Random Sets

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May 2012, ETH-Zürich
Lecture #2

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \ldots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \ldots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \ldots \xi^\nu$ sample P^ν (random) empirical measure

approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

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$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

SAA-applies without 'any' restrictions

f on $\Xi \times E$, random lsc fcn (loc. inf-$\int$), $\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \dots, \}$ iid

Then $E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot\,) = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\}$

ε-argmin $E^\nu f \Rightarrow_v \varepsilon$-argmin Ef, $\forall\, \varepsilon \geq 0$

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$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

Stochastic Programming (with recourse)

$f(\xi, x) = f_{01}(x) + Q(\xi, x), \;\; Q(\xi, x) = \inf_y \left\{ f_{02}(\xi, y) \,\middle|\, y \in C_2(\xi, x) \right\}$

SAA-problem: $\min \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x) \xrightarrow{e} Ef(x) = \mathbb{E}\{f(\xi, x)\}$

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Statistical Estimation (fusion of hard & soft information)

$L(\xi, h) = -\ln h(\xi)$ if $h \geq 0, \int h = 1, h \in A^{\mathrm{soft}} \subset E$

Then, estimate $h^\nu \in \mathrm{argmin}_E\, \mathbb{E}^\nu\{L(\boldsymbol{\xi}, h)\} \to h^{\mathrm{true}} = \mathrm{argmin}\, \mathbb{E}\{L(\boldsymbol{\xi}, h)\}$

example: Normal density

mean = (0,0) ... data samples correlated

covariance: MDM^T, $D=$ diag(4,1), $M=\begin{pmatrix} \cos(\pi/6) & \cos(2\pi/3) \\ \sin(\pi/6) & \sin(2\pi/3) \end{pmatrix}$

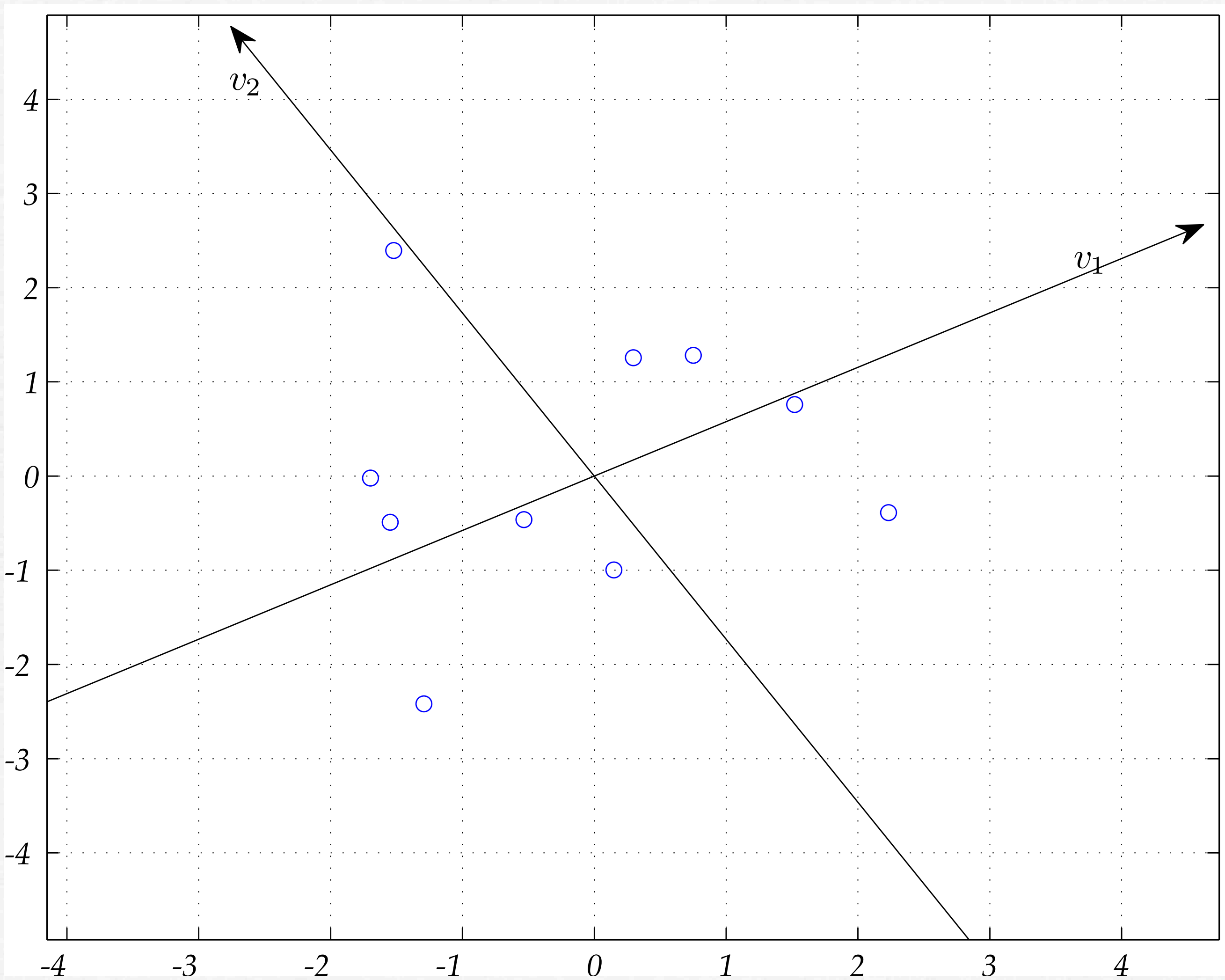
samples: $\nu = 10$,

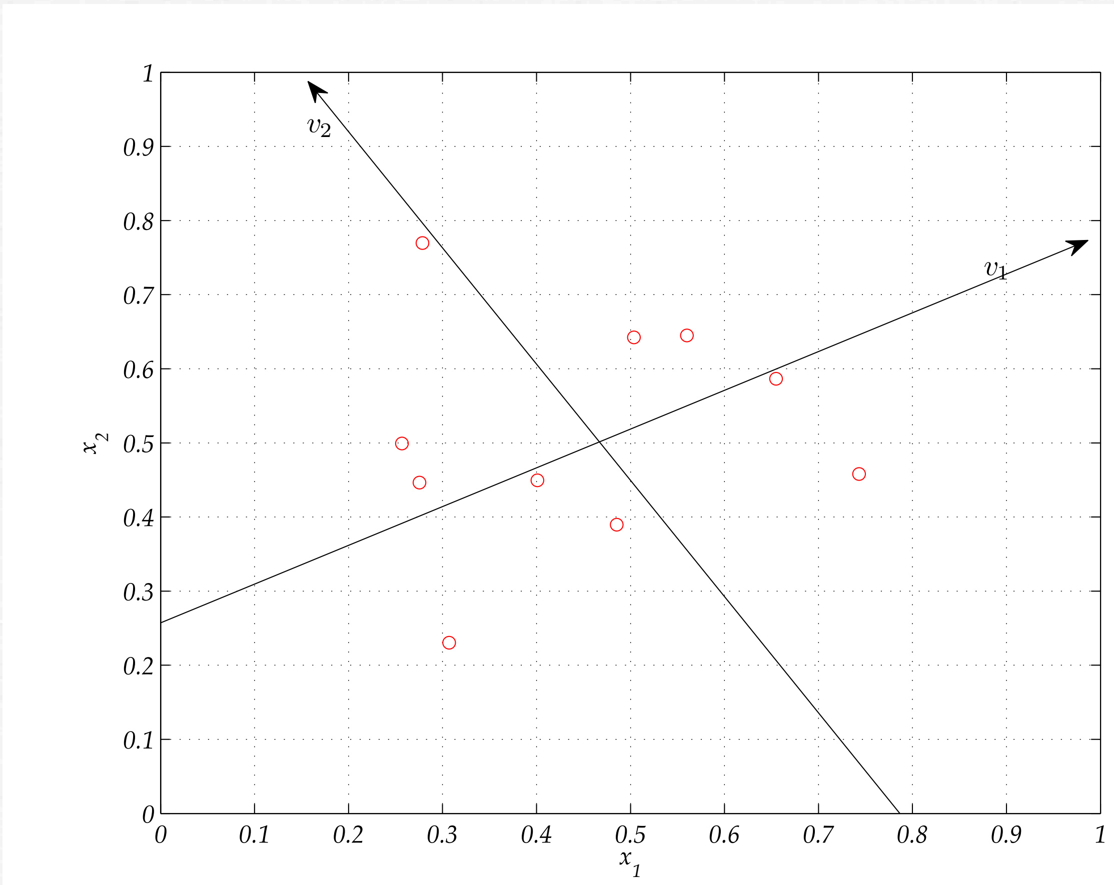
"soft" information: h unimodal

Results:

$$\left\| h^{true} - h^{est} \right\|_2^2 = 0.028, \quad \left\| h^{true} - h^{est} \right\|_\infty = 0.006$$

