\,\mathrm{d}t = (\mathrm{d}W, \chi)\ \forall \chi \in S_h,\ t > 0$
- $\Lambda_h \colon S_h \to S_h$, discrete Laplacian, $(\Lambda_h \psi, \chi) = (\nabla \psi, \nabla \chi)\ \forall \psi, \chi \in S_h$
- $A_h = \Lambda_h$
- $P_h \colon L_2 \to S_h$, orthogonal projection, $\quad (P_h f, \chi) = (f, \chi)\ \forall \chi \in S_h$

$$\begin{cases} X_h(t) \in S_h, \quad X_h(0) = P_h X_0 \\ \mathrm{d}X_h + A_h X_h\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t > 0 \end{cases}$$

$P_h W(t)$ is Q_h-Wiener process with $Q_h = P_h Q P_h$.

Mild solution, with $E_h(t) = e^{-tA_h}$,

$$X_h(t) = E_h(t) P_h X_0 + \int_0^t E_h(t-s) P_h\,\mathrm{d}W(s)$$

Regularity and strong convergence

$$|v|_\beta = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^{\beta} (v, \phi_j)^2 \Big)^{1/2}, \quad \dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

Theorem. If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0, r]$, then

$$\|X(t)\|_{L_2(\Omega, \dot{H}^\beta)} \le C\Big(\|X_0\|_{L_2(\Omega, \dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} \Big)$$

$$\|X_h(t) - X(t)\|_{L_2(\Omega, H)} \le C h^\beta \Big(\|X_0\|_{L_2(\Omega, \dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} \Big)$$

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$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \le Ch^\beta \Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} \Big)$$

Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ for $\beta < 1/2$.

Proofs

The proofs are based on

- Itô isometry

$$\mathbf{E}\Big\| \int_0^t F(s)\,\mathrm{d}W(s)\Big\|^2 = \mathbf{E}\int_0^t \|F(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s$$

- Smoothing property

$$\int_0^t \|\Lambda^{1/2}E(s)v\|^2\,\mathrm{d}s \le C\|v\|^2$$

- Error estimates for the approximation of the semigroup

Approximation of the semigroup

$$\begin{cases} u_t + \Lambda u = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} u_{h,t} + \Lambda_h u_h = 0, & t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad u_h(t) = E_h(t)P_h v$$

Denote

$$F_h(t)v = E_h(t)P_h v - E(t)v, \quad |v|_\beta = \|\Lambda^{\beta/2} v\|.$$

We have, for $0 \le \beta \le r$,

- $\|F_h(t)v\| \le Ch^\beta |v|_\beta, \quad t \ge 0$
- $\Big(\int_0^t \|F_h(s)v\|^2 \, \mathrm{d}s \Big)^{1/2} \le Ch^\beta |v|_{\beta-1}, \quad t \ge 0$

V. Thomée, Galerkin Finite Element Methods for Parabolic Problems

The stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

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$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W,$$

$$X = \begin{bmatrix} u \\ u_t \end{bmatrix},\ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{U} = \dot{H}^0 = L_2(\mathcal{D})$$

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$$\mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}, \quad \mathcal{H} = \mathcal{H}^0 = \dot{H}^0 \times \dot{H}^{-1}, \quad D(A) = \mathcal{H}^1$$

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Here

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})(v,\phi_j)\phi_j, \quad (\lambda_j,\phi_j) \text{ are eigenpairs of } \Lambda$$

Regularity

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Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique weak solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s) B \, \mathrm{d}W(s) = \begin{bmatrix} \displaystyle\int_0^t \Lambda^{-1/2} \sin\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \\ \displaystyle\int_0^t \cos\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega,\mathcal{H}^\beta)} \leq C(t) \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}, \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}.$$

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The mild solution is:

$$
\begin{aligned}
X_h(t) &= \begin{bmatrix} X_{h,1}(t) \\ X_{h,2}(t) \end{bmatrix} \\
&= \int_0^t E_h(t-s) B_h \,\mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda_h^{-1/2} \sin\left((t-s)\Lambda_h^{1/2}\right) P_h \,\mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda_h^{1/2}\right) P_h \,\mathrm{d}W(s) \end{bmatrix}
\end{aligned}
$$

where, for example,

$$
\cos(t\Lambda_h^{1/2})v = \sum_{j=1}^{N_h} \cos\left(t\sqrt{\lambda_{h,j}}\right)(v, \phi_{h,j})\phi_{h,j},
$$

and $(\lambda_{h,j}, \phi_{h,j})$ are eigenpairs of Λ_h.

Spatially semidiscrete: approximation of the semigroup

$$\begin{cases} v_{tt}(t) + \Lambda v(t) = 0, \ t > 0 \\ v(0) = 0, \ v_t(0) = f \end{cases} \qquad \Rightarrow v(t) = \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right) f$$

$$\begin{cases} v_{h,tt}(t) + \Lambda_h v_h(t) = 0, \ t > 0 \\ v_h(0) = 0, \ v_{h,t}(0) = P_h f \end{cases} \qquad \Rightarrow v_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h f$$

We have, for $K_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h - \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right)$ and $r = 2$,

$\|K_h(t)f\| \le C(t)h^2 \|f\|_{\dot{H}^2}$ "initial regularity of order 3"

$\|K_h(t)f\| \le 2\|f\|_{\dot{H}^{-1}}$ "initial regularity of order 0" (stability)

$\|K_h(t)f\| \le C(t)h^{\frac{2}{3}\beta} \|f\|_{\dot{H}^{\beta-1}}, \quad 0 \le \beta \le 3$

Note: $\|v(t)\|_{\dot{H}^2} \le \|f\|_{\dot{H}^1}$ "initial regularity of order 2"

Spatially semidiscrete: Strong convergence

Theorem. Let $X_0 = 0$ and $r = 2$. If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0, 3]$, then

$$\|X_{h,1}(t) - X_1(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C(t)\, h^{\frac{2}{3}\beta} \left\|\Lambda^{(\beta-1)/2} Q^{1/2}\right\|_{\mathrm{HS}}$$

Higher order FEM: $\quad O(h^{\frac{r}{r+1}\beta}), \quad \beta \in [0, r+1].$

Weak convergence

The law of $X_h(T)$:

$$\mu_{X_h(T)} = \mathbf{P} \circ X_h(T)^{-1}$$

converges weakly to the law of $X(T)$, if

$$\langle \mu_{X_h(T)}, \varphi \rangle \to \langle \mu_{X(T)}, \varphi \rangle \quad \text{as } h \to 0 \quad \forall \varphi \in \mathcal{C}_\mathrm{b}(H, \mathbf{R})$$

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Since

$$\langle \mu_{X_h(T)}, \varphi \rangle = \int_H \varphi(x) \, \mathrm{d}\mu_{X_h(T)}(x) = \int_\Omega \varphi(X_h(T, \omega)) \, \mathrm{d}\mathbf{P}(\omega) = \mathbf{E}\big[\varphi(X_h(T))\big],$$

this means

$$\mathbf{E}\big[\varphi(X_h(T))\big] \to \mathbf{E}\big[\varphi(X(T))\big] \quad \text{as } h \to 0 \quad \forall \varphi \in \mathcal{C}_{\mathrm{b}}(H, \mathbf{R}).$$

Weak convergence

Test functions:

$$\varphi \in \mathcal{C}_{\mathrm{b}}(H, \mathbf{R}) = \text{continuous and bounded functions}$$

But we will use

$$\varphi \in \mathcal{C}^2_{\mathrm{b}}(H, \mathbf{R}) = \text{not necessarily bounded but with}$$
$$\text{continuous and bounded derivatives } D\varphi \text{ and } D^2\varphi$$

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It can be shown that $\mathcal{C}_{\mathrm{b}}^2(H, \mathbf{R})$ is dense in $\mathcal{C}_{\mathrm{b}}(H, \mathbf{R})$ for $H = L_2(\mathcal{D})$ [Grorud and Pardoux(1992)].

By a modification of the Portmanteau theorem, it follows that it is sufficient to use test functions in $\mathcal{C}_{\mathrm{b}}^2(H, \mathbf{R})$.

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Our goal is now to show

$$\mathbf{E}\big[G(X_h(T))\big] - \mathbf{E}\big[G(X(T))\big] = O(h^{2\beta}) \quad \text{as } h \to 0 \quad \forall G \in \mathcal{C}_{\mathrm{b}}^2(H, \mathbf{R}).$$

The weak rate is twice the strong rate of convergence.