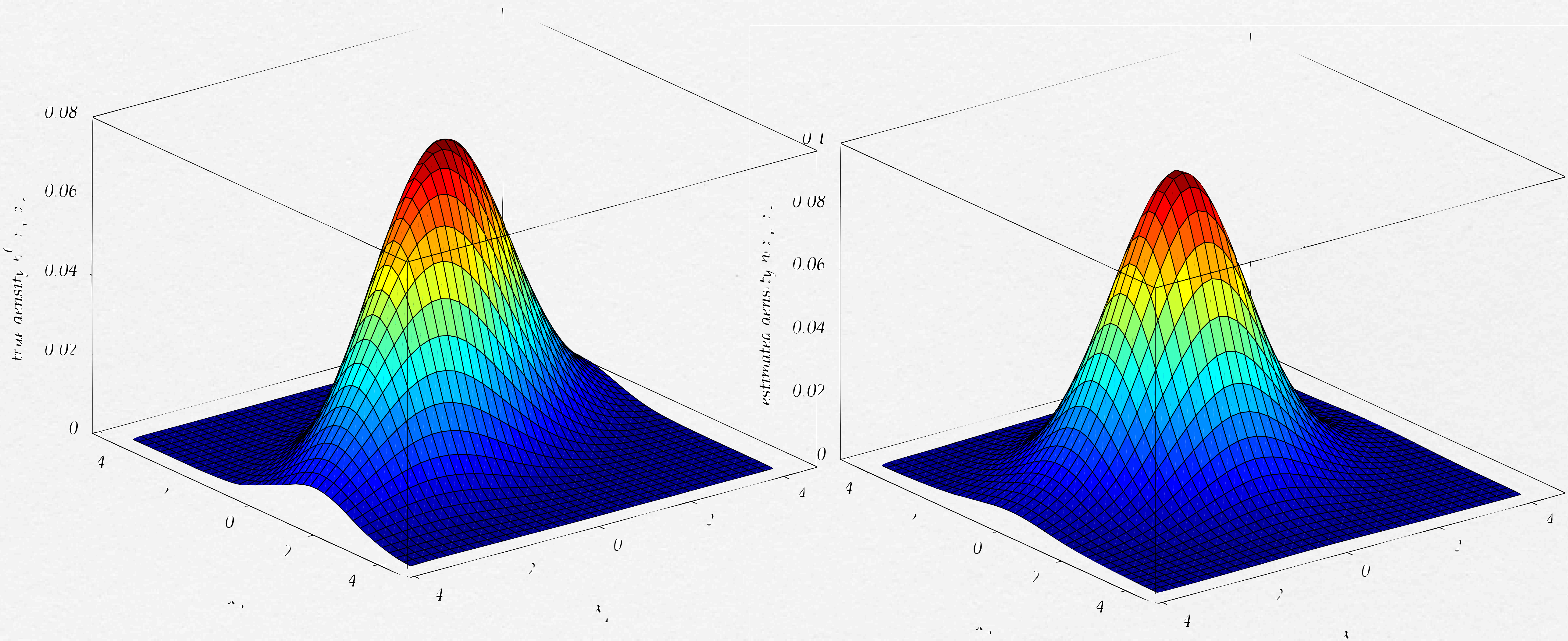
Sampled data



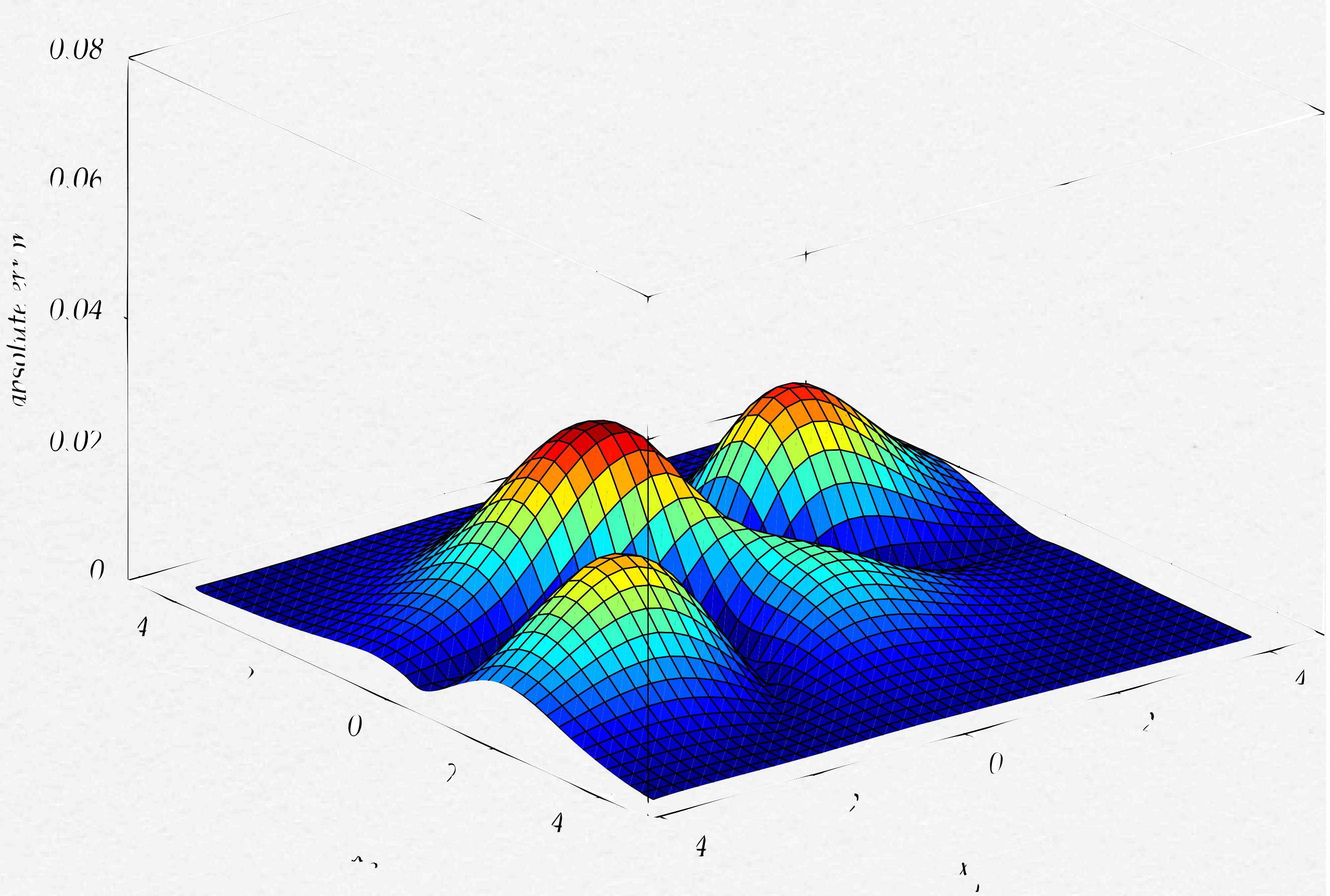


normalized

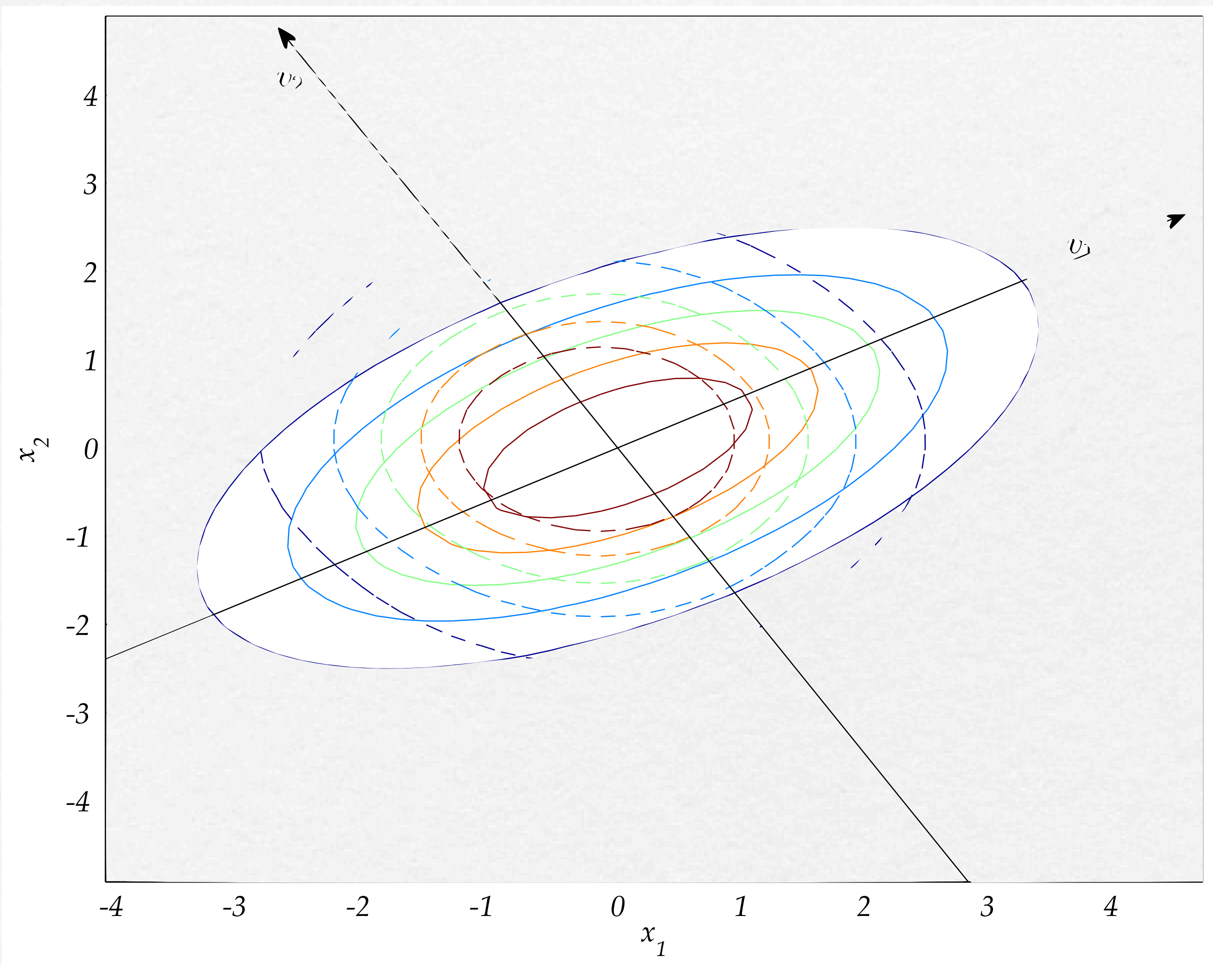
True & Estimated density



Measurement Errors



Absolute Error



Level curves: true & estimate

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$$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0 \text{ “approximated” by } G^\nu(x) = 0$$
$$\xi^1, \dots, \xi^\nu \text{ sample}, \quad G^\nu(x) = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} G(\xi^l, x)$$

$$G : \Xi \times D \rightrightarrows E, \text{ set-valued } G(\xi, x) \subset E, \text{ inclusion } \mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$$
$$\xi^1, \dots, \xi^\nu \text{ sample, approximation } \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$$

Pricing contingent claims

claims $\left\{G^t(\vec{\boldsymbol{\xi}}^t)\right\}$, instrum. prices $\left\{S^t(\vec{\boldsymbol{\xi}}^t)\right\}_t$, invest. $\left\{X^t(\vec{\boldsymbol{\xi}}^t)\right\}$

$\max \mathbb{E}\{\langle S^T, X^T\rangle\}$ s.t. $\langle S^t, X^{t-1}\rangle \le G^t + \langle S^t, X^{t-1}\rangle$ + end conditions.

Use ‘improved estimation’ & sampling: $\max \sum p_\xi \langle S^T(\xi), X^T(\xi)\rangle$

Correct pricing = well regulated market??

Wednesday, May 16, 2012

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Stochastic homogenization: Variational formulation

given $u(\xi, x) \in \operatorname{argmin}_{H_0^1(\Omega)} g(\xi, u) = \frac{1}{2}\int_\Omega a(\xi, x)|\nabla u|^2\, dx - \langle h, u\rangle$

find g^{hom} such that $\mathbb{E}\{u(\boldsymbol{\xi}, \cdot)\} \in \operatorname{argmin} g^{\text{hom}}$

via Ergodic Thm: $g^{\text{hom}} = \Big(\text{epi}_w\text{-}\lim)\nu \frac{1}{\nu}\sum l = 1^\nu g^*(\boldsymbol{\xi}^l, \cdot)\Big)^*$

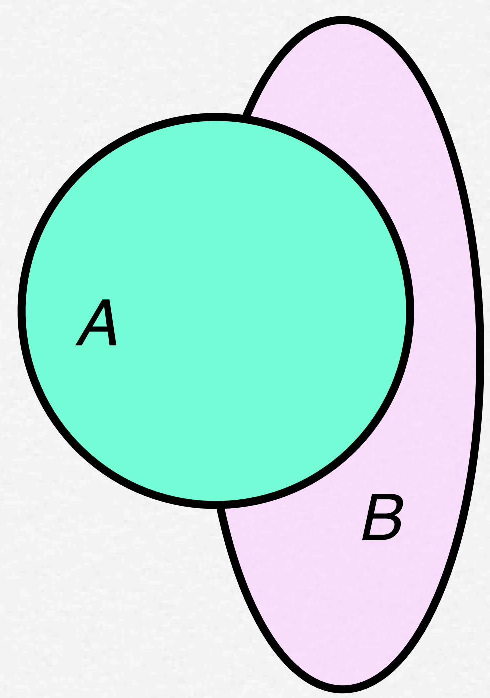
Topology of Hyperspaces

Painlevé, Pompeiu, Zoretti
Zarankiewicz, Hausdorff, Lubben, Moore
Choquet, Vietoris, Fell, Attouch-Wets, Beer, ...

Hyperspace: *sets(E)*

- (E,d) always a Polish space
- $C \subset E,\ d(x,C) = \inf\{d(z,x) \mid z \in C\}, \quad d(x,\emptyset) = \infty$
- $\text{cl-sets}(E) = \{\text{all closed subsets of } E\}, \quad \emptyset, E \in \text{cl-sets}(E)$
- $dl(A,B) =$ distance between A & B, metric(?) on cl-sets(E)
- $(\text{cl-sets}(E), dl)$ Polish space = complete separable metric ??
- $dl(C^\nu, C) \to 0$ means $C^\nu \to C$ (set-convergence)

Pompeiu-Hausdorff distance



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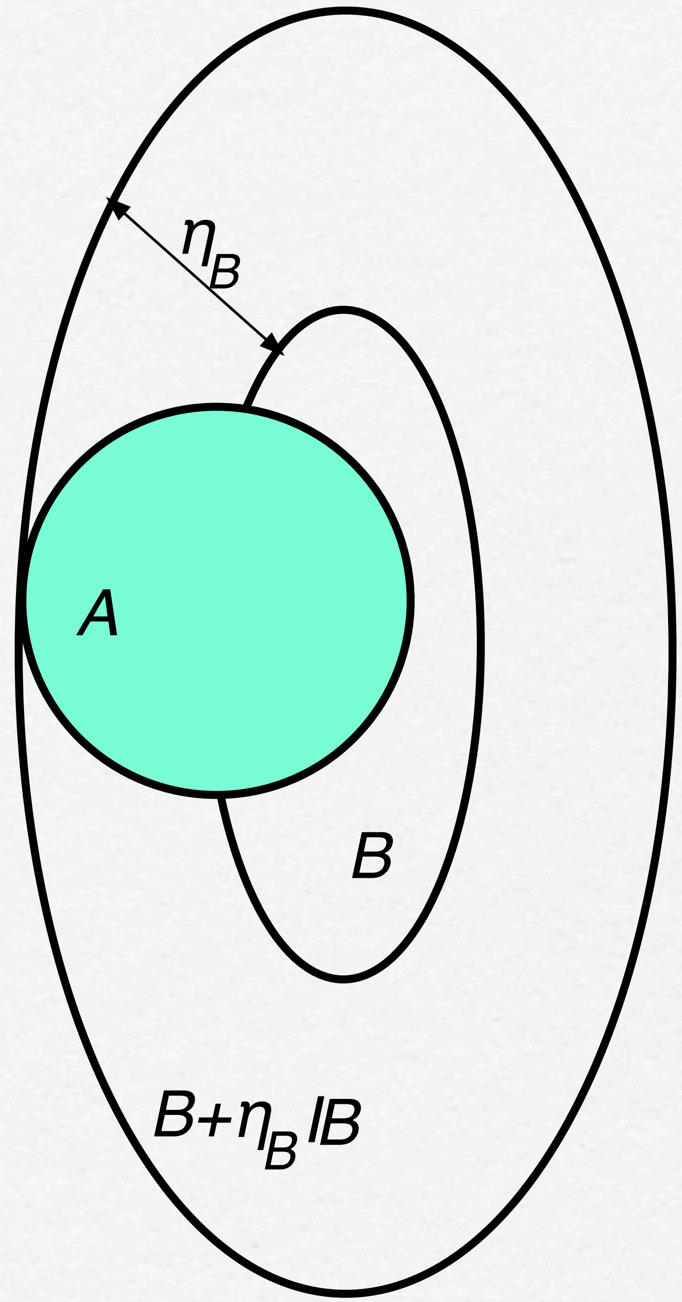

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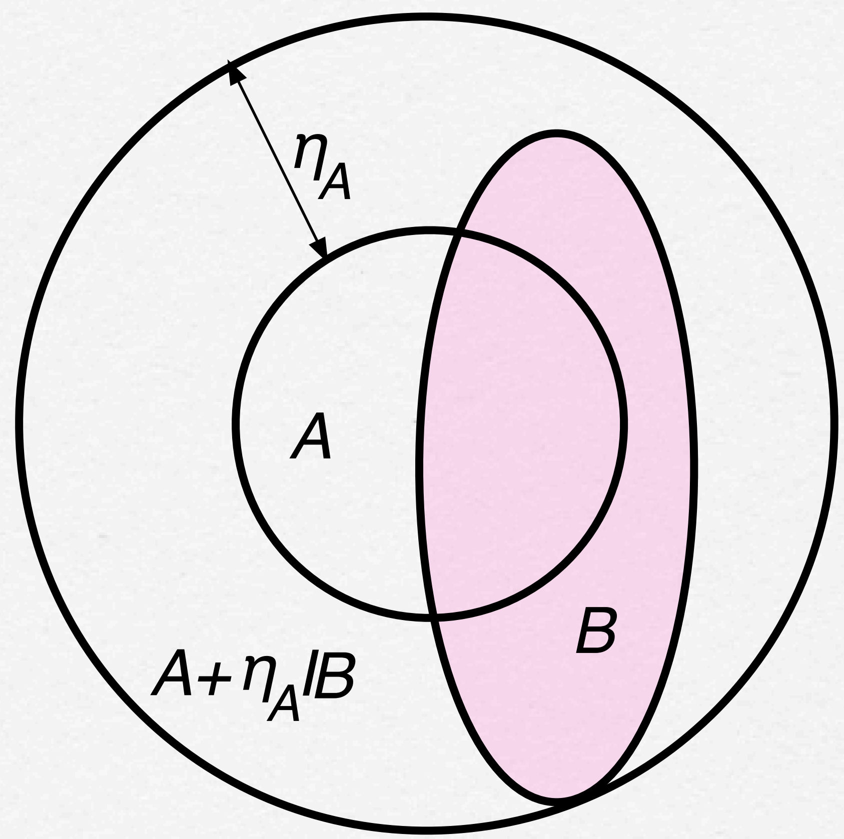
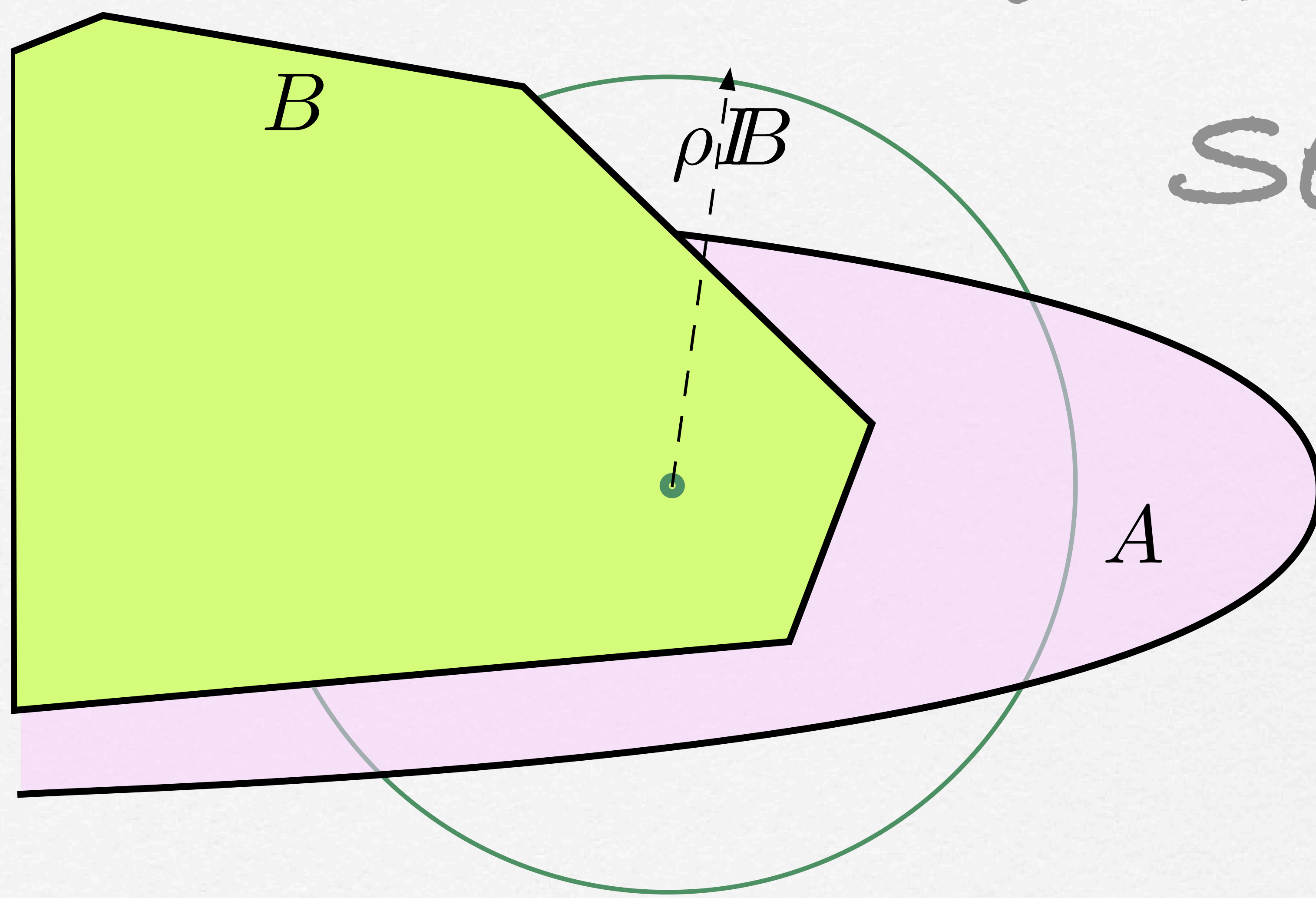