Weak convergence

We will prove this for linear problems (heat and wave equations).
But first we will perform formal calculations for the nonlinear problem

$$\mathrm{d}X(t) + [AX(t) - f(X(t))]\,\mathrm{d}t = g(X(t))\,\mathrm{d}W(t),\ t \in (0,T]; \quad X(0) = X_0,$$

or in mild form

$$\begin{aligned} X(t) = E(t)X_0 &+ \int_0^t E(t-s)f(X(s))\,\mathrm{d}s \\ &+ \int_0^t E(t-s)g(X(s))\,\mathrm{d}W(s), \quad t \in [0,T]. \end{aligned}$$

The semidiscrete approximation is

$$\begin{aligned} X_h(t) = E_h(t)P_hX_0 &+ \int_0^t E_h(t-s)P_hf(X_h(s))\,\mathrm{d}s \\ &+ \int_0^t E_h(t-s)P_hg(X_h(s))\,\mathrm{d}W(s), \quad t \in [0,T]. \end{aligned}$$

Weak error representation: preliminaries

$$\mathrm{d}X(t) + [AX(t) - f(X(t))]\,\mathrm{d}t = g(X(t))\,\mathrm{d}W(t),\ t \in (0,T]; \quad X(0) = X_0,$$

Auxiliary process $Z(s) = Z(s;t,\xi)$: if ξ is $\mathcal{F}_t$-measurable and $0 \le t \le s \le T$

$$Z(s) = E(s-t)\xi + \int_t^s E(s-r) f(Z(r))\,\mathrm{d}r + \int_t^s E(s-r) g(Z(r))\,\mathrm{d}W(r)$$

Define $u : H \times [0,T] \to \mathbf{R}$ by

$$u(x,t) = \mathbf{E}\Big[G\big(Z(T;t,x)\big)\Big].$$

If $G \in C_{\mathrm{b}}^2(H,\mathbf{R})$, then u is a solution to Kolmogorov's equation

$$\begin{cases} u_t(x,t) - \langle u_x(x,t), Ax - f(x)\rangle + \frac{1}{2}\operatorname{Tr}\left(u_{xx}(x,t) g(x) Q g(x)^*\right) = 0, \\ \hfill t \in [0,T),\ x \in D(A), \\ u(x,T) = G(x) \end{cases}$$

Weak convergence

If ξ is $\mathcal{F}_t$-measurable and $0 \le t \le s \le T$:

$$Z(s; t, \xi) = E(s-\tau)\xi + \int_t^s E(s-r)f(X(r))\,\mathrm{d}r + \int_t^s E(t-r)g(X(r))\,\mathrm{d}W(r)$$

Define $u : H \times [0, T] \to \mathbb{R}$ by

$$u(x,t) = \mathbf{E}\Big[G\big(Z(T;t,x)\big)\Big].$$

With random $\mathcal{F}_t$-measurable input ξ:

$$u(\xi,t) = \mathbf{E}\Big[G\big(Z(T;t,\xi)\big)|\mathcal{F}_t\Big]$$

Hence

$$\mathbf{E}[u(\xi,t)] = \mathbf{E}\Big[\mathbf{E}\Big[G\big(Z(T;t,\xi)\big)|\mathcal{F}_t\Big]\Big] = \mathbf{E}\Big[G\big(Z(T;t,\xi)\big)\Big].$$

Weak convergence

So we have

$$\mathbf{E}[u(\xi, t)] = \mathbf{E}\Big[G\big(Z(T; t, \xi)\big)\Big].$$

Note also

$$Z(T; t, \xi) = Z(T; s, Z(s; t, \xi))$$

Then

$$\begin{aligned}\mathbf{E}[u(\xi, t)] &= \mathbf{E}\Big[G\big(Z(T; t, \xi)\big)\Big] \\ &= \mathbf{E}\Big[G\big(Z(T; s, Z(s; t, \xi))\big)\Big] = \mathbf{E}\Big[u(s, Z(s; t, \xi))\Big],\end{aligned}$$

that is, the expected value of u is constant along trajectories

$$y = Z(s; t, \xi), \quad s \in [t, T].$$

Weak convergence

Assume $X_h(0) = X(0)$ for simplicity.

$$\mathbf{E}\Big(G(X_h(T)) - G(X(T))\Big) = \mathbf{E}\Big(u(X_h(T), T) - u(X(T), T)\Big)$$

$$= \mathbf{E}\Big(u(X_h(T), T) - u(X(0), 0)\Big) = \mathbf{E}\Big(u(X_h(T), T) - u(X_h(0), 0)\Big)$$

Itô's formula: $= \mathbf{E}\int_0^T \langle u_x, \mathrm{d}X_h\rangle + \frac{1}{2} u_{xx} \mathrm{d}[X_h, X_h]$

$$= \mathbf{E}\int_0^T \Big\{ u_t(X_h(t), t) - \langle u_x(X_h(t), t), A_h X_h(t) - P_h f(X_h(t))\rangle$$

$$+ \tfrac{1}{2}\operatorname{Tr}[u_{xx}(X_h(t), t) P_h g(X_h(t)) Q g(X_h(t))^* P_h]\Big\}\,\mathrm{d}t$$

Kolm. eq: $u_t(X_h(t), t) = \langle u_x(X_h(t), t), AX_h(t) - f(X_h(t))\rangle$

$-\frac{1}{2}\operatorname{Tr}[u_{xx}(X_h(t), t) g(X_h(t)) Q g(X_h(t))^*]$

$$= \mathbf{E}\int_0^T \Big\{ - \langle u_x(\cdot, t), (A_h - A) X_h(t) - (P_h - I) f(X_h(t))\rangle$$

$$+ \tfrac{1}{2}\operatorname{Tr}\Big[u_{xx}(\cdot, t)[P_h g(\cdot) Q g(\cdot)^* P_h - g(\cdot) Q g(\cdot)^*]\Big]\Big\}\,\mathrm{d}t,$$

where $\cdot = X_h(t)$.

Weak convergence

The first term:

$$\mathbf{E}\int_0^T -\langle u_x(X_h(t),t),(A_h-A)X_h(t)-(P_h-I)f(X_h(t))\rangle\,\mathrm{d}t,$$

Here $(A_hX_h - P_hf(X_h))\,\mathrm{d}t = -\mathrm{d}X_h + P_hg\,\mathrm{d}W$, so we get

$$\mathbf{E}\int_0^T \langle u_x(X_h(t),t),\mathrm{d}X_h(t)+(AX_h(t)-f(X_h(t))\,\mathrm{d}t\rangle.$$

We identify the residual of X_h: $\mathrm{d}X_h(t)+[AX_h(t)-f(X_h(t)]\,\mathrm{d}t$.

Weak convergence

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$$\mathbf{E}\int_0^T \langle u_x(X_h(t),t),\mathrm{d}X_h(t)+(AX_h(t)-f(X_h(t))\,\mathrm{d}t\rangle.$$

We identify the residual of X_h: $\mathrm{d}X_h(t)+[AX_h(t)-f(X_h(t)]\,\mathrm{d}t$.
Related to a posteriori error analysis?

Weak convergence

$$u(x,t) = \mathbf{E}\Big[G\big(Z(T;t,x)\big)\Big].$$

The derivative $u_x(x,t) \in H$ is given by

$$\begin{aligned}\langle u_x(x,t), \phi\rangle &= \mathbf{E}\Big[\langle G'\big(Z(T;t,x)\big), Z_x'(T;t,x)\phi\rangle\Big] \\ &= \mathbf{E}\Big[\langle Z_x'(T;t,x)^* G'\big(Z(T;t,x)\big), \phi\rangle\Big]\end{aligned}$$

So, in order to bound norms of $u_x(x,t) = \mathbf{E}\big[Z_x'(T;t,x)^* G'\big(Z(T;t,x)\big)\big]$, we must study the linearized adjoint equation:

$$\begin{aligned}\eta(s) = E(T-s)G'(Z(T;t,x)) &+ \int_s^T E(T-r)f'(Z(r;t,x))\eta(r)\,\mathrm{d}r \\ &+ \int_s^T E(T-r)\big[g'(Z(r;t,x))\eta(r)\big]\,\mathrm{d}W(r)\end{aligned}$$

The second derivative is related to the second adjoint variation.

Weak convergence

Let us compute $u_x(x,t)$ in the simplest case, the linear case:

$$u(x,t) = \mathbf{E}\Big[G\big(Z(T;t,x)\big)\Big] = \mathbf{E}\Big[G\Big(E(T-t)x + \int_t^T E(T-s)B\,\mathrm{d}W(s)\Big)\Big]$$

Then

$$\begin{aligned}\langle u_x(x,t),\phi\rangle &= \mathbf{E}\Big[\Big\langle G'\Big(E(T-t)x + \int_t^T E(T-s)B\,\mathrm{d}W(s)\Big), E(T-t)\phi\Big\rangle\Big] \\ &= \mathbf{E}\Big[\langle E(T-t)^* G'\big(Z(T;t,x)\big),\phi\rangle\Big]\end{aligned}$$

so that $u_x(x,t) = \mathbf{E}\Big[E(T-t)^* G'\big(Z(T;t,x)\big)\Big]$. This is $\eta(t) = \eta(t;t,x)$, where $\eta(s) = \eta(s;t,x) = \mathbf{E}\Big[E(T-s)^* G'\big(Z(T;t,x)\big)\Big]$ is the solution of the adjoint equation, recall $E(T-s)^* = \mathrm{e}^{-(T-s)A^*}$,

$$\dot{\eta}(s) - A^*\eta(s) = 0,\ s \le T; \quad \eta(T) = G'\big(Z(T;t,x)\big).$$

Similarly, we have $u_{xx}(x,t) = \mathbf{E}\Big[E(T-t)^* G''\big(Z(T;t,x)\big)E(T-t)\Big]$.

Weak convergence

Another difficulty: the Kolmogorov equation is proved only for $x \in D(A)$.

$$\begin{cases} u_t(x,t) - \langle u_x(x,t), Ax - f(x)\rangle + \frac{1}{2}\operatorname{Tr}\left(u_{xx}(x,t)g(x)Qg(x)^*\right) = 0, \\ \hfill t \in [0,T),\ x \in D(A), \\ u(x,T) = G(x) \end{cases}$$

Project onto the eigenspaces of A. Auxiliary process $Z_m(s) = Z_m(s;t,x)$:

$$\begin{aligned} Z_m(s) = E_m(s-t)P_m\xi + \int_t^s E_m(s-r)P_m f(Z_m(r))\,\mathrm{d}r \\ + \int_t^s E_m(s-r)P_m g(Z_m(r))\,\mathrm{d}W(r). \end{aligned}$$

Define $u_m : H \times [0,T] \to \mathbf{R}$ by

$$u_m(x,t) = \mathbf{E}\Big[G\big(Z_m(T;t,x)\big)\Big].$$

Then $u_m(x,t) = u_m(P_m x, t)$, to be used with $x = X_h(t)$.
The Kolmogorov equation is now well-defined.
Must verify that additional terms vanish as $m \to \infty$.

Weak convergence

The first term again:

$$\mathbf{E}\int_0^T -\langle u_x(X_h(t),t),(A_h-A)X_h(t)-(P_h-I)f(X_h(t))\rangle\,\mathrm{d}t,$$

For the heat equation, we have here $A=\Lambda$, $A_h=\Lambda_h$, so that

$$\begin{aligned}\langle u_x,(A_h-A)X_h\rangle &= \langle (A_hP_h-A)u_x,X_h\rangle\\ &= \langle A_hP_h(A^{-1}-A_h^{-1}P_h)Au_x,X_h\rangle\\ &= \langle (A^{-1}-A_h^{-1}P_h)Au_x,A_hX_h\rangle\end{aligned}$$

Related to the "elliptic" error $(A^{-1}-A_h^{-1}P_h)$. But the norms are badly distributed between the factors. For the heat equation this can be handled (to some extent) by rewriting by means of Malliavin calculus.

Weak convergence

Here we try to explain why the norms on the previous slide are "badly distributed". We compute for the linear heat equation:

$$
\begin{aligned}
\langle u_x, (A_h - A)X_h \rangle &= \langle (A^{-1} - A_h^{-1}P_h)Au_x, A_hX_h \rangle, \\
u_x(x,t) &= \mathbf{E}\Big[E(T-t)G'\big(Z(T;t,x)\big)\Big], \\
Au_x(X_h(t),t) &= \mathbf{E}\big[AE(T-t)G'\big(Z(T;t,X_h(t))\big)|\mathcal{F}_t\big], \\
\|A^{-1} - A_h^{-1}P_h\|_{\mathcal{L}(H)} &\le Ch^2, \\
\|AE(T-t)\|_{\mathcal{L}(H)} &\le C(T-t)^{-1}.
\end{aligned}
$$

Weak convergence

Hence, the bad term becomes

$$
\begin{aligned}
&\Big|\mathbf{E}\int_0^T \langle u_x(X_h(t),t),(A_h-A)X_h(t)\rangle\,\mathrm{d}t\Big|\\
&=\Big|\mathbf{E}\int_0^T \Big\langle (A^{-1}-A_h^{-1}P_h)Au_x(X_h(t),t),A_hX_h(t)\Big\rangle\,\mathrm{d}t\Big|\\
&=\Big|\mathbf{E}\int_0^T \Big\langle (A^{-1}-A_h^{-1}P_h)\mathbf{E}\big[AE(T-t)G'\big(Z(T;t,X_h(t))\big)|\mathcal{F}_t\big],A_hX_h(t)\Big\rangle\,\mathrm{d}t\Big|\\
&\le C\int_0^T \|A^{-1}-A_h^{-1}P_h\|_{\mathcal{L}(H)}\|AE(T-t)\|_{\mathcal{L}(H)}\sup_{x\in H}\|G'(x)\|_H\\
&\qquad\qquad \times\|A_hX_h(t)\|_{L_2(\Omega,H)}\,\mathrm{d}t\\
&\le Ch^2\int_0^T (T-t)^{-1}\,\mathrm{d}t\,|G|_{\mathcal{C}_\mathrm{b}^1}\sup_{t\in[0,T]}\|A_hX_h(t)\|_{L_2(\Omega,H)}.
\end{aligned}
$$

Here: $\sup_{t\in[0,T]}\|A_hX_h(t)\|_{L_2(\Omega,H)}<\infty$ if $\|A^{\frac{\beta-1}{2}}Q^{\frac12}\|_{\mathrm{HS}}$ with $\beta=2$ (regularity and strong convergence of order $\beta=2$). But the rate is only $h^2=h^\beta$, not $h^{2\beta}$.

Weak convergence

Here we have not been able to exploit the possibility for the integral to absorb a singularity at $t = 0$, i.e.,

$$\int_0^T (T - t)^{-1} t^{-1} \, \mathrm{d}t \quad \text{(almost convergent).}$$

This can be achieved by an integration by parts from the Malliavin calculus.

Weak convergence: the linear case

This explains some difficulties encountered in connection with the nonlinear problem.

The story is more complete for the linear problem:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

I will present this now for the heat and wave equations.
(We have also studied the linearized Cahn-Hilliard-Cook equation.)

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We use a trick introduced by
De Bouard and Debussche (nonlinear Schrödinger equation) [de Bouard and Debussche(2006)].
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The trick is: Remove the troublesome term $(A_h - A)X_h$ by means of an integrating factor.

Weak error representation: preliminaries

Apply the integrating factor $E(T-t)$ to get $Y(t) = E(T-t)X(t)$:

$$\mathrm{d}Y(t) = E(T-t)B\,\mathrm{d}W(t), \ t \in (0,T]; \ Y(0) = E(T)X_0,$$

with mild solution

$$Y(t) = E(T)X_0 + \int_0^t E(T-s)B\,\mathrm{d}W(s).$$

Similarly, consider

$$\mathrm{d}Y_h(t) = E_h(T-t)B\,\mathrm{d}W(t), \ t \in (0,T]; \ Y_h(0) = E_h(T)P_hX_0,$$

with mild solution

$$Y_h(t) = E_h(T)P_hX_0 + \int_0^t E_h(T-s)B_h\,\mathrm{d}W(s).$$

Note: $X(T) = Y(T)$, $X_h(T) = Y_h(T)$.
No drift term in eq. for Y and Y_h.

Weak error representation: preliminaries

Auxiliary problem: $Z(s) = Z(s; t, \xi)$, ξ is a $\mathcal{F}_t$-measurable,

$$\mathrm{d}Z(s) = E(T-s)B\,\mathrm{d}W(s),\ s \in (t, T];\ Z(t) = \xi.$$

Unique mild solution: $Z(s; t, \xi) = \xi + \int_t^s E(T-r)B\,\mathrm{d}W(r)$.

Define $u : H \times [0, T] \to \mathbf{R}$ by $u(x,t) = \mathbf{E}\left[G(Z(T; t, x))\right]$.