$$\hat{d\!l}(A,B) = \max\,[\,\eta_A, \eta_B\,]$$
$$= d\!l_\infty(A,B)$$

Unbounded Sets



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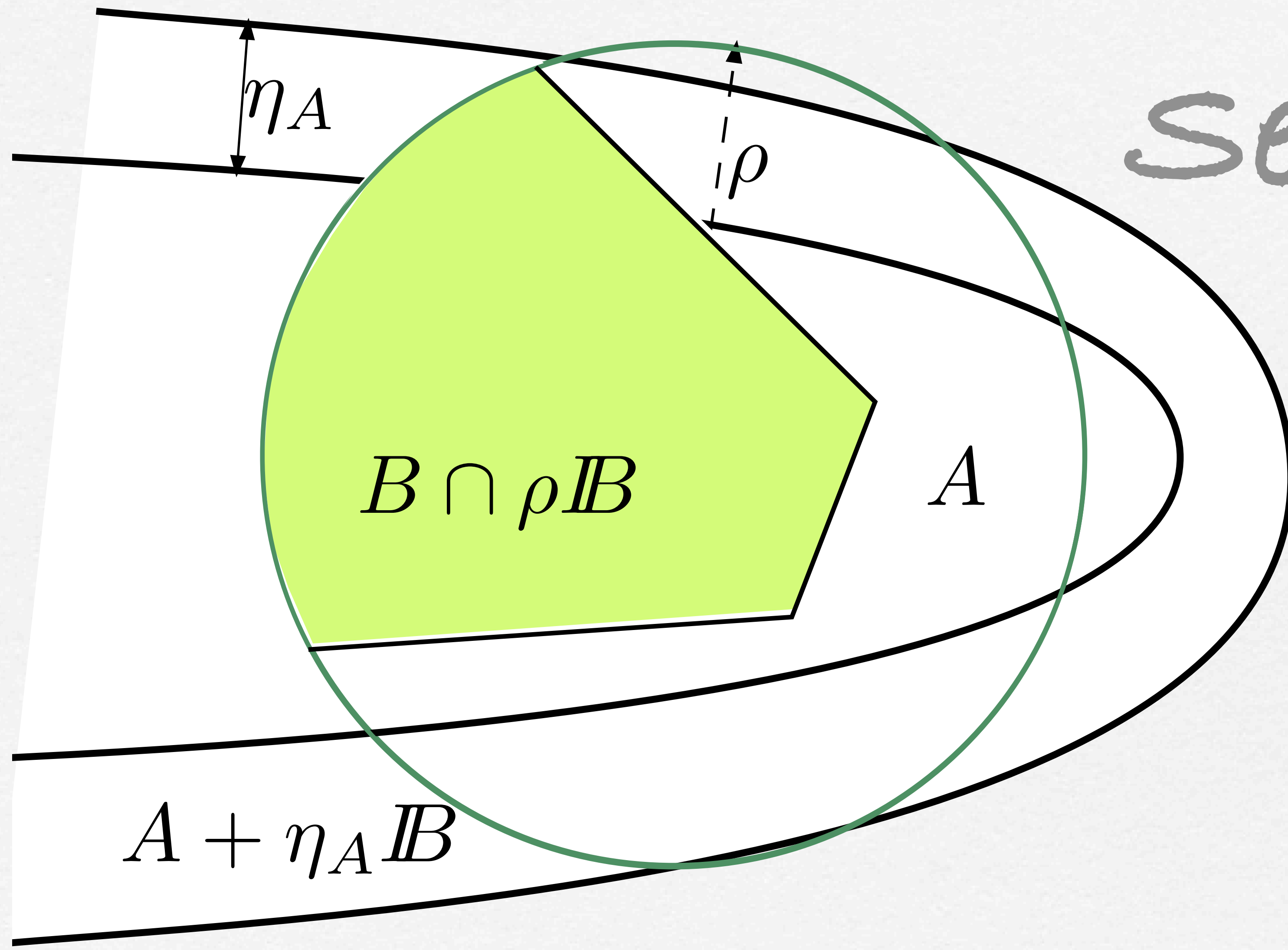

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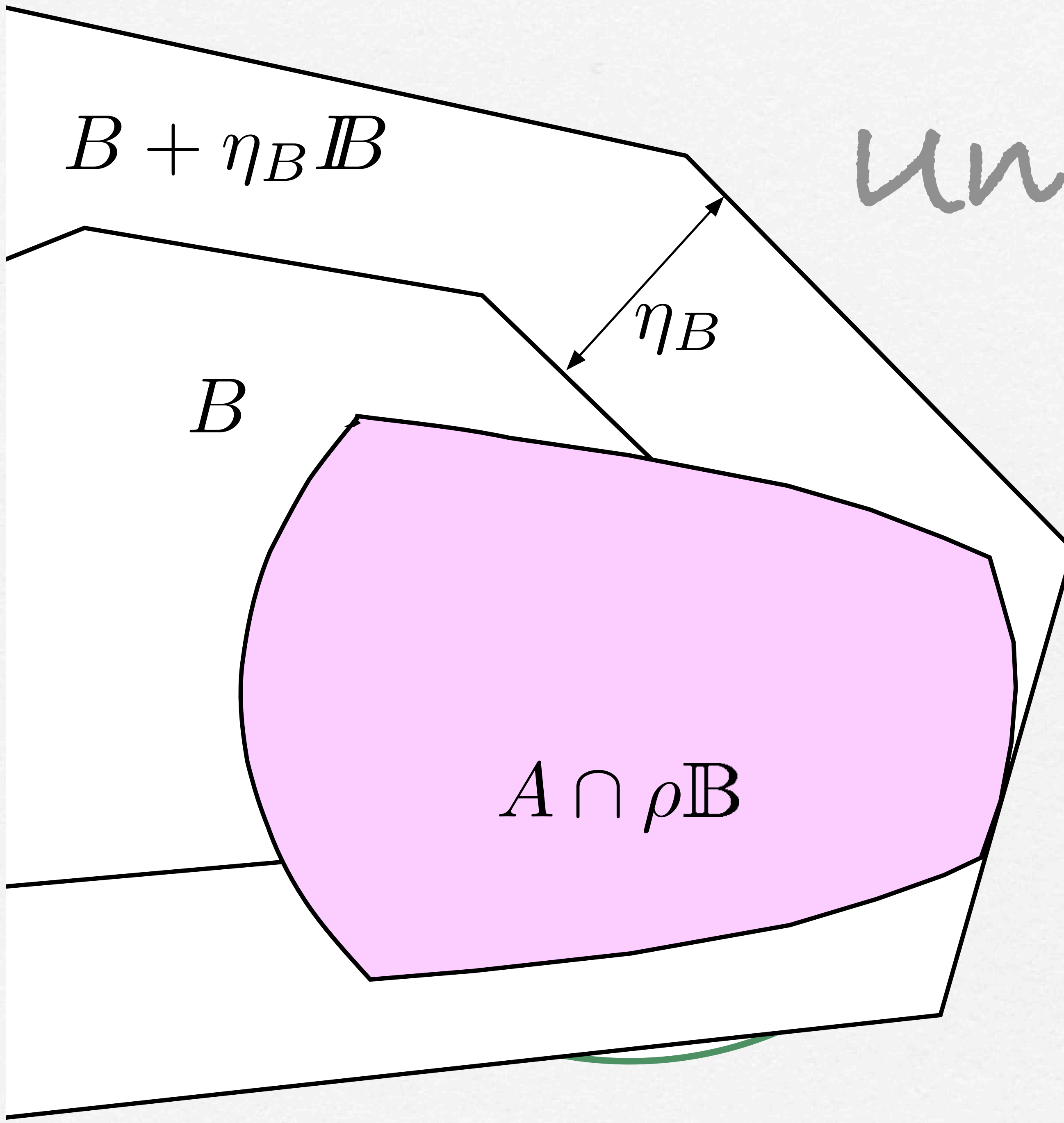

$$\hat{d\!l}_\rho(A,B) = \max\left[\,\eta_A, \eta_B\,\right]$$

set distance (~Attouch-Wets)

τ_{aw} topology

- $\hat{d\!\!l}_\rho(A,B) \geq 0,\ \hat{d\!\!l}(A,A) = 0, \triangle$ inequality
- but $\hat{d\!\!l}_\rho(A,B) = 0$ possibly when $A \neq B$
-
-
-
- $\hat{d\!\!l}_\rho(A,B) \leq d\!\!l_\rho(A,B) \leq \hat{d\!\!l}_{\rho'}(A,B) \quad \rho' \geq 2\rho + d_0$

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- but $\hat{d\!l}_\rho(A,B) = 0$ possibly when $A \neq B$
- $d\!l_\rho(A,B) = \sup_{x \in \rho \boldsymbol{B}} \big[d(x,A),\, d(x,B) \big]$
- for all $\rho \geq 0,\ d\!l_\rho$ is a pseudo-metric
- $d\!l(A,B) = \int_{\rho \geq 0} d\!l_\rho(A,B) e^{-\rho}\, d\rho,$ set-metric
- $\hat{d\!l}_\rho(A,B) \leq d\!l_\rho(A,B) \leq \hat{d\!l}_{\rho'}(A,B) \quad \rho' \geq 2\rho + d_0$

Properties of the set-distance

$$C^\nu \to C \text{ if } d\!l(C^\nu, C)) \to 0 \iff \text{for any } \bar\rho \geq 0,$$

$$\begin{cases} d\!l_\rho(C^\nu, C) \to 0 & \text{for all } \rho \geq \bar\rho \\ \hat{d\!l}_\rho(C^\nu, C) \to 0 & \text{for all } \rho \geq \bar\rho \end{cases}$$

$$(E, d) \text{ Polish} \implies (\text{cl-sets}(E), d\!l) \text{ complete, metric space}$$