The partial derivatives are:

$$u_x(x,t) = \mathbf{E}\big[G'(Z(T; t, x))\big],$$
$$u_{xx}(x,t) = \mathbf{E}\big[G''(Z(T; t, x))\big].$$

If $G \in C_{\mathrm{b}}^2(H, \mathbb{R})$, then u is a solution to Kolmogorov's equation

$$\begin{cases} u_t(x,t) + \frac{1}{2}\operatorname{Tr}\left(u_{xx}(x,t)E(T-t)BQ[E(T-t)B]^*\right) = 0,\ t \in [0, T),\ x \in H \\ u(x, T) = G(x) \end{cases}$$

Weak error representation

THEOREM. If

$$\operatorname{Tr}\Big(\int_0^T E(t)BQ[E(t)B]^*\,\mathrm{d}t\Big) < \infty$$

and $G \in C_{\mathrm{b}}^2(H,\mathbf{R})$, then the weak error

$$e_h(T) = \mathbf{E}[G(X_h(T))] - \mathbf{E}[G(X(T))]$$

has the representation

$$\begin{aligned}
e_h(T) &= \mathbf{E}\left[u(Y_h(0),0) - u(Y(0),0)\right] \\
&\quad + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times [E_h(T-t)B_h + E(T-t)B]Q[E_h(T-t)B_h - E(T-t)B]^*\Big)\,\mathrm{d}t \\
&= \mathbf{E}\left[u(Y_h(0),0) - u(Y(0),0)\right] \\
&\quad + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times [E_h(T-t)B_h - E(T-t)B]Q[E_h(T-t)B_h + E(T-t)B]^*\Big)\,\mathrm{d}t.
\end{aligned}$$

Weak convergence: proof

Use Itô formula and Kolmogorov equation as before:

$$
\begin{aligned}
&\mathbf{E}[G(X_h(T))] - \mathbf{E}[G(X(T))] \\
&\quad = \mathbf{E}[G(Y_h(T))] - \mathbf{E}[G(Y(T))] \\
&\quad = \mathbf{E}\Big[u(Y_h(T),T) - u(Y(T),T)\Big] \\
&\quad = \mathbf{E}\Big[u(Y_h(T),T) - u(Y_h(0),0)\Big] + \mathbf{E}\Big[u(Y_h(0),0) - u(Y(0),0)\Big] \\
&\quad = \mathbf{E}\left[u(Y_h(0),0) - u(Y(0),0)\right] + \mathbf{E}\int_0^T \Big\{u_t(Y_h(t),t) \\
&\qquad + \frac{1}{2}\operatorname{Tr}\Big(u_{xx}(Y_h(t),t)[E_h(T-t)B_h]Q[E_h(T-t)B_h]^*\Big)\Big\}\,\mathrm{d}t \\
&\quad = \mathbf{E}\left[u(Y_h(0),0) - u(Y(0),0)\right] + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times \Big\{[E_h(T-t)B_h]Q[E_h(T-t)B_h]^* - E(T-t)BQB^*E(T-t)^*\Big\}\Big)\,\mathrm{d}t.
\end{aligned}
$$

Weak convergence: proof

Here the expression

$$[S, T] = \mathrm{Tr}(u_{xx} SQT^*)$$

is symmetric:

$$\begin{aligned}[S, T] &= \mathrm{Tr}(u_{xx} SQT^*) = \mathrm{Tr}(SQT^* u_{xx}) \\ &= \mathrm{Tr}([SQT^* u_{xx}]^*) = \mathrm{Tr}(u_{xx} TQS^*) = [T, S],\end{aligned}$$

because Q, u_{xx} are selfadjoint and $\mathrm{Tr}(S^*) = \mathrm{Tr}(S)$, $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$.
Hence, we have a conjugate rule

$$[S + T, S - T] = [S, S] - [T, T].$$

Therefore,

$$\begin{aligned}&\mathrm{Tr}\left(u_{xx}(\xi, r)\{[E_h(s)B_h]Q[E_h(s)B_h]^* - [E(s)B]Q[E(s)B]^*\}\right) \\ &= \mathrm{Tr}\left(u_{xx}(\xi, r)[E_h(s)B_h + E(s)B]Q[E_h(s)B_h - E(s)B]^*\right) \\ &= \mathrm{Tr}\left(u_{xx}(\xi, r)[E_h(s)B_h - E(s)B]Q[E_h(s)B_h + E(s)B]^*\right).\end{aligned}$$

Note, by the way, that $B_h \in \mathcal{L}(U, H)$ with $B_h : U \to S_h$, $E_h(s) : S_h \to S_h$, and we consider $E_h(s)B_h \in \mathcal{L}(U, H)$. Hence, $[E_h(s)B_h]^* \neq B_h^* E_h(s)^*$.

Weak convergence: heat equation

Here $A = \Lambda$, $B = I$, $A_h = \Lambda_h$, $B_h = P_h$.

$$\mathrm{d}X + \Lambda X\,\mathrm{d}t = \mathrm{d}W,\ \ t > 0; \qquad X(0) = X_0, \tag{1}$$
$$\mathrm{d}X_h + \Lambda_h X_h\,\mathrm{d}t = P_h\,\mathrm{d}W,\ \ t > 0; \qquad X_h(0) = P_h X_0. \tag{2}$$

Theorem

Let X and X_h be the solutions of (1) and (2), respectively. Let $G \in C_{\mathrm{b}}^2(H, \mathbf{R})$ and assume that $\|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} = \|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in (0,1]$. Then there are $C > 0$, $h_0 > 0$, depending on G, X_0, Q, β, and T but not on h, such that for $h \leq h_0$,

$$\big|\mathbf{E}\big[G(X_h(T)) - G(X(T))\big]\big| \leq C h^{2\beta} |\log(h)|.$$

If, in addition $X_0 \in L_1(\Omega, \dot{H}^{2\beta})$, then C is independent of T as well.

Proof

$$
\begin{aligned}
e_h(T) = \mathbf{E}\left[u(Y_h(0),0) - u(Y(0),0)\right] \\
+ \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t) \\
\times [E_h(T-t)P_h + E(T-t)]Q[E_h(T-t)P_h - E(T-t)]^*\Big)\,\mathrm{d}t
\end{aligned}
$$

Approximation of the semigroup:

$$
\|(E_h(t)P_h - E(t))v\| = \|F_h(t)v\| \le Ch^s t^{-\frac{s-\gamma}{2}}|v|_\gamma, \quad 0 \le \gamma \le s \le r.
$$

Proof

In the initial error we have

$$Y_h(0) - Y(0) = E_h(T)P_hX_0 - E(T)X_0 = F_h(T)X_0,$$

so that

$$\begin{aligned}
&\mathbf{E}\left(u(Y_h(0),0) - u(Y(0),0)\right) \\
&\quad = \mathbf{E}\int_0^1 \langle u_x(Y(0) + s(Y_h(0) - Y(0)),0), Y_h(0) - Y(0)\rangle\,\mathrm{d}s \\
&\quad = \mathbf{E}\int_0^1 \langle u_x(E(T)X_0 + sF_h(T)X_0,0), F_h(T)X_0\rangle\,\mathrm{d}s.
\end{aligned}$$

Thus, recalling $u_x(x,t) = \mathbf{E}\big[G'(Z(T;t,x))\big]$,

$$\begin{aligned}
\left|\mathbf{E}\left(u(Y_h(0),0) - u(Y(0),0)\right)\right| &\le \sup_{x\in H}\|u_x(x,0)\|\,\mathbf{E}\big(\|F_h(T)X_0\|\big) \\
&\le Ch^{2\beta}T^{-\frac{2\beta-\gamma}{2}}\mathbf{E}\big(|X_0|_\gamma\big)\sup_{x\in H}\|G'(x)\|, \quad 0\le\gamma\le 2\beta.
\end{aligned}$$

If $\gamma = 2\beta$ there is no dependence on T.

Proof

The main term: use $|\operatorname{Tr}(ST)| \le \|S\|_{\mathrm{HS}}\|T\|_{\mathrm{HS}}$

$$\begin{aligned}
&\Big|\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times [E_h(T-t)P_h + E(T-t)]Q[E_h(T-t)P_h - E(T-t)]^*\Big)\,\mathrm{d}t\Big| \\
&= \Big|\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t)[E_h(T-t)P_h + E(T-t)]^* \\
&\qquad \times A^{\frac{1-\beta}{2}} A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}} Q^{\frac{1}{2}} A^{\frac{\beta-1}{2}} A^{\frac{1-\beta}{2}} F_h(T-t)\Big)\,\mathrm{d}t\Big| \\
&= \Big|\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t)(A^{\frac{1-\beta}{2}}[E_h(T-t)P_h + E(T-t)])^* \\
&\qquad \times A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}} Q^{\frac{1}{2}} A^{\frac{\beta-1}{2}} A^{\frac{1-\beta}{2}} F_h(T-t)\Big)\,\mathrm{d}t\Big| \\
&\le \mathbf{E}\int_0^T \|u_{xx}(Y_h(t),t)(A^{\frac{1-\beta}{2}}[E_h(T-t)P_h + E(T-t)])^* A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} \\
&\qquad \times \|Q^{\frac{1}{2}} A^{\frac{\beta-1}{2}} A^{\frac{1-\beta}{2}} F_h(T-t)\|_{\mathrm{HS}}\,\mathrm{d}t
\end{aligned}$$