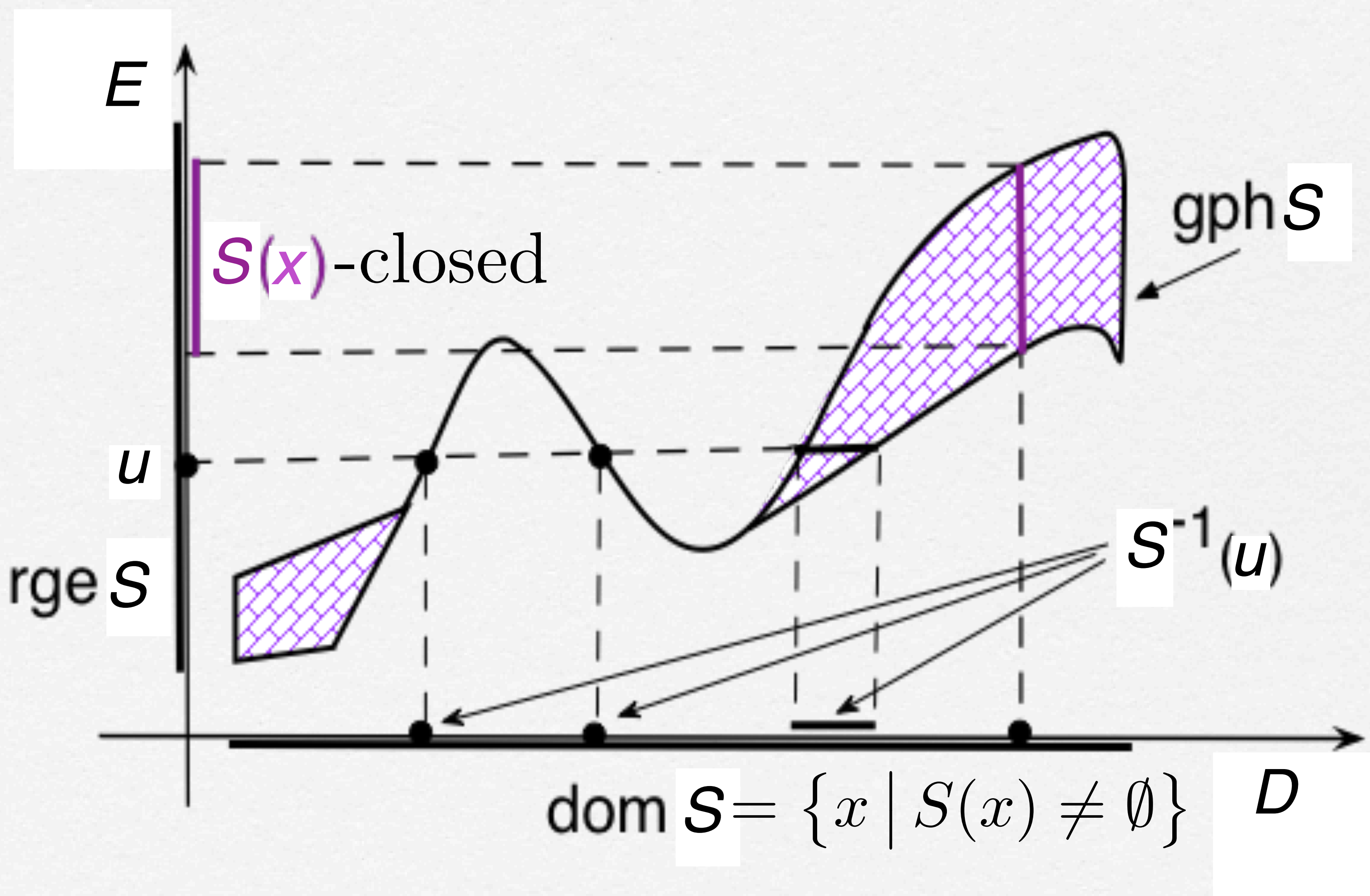
$$(\text{cl-sets}(E), d\!l) \text{ Polish} \iff E = \mathbb{R}^n$$

space of *osc-mappings*

outer semicontinuous

$$S : D \rightrightarrows E \text{ osc} \iff \text{gph}\, S \subset D \times E \text{ closed}$$

$$\text{gph}\, S = \{(x,u) \,|\, u \in S(x), x \in E\}$$



space of osc-mappings

outer semicontinuous

$$\mathbb{B} = \mathbb{B}_D \times \mathbb{B}_E \text{ (or } \mathbb{B}_{E\times D})$$

$$d\!\!l(R,S) = d\!\!l(\operatorname{gph} R, \operatorname{gph} S), \quad d\!\!l_\rho,\ \hat{d\!\!l}_\rho$$

$(\text{osc-maps}(D,E), d\!\!l)$ complete metric, Polish: $D = \mathbb{R}^n, E = \mathbb{R}^m$

$S : D \to E$ (single-valued) continuous $\Longrightarrow$ osc, ...

$$d\!\!l(f^\nu, f) \to 0 \Longrightarrow \operatorname{argmin} f^\nu \rightrightarrows_v \operatorname{argmin} f$$

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$S^{-1}(0) =$ sol'ns of $S(x) \ni 0$

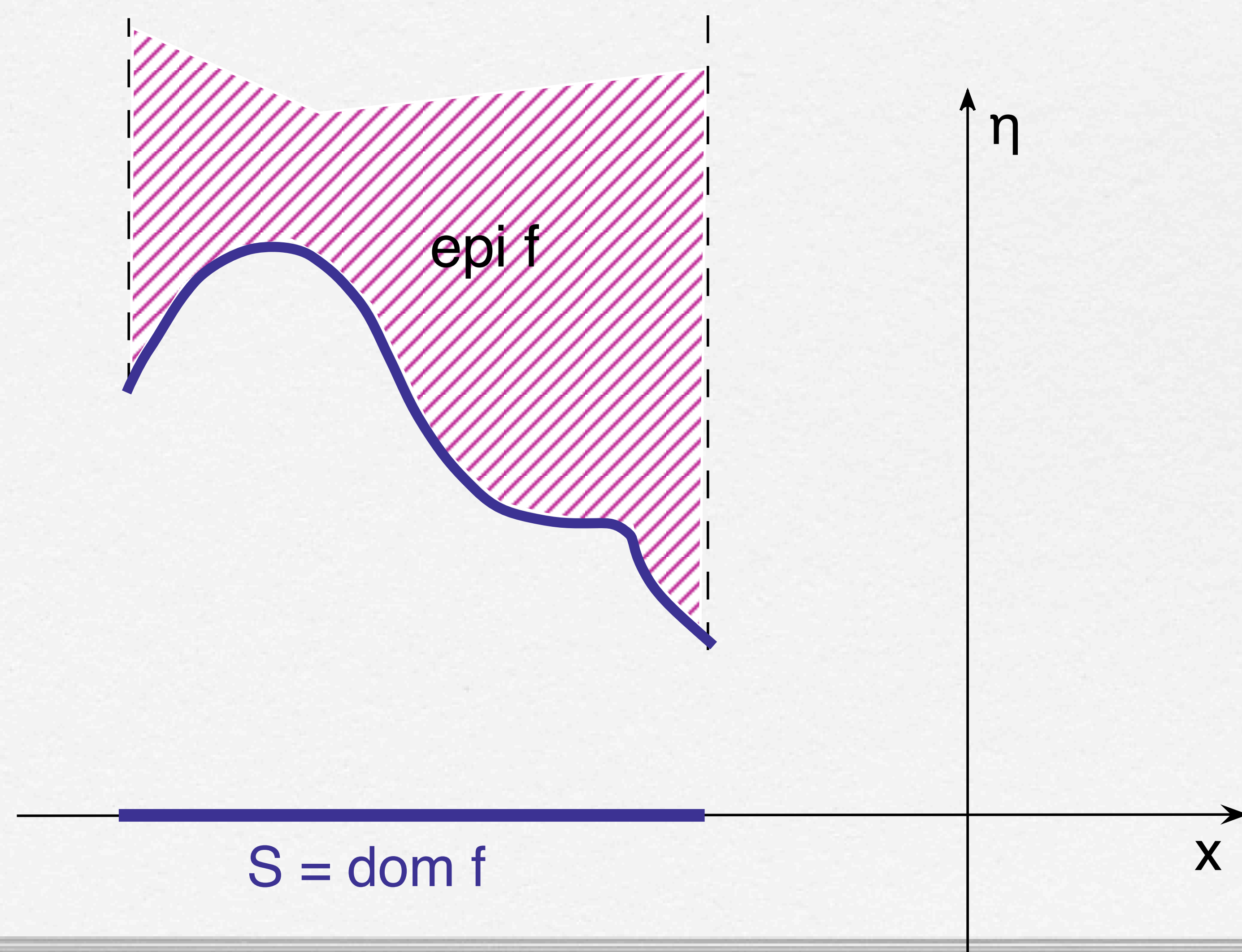
$S^\nu \to S$ uniformly $\Rightarrow d\!l(S^\nu, S) \to 0$

space of *lsc-fcns(E)*

lower semicontinuous

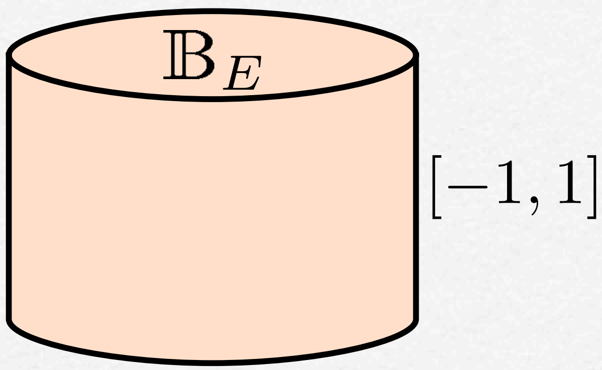
$$f : E \to \overline{\mathbb{R}} \text{ lsc} \iff \text{epi} f \subset E \times \mathbb{R} \text{ closed}$$

$$\text{epi} f = \left\{(x,\eta) \,\middle|\, \eta \geq f(x)\right\}$$



space of *lsc-fcns(E)*

lower semicontinuous



unit ball $\mathbb{B} = \mathbb{B}_E \times [-1,1]$

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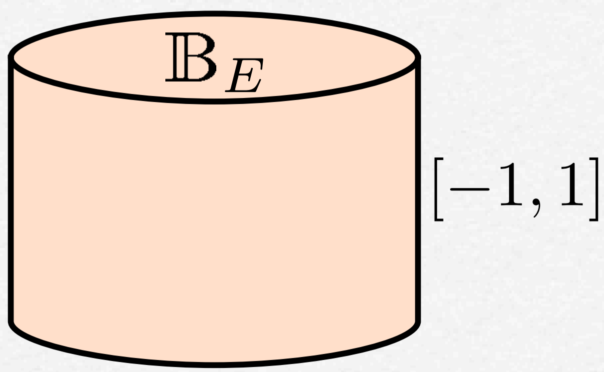

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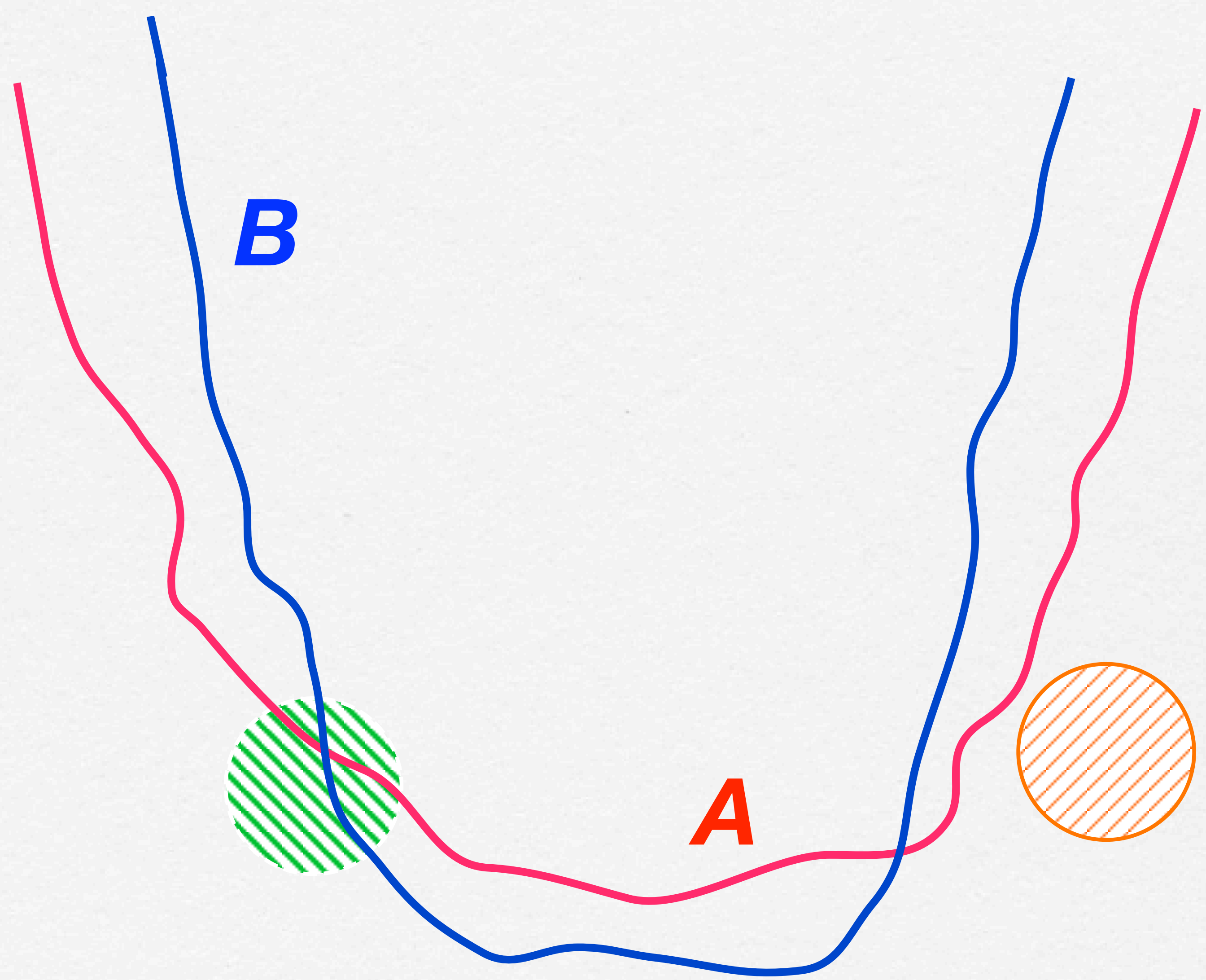
Hit-Open & Miss-Compact Sets



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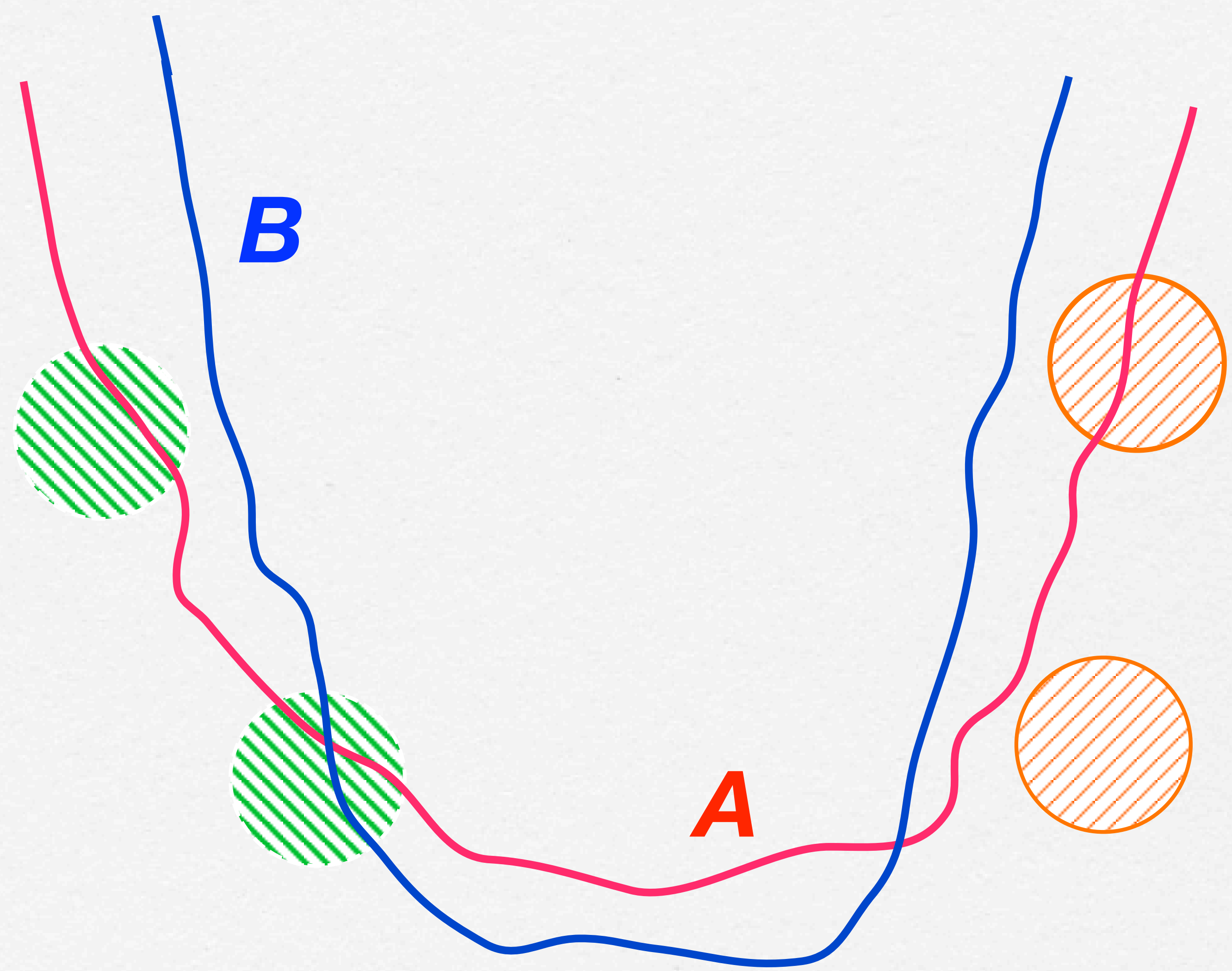


$\mathbb{R}^n$: Set-convergence ($\tau_{aw} = \tau_f$) topology

$\mathcal{F} =$ cl-sets($\mathbb{R}^n$), all closed subsets of $\mathbb{R}^n$

$\mathcal{F}^D =$ subsets $\mathbb{R}^n$ that *miss* $D = \{F \cap D = \emptyset\}$

$\mathcal{F}_D =$ subsets $\mathbb{R}^n$ that *hit* $D = \{F \cap D \neq \emptyset\}$

Hit-and-miss topology (= τ_f Fell topology)

subbase: $\{\mathcal{F}^K \mid K \text{ compact}\}$ & $\{\mathcal{F}_O \mid O \text{ open}\}$

$\mathbb{B}(x,\rho)$ closed ball, center x radius ρ, $\mathbb{B}^o(x,\rho)$ open

a subbase $\left\{\mathcal{F}^{\mathbb{B}(x,\rho)}, \mathcal{F}_{\mathbb{B}^o(x,\rho)} \;\middle|\; x \in \mathbb{Q}^d, \rho \in \mathbb{Q}_{++}\right\}$

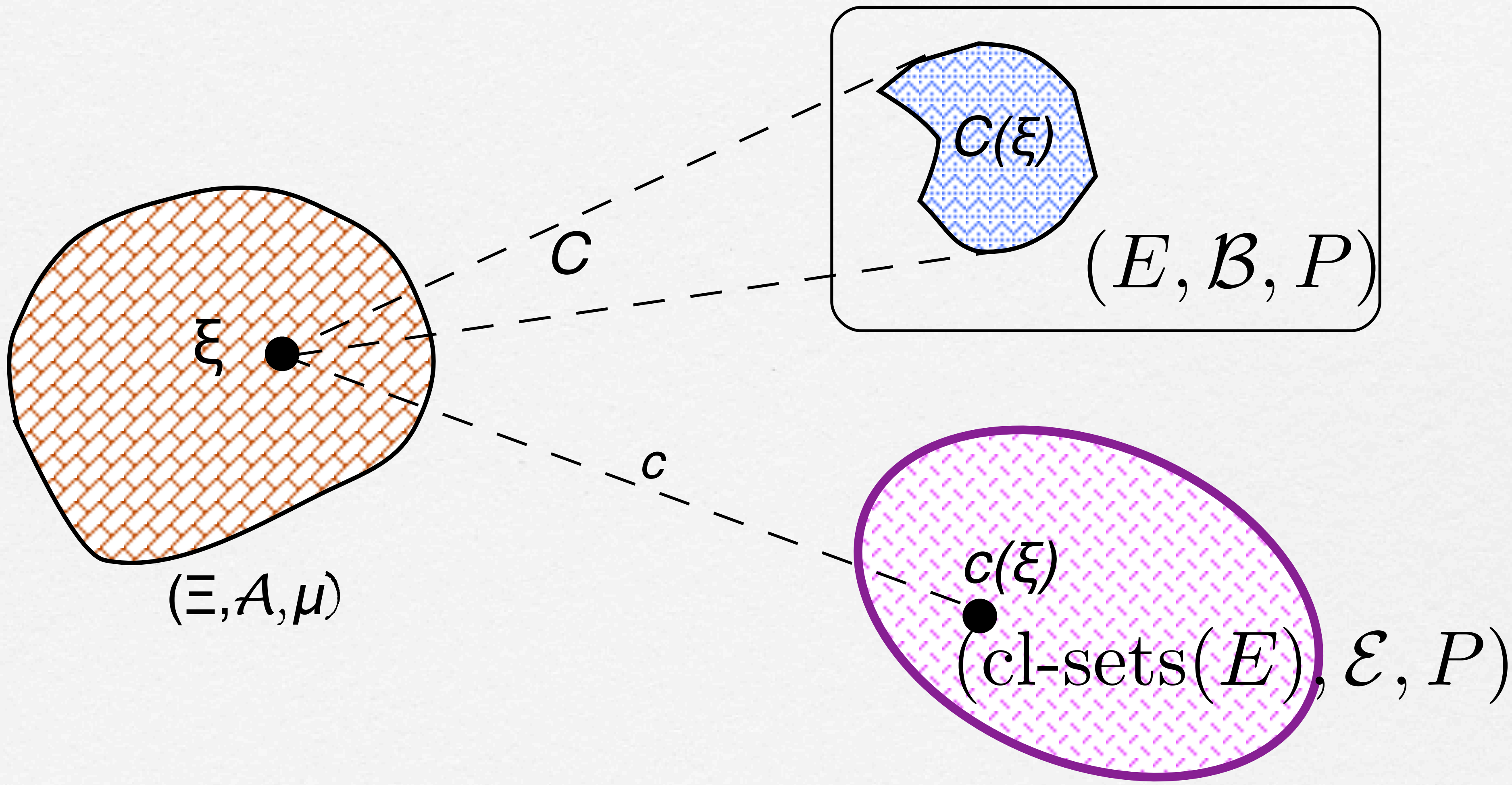
countable base: $\left\{\mathcal{F}^{\mathbb{B}(x^1,\rho_1)\cup\ldots\cup\mathbb{B}(x^r,\rho_r)} \cap \mathcal{F}_{\mathbb{B}^o(x^1,\rho_1)\cup\ldots\cup\mathbb{B}^o(x^s,\rho_s)}\right\}$