Proof

Use $\|ST\|_{\mathrm{HS}} \leq \|S\|_{\mathcal{L}}\|T\|_{\mathrm{HS}}$:

$$\begin{aligned}
\cdots &\leq \mathbf{E}\int_0^T \|u_{xx}(Y_h(t),t)(A^{\frac{1-\beta}{2}}[E_h(T-t)P_h + E(T-t)])^* A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} \\
&\qquad \times \|Q^{\frac{1}{2}} A^{\frac{\beta-1}{2}} A^{\frac{1-\beta}{2}} F_h(T-t)\|_{\mathrm{HS}}\,\mathrm{d}t \\
&\leq \sup_{(x,t)\in H\times[0,T]} \|u_{xx}(x,t)\|_{\mathcal{L}(H)} \|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \\
&\qquad \times \int_0^T \|A^{\frac{1-\beta}{2}}(E_h(t)P_h + E(t))\|_{\mathcal{L}(H)} \|A^{\frac{1-\beta}{2}} F_h(t)\|_{\mathcal{L}(H)}\,\mathrm{d}t.
\end{aligned}$$

Here

$$\sup_{(x,t)\in H\times[0,T]} \|u_{xx}(x,t)\|_{\mathcal{L}(H)} \leq \sup_{x\in H} \|G''(x)\|_{\mathcal{L}(H)}.$$

Recall

$$\begin{aligned}
&\|A^{\frac{1}{2}} v_h\| = \|\nabla v_h\| = \|A_h^{\frac{1}{2}} v_h\|, \quad v_h \in S_h, \\
&\|A^{\delta} v_h\| \leq \|A_h^{\delta} v_h\|, \quad v_h \in S_h,\ \delta \in [0, \tfrac{1}{2}], \\
&\|A^{\delta}(E_h(t)P_h + E(t))\|_{\mathcal{L}(H)} \leq C e^{-\omega t} t^{-\delta}, \quad \delta = \tfrac{1-\beta}{2} \in [0, \tfrac{1}{2}].
\end{aligned}$$

Proof

Now consider $\|A^{\frac{1-\beta}{2}} F_h(t)\|_{\mathcal{L}(H)}$. Analyticity:

$$\|A^{\delta} F_h(t)\|_{\mathcal{L}(H)} \leq Ct^{-\delta}, \quad \delta \in [0, \tfrac{1}{2}].$$

Approximation:

$$\|F_h(t)v\| \leq Ch^s t^{-\frac{s-\gamma}{2}} |v|_{\gamma}, \quad 0 \leq \gamma \leq s \leq r.$$

Hence

$$\|A^{\frac{1-\beta}{2}} F_h(t)\|_{\mathcal{L}(H)} \leq \|F_h(t)\|_{\mathcal{L}(H)}^{\beta} \|A^{\frac{1}{2}} F_h(t)\|_{\mathcal{L}(H)}^{1-\beta} \leq Ch^{2\beta} t^{-\frac{1+\beta}{2}}, \quad \beta \in [0, 1].$$

Therefore, for $\beta \in (0, 1]$ one may estimate the above integral:

$$\begin{aligned}
&\int_0^T \|A^{\frac{1-\beta}{2}} (E_h(t)P_h + E(t))\|_{\mathcal{L}(H)} \|A^{\frac{1-\beta}{2}} F_h(t)\|_{\mathcal{L}(H)} \,\mathrm{d}t \\
&= \left(\int_0^{h^2} + \int_{h^2}^T \right) \|A^{\frac{1-\beta}{2}} (E_h(t)P_h + E(t))\|_{\mathcal{L}(H)} \|A^{\frac{1-\beta}{2}} F_h(t)\|_{\mathcal{L}(H)} \,\mathrm{d}t \\
&\leq C \int_0^{h^2} t^{-\frac{1-\beta}{2}} t^{-\frac{1-\beta}{2}} \,\mathrm{d}t + C \int_{h^2}^T e^{-\omega t} t^{-\frac{1-\beta}{2}} h^{2\beta} t^{-\frac{1+\beta}{2}} \,\mathrm{d}t \leq Ch^{2\beta} |\log(h)|
\end{aligned}$$

and the proof is complete.

Weak convergence: heat equation

By inspection of the above proof we see that the error estimate is

$$\begin{aligned}
&\big|\mathbf{E}\big(G(X_h(T)) - G(X(T))\big)\big| \\
&\qquad \le Ch^{2\beta} T^{-\frac{2\beta-\gamma}{2}} \mathbf{E}\big(|X_0|_\gamma\big) \sup_{x\in H} \|G'(x)\|_H \\
&\qquad\quad + Ch^{2\beta} |\log(h)| \beta^{-1} \sup_{x\in H} \|G''(x)\|_{\mathcal{L}(H)} \|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2.
\end{aligned}$$

Weak convergence: heat equation

By inspection of the above proof we see that the error estimate is

$$\begin{aligned}
&\big|\mathbf{E}\big(G(X_h(T)) - G(X(T))\big)\big| \\
&\qquad \le Ch^{2\beta} T^{-\frac{2\beta-\gamma}{2}} \mathbf{E}\big(|X_0|_\gamma\big) \sup_{x\in H} \|G'(x)\|_H \\
&\qquad\quad + Ch^{2\beta} |\log(h)|\beta^{-1} \sup_{x\in H} \|G''(x)\|_{\mathcal{L}(H)} \|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 .
\end{aligned}$$

The previous theorem does not allow $\beta > 1$.
This is satisfactory if the order of the FEM is $r = 2$.
Under a slightly stronger condition on A and Q we now extend the result to the case $\beta > 1$.

Weak convergence: heat equation

Theorem

Let X and X_h be the solutions of (1) *and* (2)*, respectively. Let $G \in C_{\mathrm{b}}^2(H, \mathbf{R})$ and assume that $\|A^{\beta-1}Q\|_{\mathsf{Tr}} = \|\Lambda^{\beta-1}Q\|_{\mathsf{Tr}} < \infty$ for some $\beta \in [1, \frac{r}{2}]$. Then there are $C > 0$, $h_0 > 0$, depending on G, X_0, Q, β, and T but not on h, such that for $h \le h_0$,*

$$\left|\mathbf{E}\big(G(X_h(T)) - G(X(T))\big)\right| \le Ch^{2\beta}|\log(h)|.$$

If, in addition $X_0 \in L_1(\Omega, \dot{H}^{2\beta})$, then C is independent of T as well.

This theorem differs in the assumption about Q. According to the theorem on "alternative conditions" in the first part of my lectures we have

$$\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathsf{HS}}^2 \le \|\Lambda^{\beta-1}Q\|_{\mathsf{Tr}}.$$

Thus, the new condition implies the previous one. If Λ and Q commute, then they coincide.

Proof

The initial error term is treated as before. For the main term we distribute factors differently:

$$
\begin{aligned}
&\Big|\mathbf{E}\int_0^T \mathrm{Tr}\,\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times [E_h(T-t)P_h - E(T-t)]Q[E_h(T-t)B_h + E(T-t)B]^*\Big)\,\mathrm{d}t\Big| \\
&\quad = \Big|\mathbf{E}\int_0^T \mathrm{Tr}\,\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times F_h(t)A^{1-\beta}A^{\beta-1}Q[E_h(T-t)P_h + E(T-t)]^*\Big)\,\mathrm{d}t\Big| \\
&\quad \le C \sup_{(x,t)\in H\times[0,T]} \|u_{xx}(x,t)\|_{\mathcal{L}(H)}\|A^{\beta-1}Q\|_{\mathrm{Tr}}\int_0^T \|F_h(t)A^{\beta-1}\|_{\mathcal{L}(H)}e^{-\omega t}\,\mathrm{d}t.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_0^T \|F_h(t)A^{1-\beta}\|_{\mathcal{L}(H)}e^{-\omega t}\,\mathrm{d}t &= \Big(\int_0^{h^{2\beta}} + \int_{h^{2\beta}}^T\Big)\|F_h(t)A^{1-\beta}\|_{\mathcal{L}(H)}e^{-\omega t}\,\mathrm{d}t \\
&\le C\int_0^{h^{2\beta}}\mathrm{d}t + Ch^{2\beta}\int_{h^{2\beta}}^T t^{-1}e^{-\omega t}\,\mathrm{d}t \le Ch^{2\beta}|\log(h)|.
\end{aligned}
$$

Weak convergence: the wave equation

Recall the notation:

$$A := \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad X := \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad X_0 := \begin{bmatrix} X_{0,1} \\ X_{0,2} \end{bmatrix},$$

$$E(t) = \mathrm{e}^{-tA} = \begin{bmatrix} C(t) & \Lambda^{-1/2} S(t) \\ -\Lambda^{1/2} S(t) & C(t) \end{bmatrix},$$

where $C(t) = \cos(t\Lambda^{1/2})$ and $S(t) = \sin(t\Lambda^{1/2})$.
Spatially discrete:

$$A_h := \begin{bmatrix} 0 & -I \\ \Lambda_h & 0 \end{bmatrix}, \quad B_h := \begin{bmatrix} 0 \\ P_h \end{bmatrix}, \quad X_{h0} = P_h X_0.$$

$$E_h(t) = \mathrm{e}^{-tA_h} = \begin{bmatrix} C_h(t) & \Lambda_h^{-1/2} S_h(t) \\ -\Lambda_h^{1/2} S_h(t) & C_h(t) \end{bmatrix},$$

with $C_h(t) = \cos(t\Lambda_h^{1/2})$, $S_h(t) = \sin(t\Lambda_h^{1/2})$.