(cl-sets($\mathbb{R}^n$), τ_{aw}) Polish space (separable, complete metric)

Random Sets

Mattheron, Choquet
Salinetti-Wets, Castaing, Valadier, Hess, Stoyan, ...

Random sets



Random Closed Sets

$(\Xi, \mathcal{A}, P), \quad \Xi \subset \mathbb{R}^N$ & E Polish, for example $\mathbb{R}^n$

$C : \Xi \rightrightarrows E,\ C(\xi) \subset E$ closed set for all $\xi \in \Xi$

& $C^{-1}(O) = \left\{ \xi \,\middle|\, C(\xi) \cap O \neq \emptyset \right\} \in \mathcal{A},\ \forall O \subset E$, open

$\Rightarrow$ $\operatorname{dom} C = C^{-1}(E) \in \mathcal{A}$, measurability ~ hit open sets

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$c : \Xi \to$ cl-sets(E), $c(\xi) \sim C(\xi)$, $\mathcal{F}_O = \{F \subset E \text{ closed} \,|\, F \cap O \neq \emptyset\}$

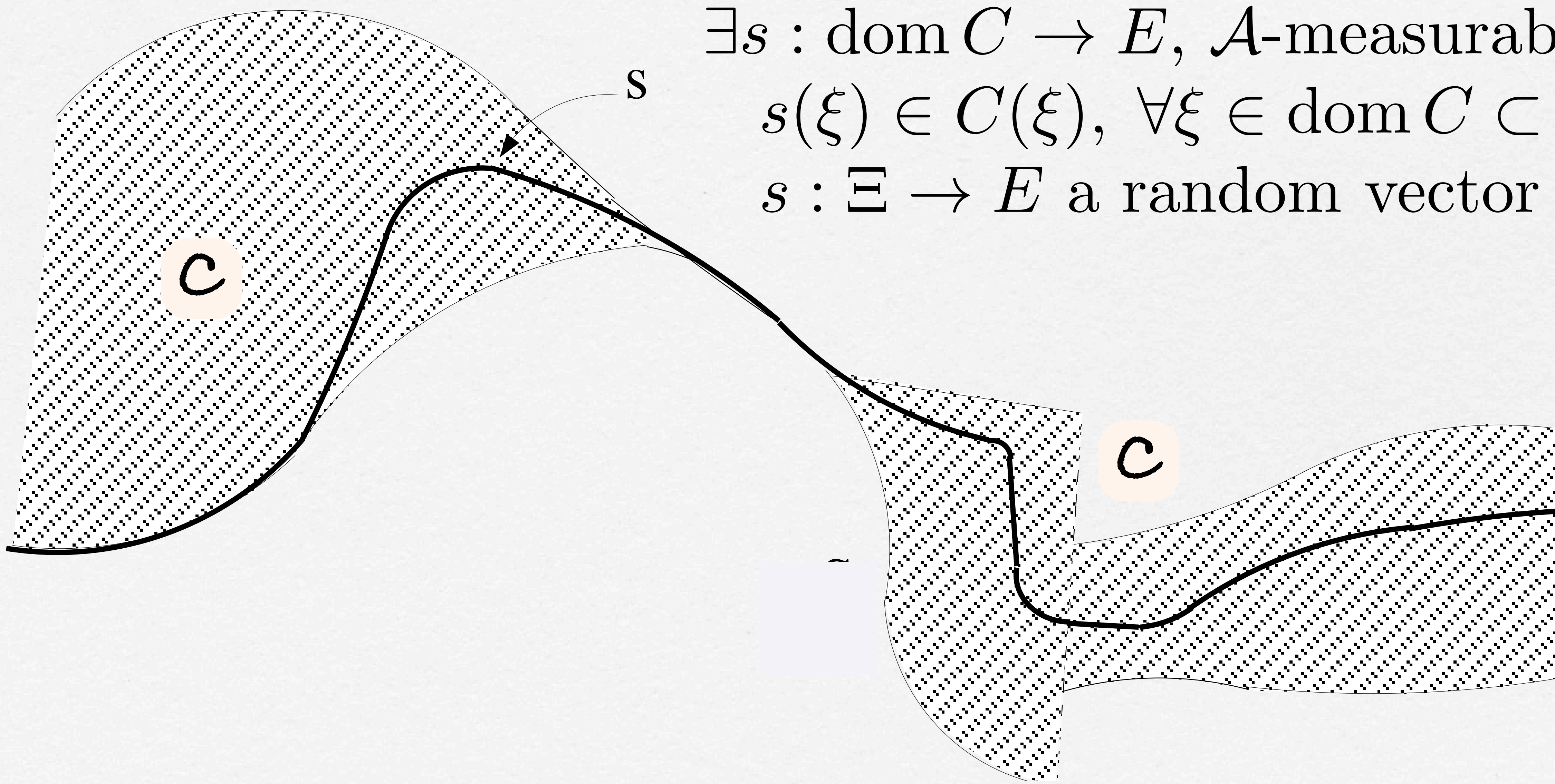
$(\text{sets}(E), \mathcal{E})$, $\mathcal{E}$ Effrös field = σ-$\{\mathcal{F}_O \in \text{sets}(\mathbb{R}^n), O \text{ open}\}$,

C measurable $\Leftrightarrow c$ measurable $[c^{-1}(\mathcal{F}_O) \in \mathcal{A}]$

$\mathcal{E} = \mathcal{B}$ Borel field when E Polish (complete separable metric space)

Measurable selection

- a random closed set C always admits a measurable selection!



$$\exists s : \operatorname{dom} C \to E,\ \mathcal{A}\text{-measurable},$$
$$s(\xi) \in C(\xi),\ \forall \xi \in \operatorname{dom} C \subset \Xi$$
$$s : \Xi \to E \text{ a random vector}$$

Castaing Representation

- C is a random closed set (& dom C measurable) ⇔ it admits a Castaing representation: ∃ a <u>countable</u> family

$$\left\{ s^\nu : \text{ dom } C \to E, \text{ meas.-selections} \right\}$$

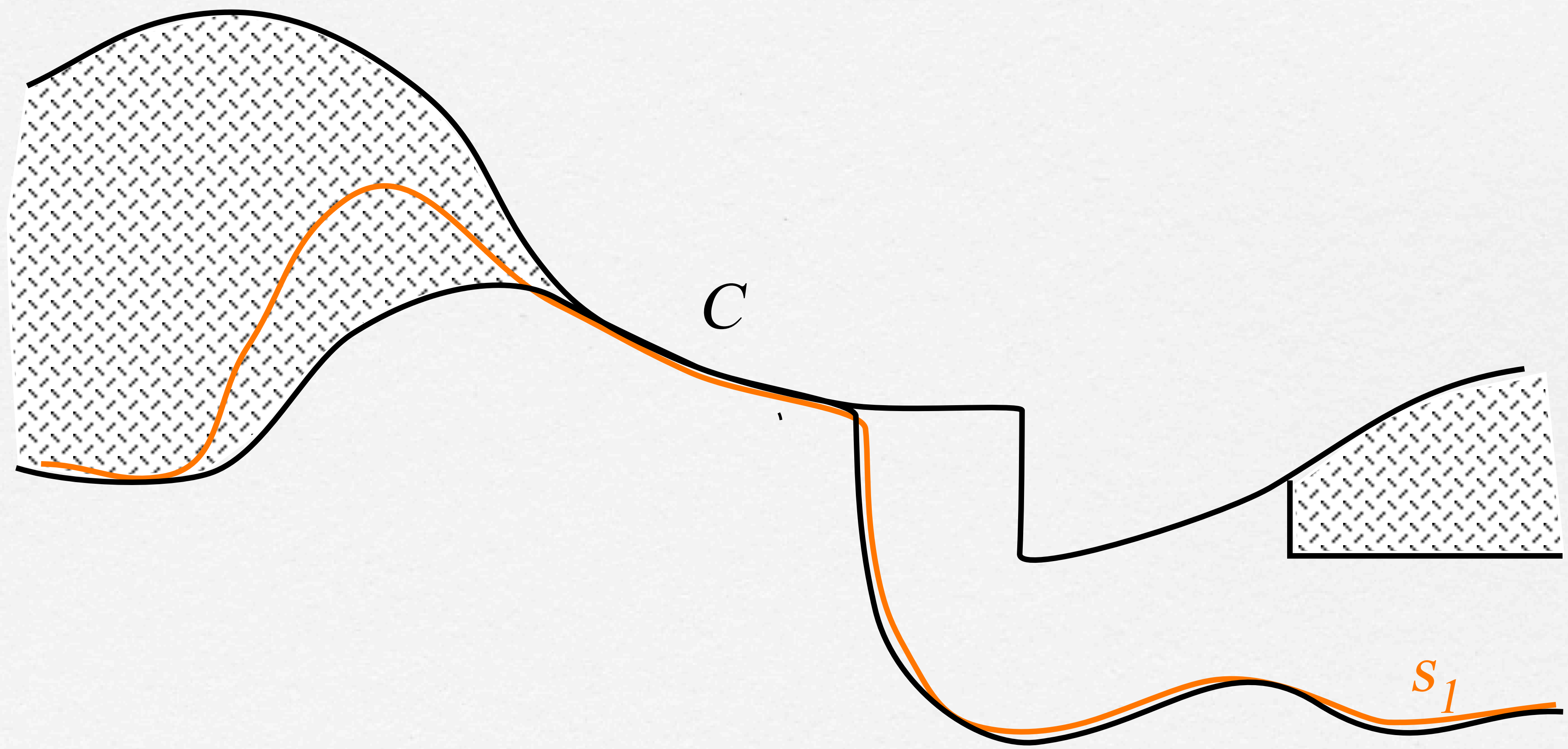
$$\text{cl} \bigcup_{\nu \in \mathbb{N}} s^\nu(\xi) = C(\xi), \forall \xi \in \text{ dom } C \subset \Xi$$