Weak convergence: the wave equation

Theorem

Let $g \in C_{\mathrm{b}}^2(\dot{H}^0, \mathbf{R})$ and assume that $\|\Lambda^{\beta-\frac{1}{2}} Q \Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}} < \infty$ and that $X_0 \in L_1(\Omega, H^{2\beta})$ for some $\beta \in [0, \frac{r+1}{2}]$. Then, there are $C > 0$, $h_0 > 0$, depending on g, X_0, Q, and T but not on h, such that for $h \le h_0$,

$$\left|\mathbf{E}\big(g(X_{h,1}(T)) - g(X_1(T))\big)\right| \le C h^{\frac{r}{r+1} 2\beta}.$$

Note: the test function g depends on the first component X_1 only. Again the new condition on Q implies the previous one:

$$\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \le \|\Lambda^{\beta-\frac{1}{2}} Q \Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}}.$$

Therefore, the rate of weak convergence is twice the rate of strong convergence.

Proof

We only make a brief discussion of the main term.
The error operator for the first component is

$$K_h(t) := \Lambda_h^{-\frac{1}{2}} S_h(t) P_h - \Lambda^{-\frac{1}{2}} S(t).$$

We have

$$\|K_h(t)w\| \leq C(T) h^{\frac{r}{r+1}s} |w|_{s-1}, \quad w \in \dot{H}^{s-1},\ s \in [0, r+1],$$

or

$$\|K_h(t)\Lambda^{\frac{1-s}{2}} v\| \leq C(T) h^{\frac{r}{r+1}s} \|v\|, \quad v \in \dot{H}^{1-s}.$$

We use $s = 2\beta$:

$$\|K_h(t)\Lambda^{\frac{1}{2}-\beta}\|_{\mathcal{L}(\dot{H}^0)} \leq C(T) h^{\frac{r}{r+1}2\beta}, \quad t \in [0, T],\ 2\beta \in [0, r+1].$$

We use a test function of the form

$$G(x) := g(P_1 x) = g(x_1), \quad \text{for } x = [x_1, x_2]^\top \in \mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}.$$

Proof

The main term is

$$\Big|\mathbf{E}\Big(\int_0^T \mathrm{Tr}\,\big(u_{xx}(Y_h(t),t) \\ \times [E_h(T-t)B_h + E(T-t)B]Q[E_h(T-t)B_h - E(T-t)B]^*\big)\,\mathrm{d}t\Big)\Big|$$

The integrand simplifies to (with $s = T - t$)

$$\begin{aligned}
&\Big|\mathbf{E}\Big(\mathrm{Tr}\,\big(u_{xx}(Y_h(t),t)[E_h(s)B_h + E(s)B]Q[E_h(s)B_h - E(s)B]^*\big)\Big)\Big| \\
&\quad = \Big|\mathbf{E}\Big(\mathrm{Tr}\,\big([E_h(s)B_h - E(s)B]Q[E_h(s)B_h + E(s)B]^* u_{xx}(Y_h(t),t)^*\big)\Big)\Big| \\
&\quad = \Big|\mathbf{E}\Big(\mathrm{Tr}\,\big(K_h(s)Q[\Lambda_h^{-\frac{1}{2}}S_h(s)P_h + \Lambda^{-\frac{1}{2}}S(s)]g''(P_1 Z(T;t,Y_h(t)))\big)\Big)\Big| \\
&\quad \le \|K_h(s)\Lambda^{\frac{1}{2}-\beta}\|_{\mathcal{L}(\dot{H}^0)}\|\Lambda^{\beta-\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}} \\
&\qquad \times \|\Lambda^{\frac{1}{2}}[\Lambda_h^{-\frac{1}{2}}S_h(s)P_h + \Lambda^{-\frac{1}{2}}S(s)]\|_{\mathcal{L}(\dot{H}^0)} \sup_{x\in\dot{H}^0}\|g''(x)\|_{\mathcal{L}(\dot{H}^0)}. \\
&\quad \le C(T)h^{\frac{r}{r+1}2\beta}\|\Lambda^{\beta-\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}} \sup_{x\in\dot{H}^0}\|g''(x)\|_{\mathcal{L}(\dot{H}^0)}.
\end{aligned}$$

Weak convergence: completely discrete

This weak error representation formula has been generalized so that it applies to completely discrete approximations. Recall

$$
\begin{aligned}
X(t) &= E(t)X_0 + \int_0^t E(t-s)B\,\mathrm{d}W(s), \\
Y(t) &= E(T-t)X(t) = E(T)X_0 + \int_0^t E(T-s)B\,\mathrm{d}W(s), \\
X(T) &= Y(T).
\end{aligned}
$$

Assume that $\tilde{X}(T)$ is the result of some temporal and spatial approximation. Construct a process $\{\tilde{Y}(t)\}_{t\in[0,T]}$ of the form

$$
\tilde{Y}(t) = \tilde{E}(T)\tilde{X}_0 + \int_0^t \tilde{E}(T-s)\tilde{B}\,\mathrm{d}W(s) \quad \text{with } \tilde{X}(T) = \tilde{Y}(T).
$$

Here $\{\tilde{E}(t)\}_{t\in[0,T]} \subset \mathcal{B}(\mathcal{S},\mathcal{S})$ and $\tilde{B} \in \mathcal{B}(\mathcal{U},\mathcal{S})$, where $\mathcal{S}$ is a Hilbert subspace of $\mathcal{H}$ with the same norm (typically $\mathcal{S} = \mathcal{H}$ or $\mathcal{S}$ is a finite-dimensional subspace of $\mathcal{H}$). $\tilde{E}(t)$ can be obtained by time interpolation of the time stepping operator.

Weak convergence

Theorem

If $G \in \mathcal{C}^2_{\mathrm{b}}(\mathcal{H}, \mathbf{R})$, then the weak error $e(T)$ has the representation

$$e(T) = \mathbf{E}\left[u(\tilde{Y}(0), 0) - u(Y(0), 0)\right] + \tfrac{1}{2}\mathbf{E}\int_0^T \mathrm{Tr}\left(u_{xx}(\tilde{Y}(t), t)\mathcal{O}(t)\right)\mathrm{d}t,$$

where

$$\mathcal{O}(t) = \big(\tilde{E}(T-t)\tilde{B} + E(T-t)B\big)Q\big(\tilde{E}(T-t)\tilde{B} - E(T-t)B\big)^*,$$

or

$$\mathcal{O}(t) = \big(\tilde{E}(T-t)\tilde{B} - E(T-t)B\big)Q\big(\tilde{E}(T-t)\tilde{B} + E(T-t)B\big)^*.$$

This has been applied to fully discrete schemes for the linear heat, wave and Cahn-Hilliard-Cook equations, [Debussche and Printems(2009)], [Kovács et al.(2012a)], [Kovács et al.(2012b)], [Lindner and Schilling(2012)].

Weak convergence: Malliavin calculus

I will now explain how the integration by parts from the Malliavin calculus can be used. As we have seen this is not needed for linear problems, but the main difficulty occurs already there, so I will present the argument for the linear heat equation.

Assume for simplicity that $X_0 = 0$, so that

$$
\begin{aligned}
X(t) &= \int_0^t E(t-s)\,\mathrm{d}W(s),\\
X_h(t) &= \int_0^t E_h(t-s)P_h\,\mathrm{d}W(s)
\end{aligned}
$$

and the weak error

$$
\begin{aligned}
&\mathbf{E}\Big[G(X_h(T)) - G(X(T))\Big]\\
&\quad = \mathbf{E}\int_0^T \Big\{ -\langle u_x(X_h(t),t), (A_h - A)X_h(t)\rangle\\
&\qquad + \tfrac{1}{2}\operatorname{Tr}\big[u_{xx}(X_h(t),t)[P_hQP_h - Q]\big]\Big\}\,\mathrm{d}t.
\end{aligned}
$$

Weak convergence: Malliavin

We assume as usual, for some $\beta \in [0, r/2]$,

$$\|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} = \|A^{\frac{\beta-1}{2}}\|_{\mathcal{L}_2^0} < \infty.$$

To be specific, let $\beta = 1$:

$$\|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} = \|Q^{\frac{1}{2}}\|_{\mathrm{HS}} = \|I\|_{\mathcal{L}_2^0} = \mathrm{Tr}(Q)^{\frac{1}{2}} < \infty.$$

We want to obtain weak order $h^{2\beta-\epsilon} = h^{2-\epsilon}$.

Malliavin calculus

Theorem

For any random variable $F \in \mathbf{D}^{1,2}(H)$ *and any predictable process* $\Phi \in L_2([0,T] \times \Omega, \mathcal{L}_2^0)$ *the following integration by parts formula is valid.*

$$\mathbf{E}\Big[\Big\langle F, \int_0^t \Phi(s)\,\mathrm{d}W(s)\Big\rangle_H\Big] = \mathbf{E}\Big[\int_0^t \big\langle D_s F, \Phi(s)\big\rangle_{\mathcal{L}_2^0}\,\mathrm{d}s\Big].$$

We will use this (essentially) with $\Phi(s) = E_h(t-s)P_h$ and

$$F = u_x(X_h(t),t), \quad D_s F = D_s u_x(X_h(t),t) = u_{xx}(X_h(t),t) D_s X_h(t),$$

where

$$X_h(t) = \int_0^t E_h(t-s)P_h\,\mathrm{d}W(s), \quad D_s X_h(t) = E_h(t-s)P_h,$$

and

$$u_{xx}(X_h(t),t) = \mathbf{E}\big[E(T-t)G''\big(Z(T;t,X_h(t))E(T-t)\big)|\mathcal{F}_t\big].$$

Malliavin

The difficult term:

$$
\begin{aligned}
&\left|\mathbf{E}\int_0^T \langle u_x(X_h(t),t),(A_h-A)X_h(t)\rangle \,\mathrm{d}t\right| \\
&= \left|\mathbf{E}\int_0^T \langle (A^{-1}-A_h^{-1}P_h)Au_x, A_hX_h\rangle \,\mathrm{d}t\right| \quad \left[K_h = A^{-1}-A_h^{-1}P_h\right] \\
&= \left|\int_0^T \mathbf{E}\Big[\Big\langle K_hAu_x(X_h(t),t), A_h\int_0^t E_h(t-s)P_h\,\mathrm{d}W(s)\Big\rangle\Big]\,\mathrm{d}t\right| \\
&= \left|\int_0^T \mathbf{E}\Big[\int_0^t \Big\langle K_hAD_su_x(X_h(t),t), A_hE_h(t-s)P_h\Big\rangle_{\mathcal{L}_2^0}\Big]\,\mathrm{d}s\,\mathrm{d}t\right| \\
&= \left|\int_0^T \mathbf{E}\Big[\int_0^t \langle K_hAu_{xx}(X_h(t),t)D_sX_h(t), A_hE_h(t-s)P_h\rangle_{\mathcal{L}_2^0}\Big]\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\le \int_0^T \mathbf{E}\Big[\int_0^t \|K_hAu_{xx}(X_h(t),t)D_sX_h(t)\|_{\mathcal{L}_2^0}\|A_hE_h(t-s)P_h\|_{\mathcal{L}_2^0}\Big]\,\mathrm{d}s\,\mathrm{d}t
\end{aligned}
$$

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$$\cdots \leq \Big| \int_0^T \mathbf{E}\Big[\int_0^t \|K_h A u_{xx}(X_h(t),t) D_s X_h(t)\|_{\mathcal{L}_2^0} \|A_h E_h(t-s) P_h\|_{\mathcal{L}_2^0} \Big] \, \mathrm{d}s \, \mathrm{d}t \Big|$$

$$\leq \int_0^T \mathbf{E}\Big[\int_0^t \|K_h\|_{\mathcal{L}} \|A u_{xx}(X_h(t),t)\|_{\mathcal{L}} \|D_s X_h(t)\|_{\mathcal{L}} \|I\|_{\mathcal{L}_2^0} \times \|A_h E_h(t-s) P_h\|_{\mathcal{L}} \|I\|_{\mathcal{L}_2^0} \Big] \, \mathrm{d}s \, \mathrm{d}t$$

$$u_{xx}(X_h(t),t) = \mathbf{E}\big[E(T-t) G''\big(Z(T;t,X_h(t)) E(T-t)\big) | \mathcal{F}_t \big]$$

$$D_s X_h(t) = E_h(t-s) P_h$$

$$\leq \int_0^T \int_0^t \|K_h\|_{\mathcal{L}} \|A E(T-t)\|_{\mathcal{L}} |G|_{\mathcal{C}_{\mathrm{b}}^2} \|E(T-t)\|_{\mathcal{L}} \|E_h(t-s) P_h\|_{\mathcal{L}} \times \|A_h E_h(t-s) P_h\|_{\mathcal{L}} \, \mathrm{d}s \, \mathrm{d}t \, \|I\|_{\mathcal{L}_2^0}^2$$

$$\leq C h^2 \int_0^T (T-t)^{-1} \int_0^t (t-s)^{-1} \, \mathrm{d}s \, \mathrm{d}t \, \|I\|_{\mathcal{L}_2^0}^2 |G|_{\mathcal{C}_{\mathrm{b}}^2}$$

Almost convergent: lose ϵ.

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Tray again, with ϵ loss:

$$\begin{aligned}
\cdots &= \Big| \int_0^T \mathbf{E}\Big[\int_0^t \langle K_h A u_{xx}(X_h(t),t) D_s X_h(t), A_h E_h(t-s) P_h \rangle_{\mathcal{L}_2^0} \Big] \,\mathrm{d}s\,\mathrm{d}t \Big| \\
&= \Big| \int_0^T \mathbf{E}\Big[\int_0^t \langle A^{\frac{\epsilon}{2}} K_h A^{\frac{\epsilon}{2}} A^{1-\frac{\epsilon}{2}} u_{xx}(X_h(t),t) D_s X_h(t), A^{-\frac{\epsilon}{2}} A_h E_h(t-s) P_h \rangle_{\mathcal{L}_2^0} \Big] \,\mathrm{d}s\,\mathrm{d}t \Big| \\
&\le \int_0^T \int_0^t \|A^{\frac{\epsilon}{2}} K_h A^{\frac{\epsilon}{2}}\|_{\mathcal{L}} \|A^{1-\frac{\epsilon}{2}} E(T-t)\|_{\mathcal{L}} |G|_{\mathcal{C}_\mathrm{b}^2} \|E(T-t)\|_{\mathcal{L}} \|E_h(t-s)P_h\|_{\mathcal{L}} \\
&\quad \times \|A^{-\frac{\epsilon}{2}} A_h^{\frac{\epsilon}{2}}\|_{\mathcal{L}} \|A_h^{1-\frac{\epsilon}{2}} E_h(t-s) P_h\|_{\mathcal{L}} \,\mathrm{d}s\,\mathrm{d}t \, \|I\|^2_{\mathcal{L}_2^0} \\
&\le C h^{2-2\epsilon} \int_0^T (T-t)^{-1+\frac{\epsilon}{2}} \int_0^t (t-s)^{-1+\frac{\epsilon}{2}} \,\mathrm{d}s\,\mathrm{d}t \, \|I\|^2_{\mathcal{L}_2^0} |G|_{\mathcal{C}_\mathrm{b}^2} \le C h^{2-2\epsilon}.
\end{aligned}$$

Here $\|A^{-\frac{\epsilon}{2}} A_h^{\frac{\epsilon}{2}}\|_{\mathcal{L}} \le C$, for example, if we have a quasi-uniform mesh family.

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In the nonlinear case we do not have formulas for $u_{xx}(X_h(t), t)$ and $D_s X_h(t)$ and so we must write down the equations that they satisfy and prove bounds for

$$\|A^{1-\frac{\epsilon}{2}} u_{xx}(X_h(t), t)\|_{\mathcal{L}}, \quad \|D_s X_h(t)\|_{\mathcal{L}_2^0}.$$

The remaining term is easier:

$$\begin{aligned}
&\left|\mathbf{E} \int_0^T \operatorname{Tr}\left[u_{xx}(X_h(t), t)[P_h Q P_h - Q]\right] \mathrm{d}t\right| \\
&= \left|\mathbf{E} \int_0^T \operatorname{Tr}\left[u_{xx}(X_h(t), t)[(P_h + I) Q (P_h - I)]\right] \mathrm{d}t\right| \\
&= \left|\mathbf{E} \int_0^T \operatorname{Tr}\left[A^{1-\frac{\epsilon}{2}} u_{xx}(X_h(t), t)[(P_h + I) Q (P_h - I) A^{-1+\frac{\epsilon}{2}}]\right] \mathrm{d}t\right| \\
&\le \mathbf{E} \int_0^T \|A^{1-\frac{\epsilon}{2}} u_{xx}(X_h(t), t)\|_{\mathcal{L}} \|P_h + I\|_{\mathcal{L}} \operatorname{Tr}(Q) \|(P_h - I) A^{-1+\frac{\epsilon}{2}}\|_{\mathcal{L}} \,\mathrm{d}t \\
&\le C h^{2-\epsilon} \int_0^T (T - t)^{-1+\frac{\epsilon}{2}} \,\mathrm{d}t \ \operatorname{Tr}(Q).
\end{aligned}$$

Weak convergence: Malliavin

The above argument is not rigorous because the Kolmogorov equation is not valid for $x \in H$. To handle this we project onto the eigenspaces of A in order to get a finite dimensional Kolmogorov equation. Auxiliary process $Z_m(s) = Z_m(s; t, x)$:

$$Z_m(s) = E_m(s-t)P_m\xi + \int_t^s E_m(s-r)P_m \, \mathrm{d}W(r).$$

Define $u_m : H \times [0, T] \to \mathbf{R}$ by

$$u_m(x,t) = \mathbf{E}\Big[G\big(Z_m(T; t, x)\big)\Big].$$

Then $u_m(x,t) = u_m(P_m x, t),$ to be used with $x = X_h(t)$. The partial derivatives are

$$u_{m,x}(x,t) = \mathbf{E}\Big[E_m(T-t)P_m G'\big(Z_m(T; t, x)\big)\Big],$$
$$u_{m,xx}(x,t) = \mathbf{E}\Big[E_m(T-t)P_m G''\big(Z_m(T; t, x)\big) E_m(T-t)P_m\Big].$$

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Auxiliary process:

$$Z_m(s) = E_m(s-t)P_m\xi + \int_t^s E_m(s-r)P_m \, \mathrm{d}W(r).$$

Define $u_m : H \times [0, T] \to \mathbf{R}$ by

$$u_m(x,t) = \mathbf{E}\Big[G\big(Z_m(T;t,x)\big)\Big].$$

Kolmogorov's equation:

$$\begin{cases} u_{m,t}(x,t) - \langle u_{m,x}(x,t), A_m x\rangle + \frac{1}{2}\,\mathrm{Tr}\,(u_{m,xx}(x,t)P_m Q P_m) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T), \;\; x \in H, \\ u(x,T) = G(P_m x) \end{cases}$$

Malliavin

This leads to the weak error formula:

$$
\begin{aligned}
&\mathbf{E}\Big[G(X_h(T)) - G(X(T))\Big] \\
&\quad = \mathbf{E}\int_0^T \Big\{ -\langle u_{m,x}(X_h(t),t),(A_h - A_m)X_h(t)\rangle \\
&\qquad + \tfrac{1}{2}\,\mathrm{Tr}\Big[u_{m,xx}(X_h(t),t)[P_hQP_h - P_mQP_m]\Big]\Big\}\,\mathrm{d}t.
\end{aligned}
$$

In the first term we write

$$
\begin{aligned}
\langle u_{m,x},(A_h - A_m)X_h\rangle &= \langle u_{m,x},(P_hA_h - A_mP_h)X_h\rangle \\
&= \langle (A_hP_h - P_hA_m)u_{m,x}, X_h\rangle \\
&= \langle A_hP_h(I - A_h^{-1}P_hA_m)u_{m,x}, X_h\rangle \\
&= \langle A_hP_h(I - P_m + A^{-1}AP_m - A_h^{-1}P_hAP_m)u_{m,x}, X_h\rangle \\
&= \langle (A^{-1} - A_h^{-1}P_h)AP_m u_{m,x}, A_hX_h\rangle \\
&\quad + \langle (I - P_m)u_{m,x}, A_hX_h\rangle.
\end{aligned}
$$

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Similar treatment of the other term:

$$\begin{aligned}
&\operatorname{Tr}\left(u_{m,xx}[P_h Q P_h - P_m Q P_m]\right) = \operatorname{Tr}\left(u_{m,xx}[P_h + P_m]Q[P_h - P_m]\right) \\
&\quad = \operatorname{Tr}\left(u_{m,xx}[P_h + P_m]Q[P_h - I + I - P_m]\right) \\
&\quad = \operatorname{Tr}\left(u_{m,xx}[P_h + P_m]Q[P_h - I]\right) + \operatorname{Tr}\left(u_{m,xx}[P_h + P_m]Q[I - P_m]\right).
\end{aligned}$$

In both cases we get an extra term containing $I - P_m$.

For fixed h, let $m \to \infty$, show that extra terms $\to 0$. Then let $h \to 0$.

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The main term is

$$\left|\mathbf{E}\int_0^T \langle (A^{-1}-A_h^{-1}P_h)AP_m u_{m,x}, A_hX_h\rangle \,\mathrm{d}t\right| \text{ Malliavin integration by parts...}$$

$$\leq \int_0^T\int_0^t \|A^{\frac{\epsilon}{2}}K_hA^{\frac{\epsilon}{2}}\|_{\mathcal{L}}\|A^{-\frac{\epsilon}{2}}A_mE_m(T-t)P_m\|_{\mathcal{L}}|G|_{\mathcal{C}^2_{\mathrm{b}}}\|E_m(T-t)P_m\|_{\mathcal{L}}$$

$$\times\|E_h(t-s)P_h\|_{\mathcal{L}}\|A^{-\frac{\epsilon}{2}}A_h^{\frac{\epsilon}{2}}\|_{\mathcal{L}}\|A_h^{1-\frac{\epsilon}{2}}E_h(t-s)P_h\|_{\mathcal{L}}\,\mathrm{d}s\,\mathrm{d}t\,\|I\|^2_{\mathcal{L}_2^0}$$

$$\leq Ch^{2-2\epsilon}\int_0^T (T-t)^{-1+\frac{\epsilon}{2}}\int_0^t (t-s)^{-1+\frac{\epsilon}{2}}\,\mathrm{d}s\,\mathrm{d}t\,\|I\|^2_{\mathcal{L}_2^0}|G|_{\mathcal{C}^2_{\mathrm{b}}} \leq Ch^{2-2\epsilon},$$

which is independent of m. The extra term becomes

$$\left|\mathbf{E}\int_0^T \langle (I-P_m)u_{m,x}, A_hX_h\rangle\,\mathrm{d}t\right|$$

$$\leq \mathbf{E}\int_0^T \|(I-P_m)A^{-1+\epsilon}\|_{\mathcal{L}}\|A^{1-\epsilon}E_m(T-t)P_m\|_{\mathcal{L}}|G|_{\mathcal{C}^1_{\mathrm{b}}}\|A_hX_h(t)\|_H\,\mathrm{d}t$$

$$\leq C\lambda_m^{-1+\epsilon}\int_0^T (T-t)^{-1+\epsilon}\,\mathrm{d}t\,|G|_{\mathcal{C}^1_{\mathrm{b}}}\|A_hP_h\|_{\mathcal{L}}\sup_{t\in[0,T]}\|X_h(t)\|_{L_2(\Omega,H)}\to 0,$$

as $m\to\infty$ for fixed h.

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More precisely,

$$\left|\mathbf{E}\Big[G(X_h(T)) - G(X(T))\Big]\right| \\ \leq Ch^{2-2\epsilon} + Ch^{-2}\lambda_m^{-1+\epsilon} + \text{ other terms of the same form.}$$

Therefore

$$\left|\mathbf{E}\Big[G(X_h(T)) - G(X(T))\Big]\right| \leq Ch^{2-2\epsilon}.$$

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More precisely,

$$\left|\mathbf{E}\Big[G(X_h(T)) - G(X(T))\Big]\right| \\ \le Ch^{2-2\epsilon} + Ch^{-2}\lambda_m^{-1+\epsilon} + \text{ other terms of the same form.}$$

Therefore

$$\left|\mathbf{E}\Big[G(X_h(T)) - G(X(T))\Big]\right| \le Ch^{2-2\epsilon}.$$

This type of analysis has been carried out for the nonlinear heat equation:

- Debussche [Debussche(2011)], multiplicative noise in 1–D, time-stepping,
- Wang and Gan [Wang and Gan(2012)], additive noise in multi–D, time-stepping,
- Andersson and L [Andersson and Larsson(2012)], additive noise in multi–D, multiplicative noise in 1–D, spatial discretization.

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