- Graph measurability

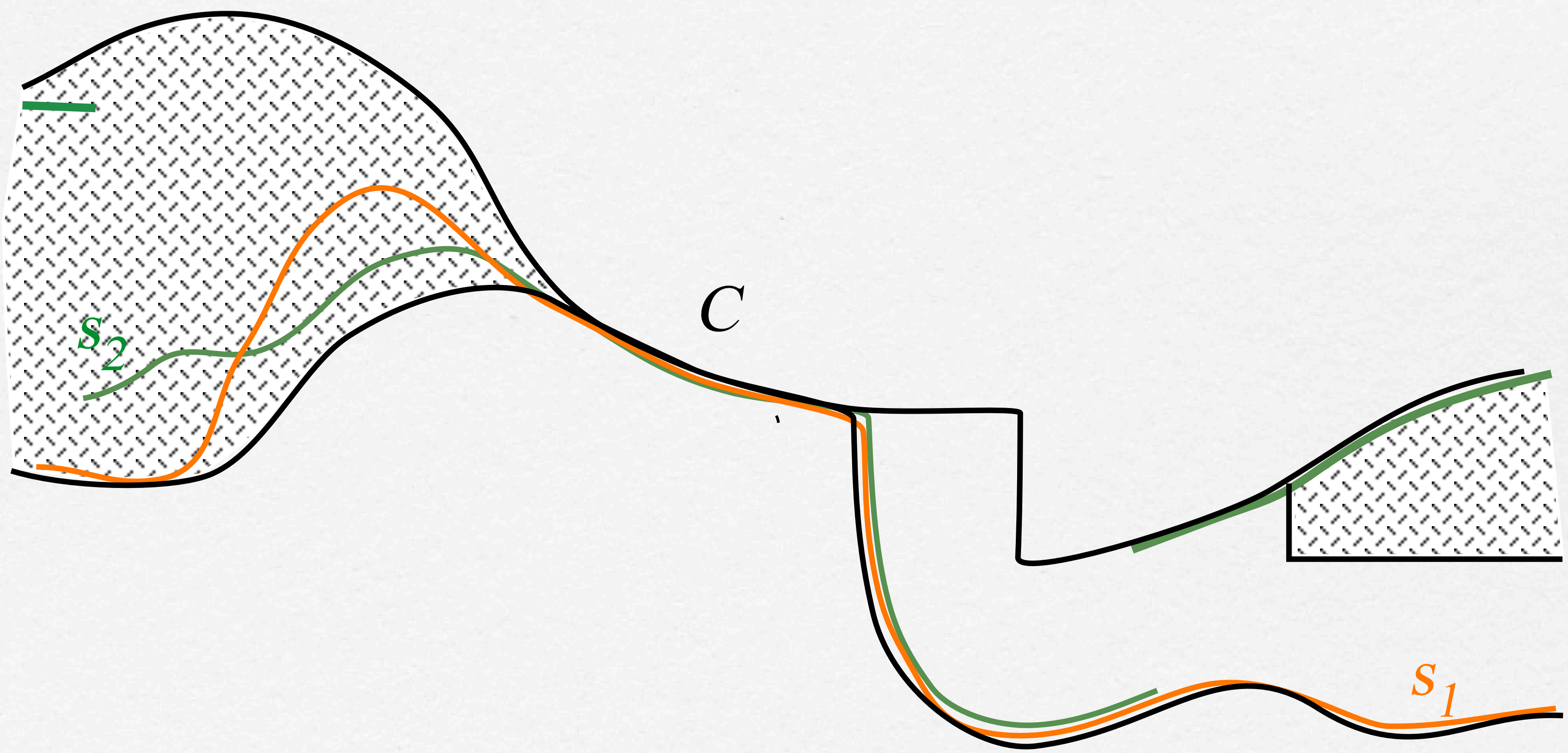
$(\Xi, \mathcal{A})$ P-complete for some P, (negligible sets are P-measurable)

C random set ⇔ gph C $\mathcal{A} \otimes \mathcal{B}_n$-measurable

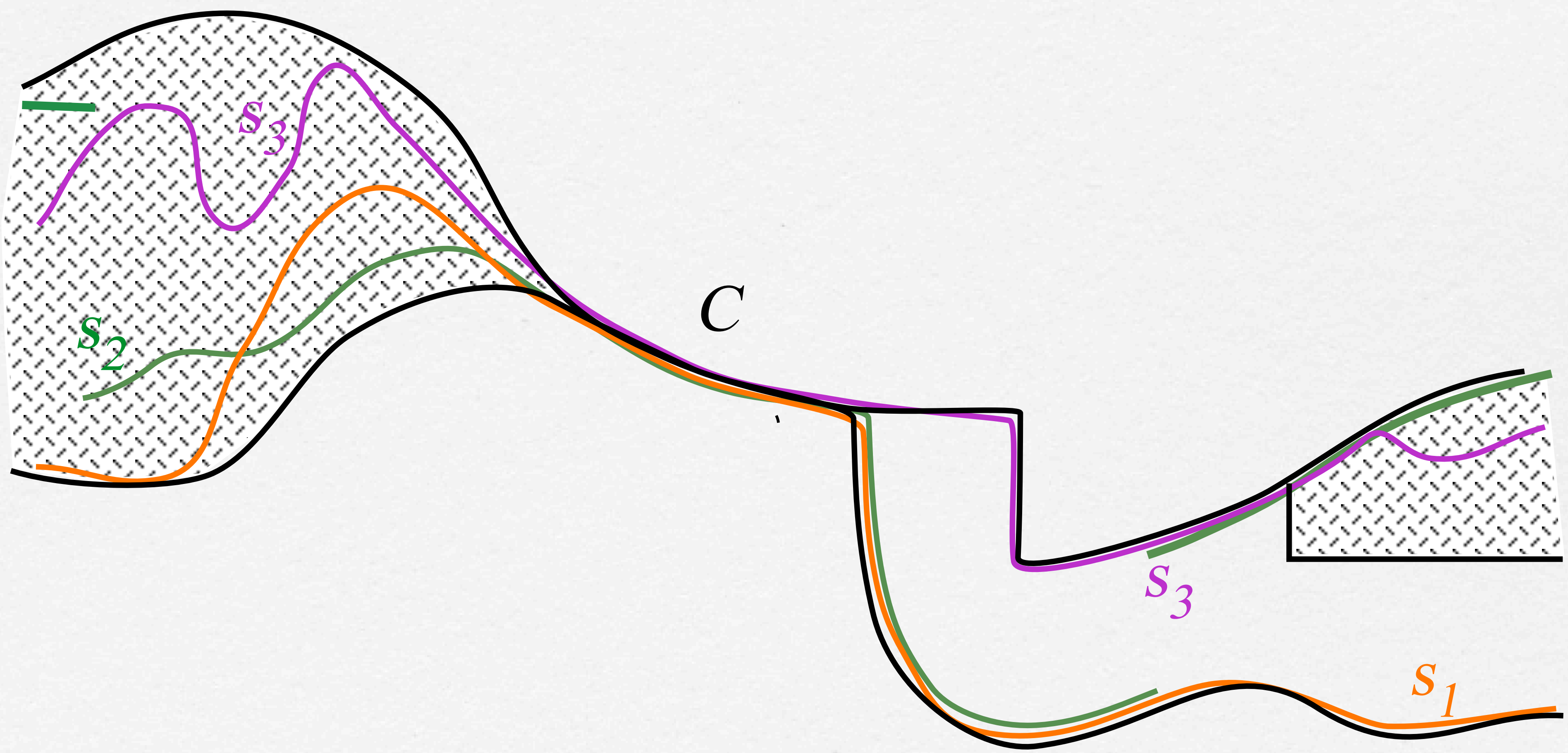
Castaing Representation



Castaing Representation



Castaing Representation



Random Elements: Convergence *(review)*

$$\boldsymbol{\xi} : (\Omega, \mathcal{F}, \mu) \to (\Xi, \mathcal{A}, P), \quad \boldsymbol{\xi}^\nu \stackrel{\star}{\to} \boldsymbol{\xi}$$

- a.s. (almost sure) convergence:

$$P\{\xi \,\big|\, \lim_\nu \boldsymbol{\xi}^\nu(\omega) = \xi \neq \boldsymbol{\xi}(\omega),\, \omega \in \Omega\} = 0$$

- convergence in probability:

$$P(|\boldsymbol{\xi}^\nu - \boldsymbol{\xi}| > \varepsilon) \to 0 \text{ for all } \varepsilon > 0$$

- convergence in distribution: $P^\nu \stackrel{\mathcal{D}}{\to} P$

a.s.-Convergence

* $\left\{C^\nu : \Xi \rightrightarrows \mathbb{R}^d, \nu \in \mathbb{N}\right\}$ random closed sets
* a.s. convergence: $dl\big(C^\nu(\xi), C(\xi)\big) \to 0$ for P-almost all $\xi \in \Xi$

  $C^\nu \to C$ a.s. $\Rightarrow C$ random closed set on $\Xi_0, \mu(\Xi_0) = 1$
* $C^\nu \to C$ P-a.s. and dom $C^\nu =$ dom C. Then,

  $\exists$ Castaing representations of $C^\nu \to$ a Castaing representation of C

  If $s : \Xi \to E$ is a measurable selection of C, then

  $\exists\, s^\nu : \Xi \to E$ selections of C^ν converging P-a.s. to s
* ('Egorov's Theorem': $C^\nu \to C$ μ-a.s. $\Leftrightarrow C^\nu \to C$ almost uniformly)

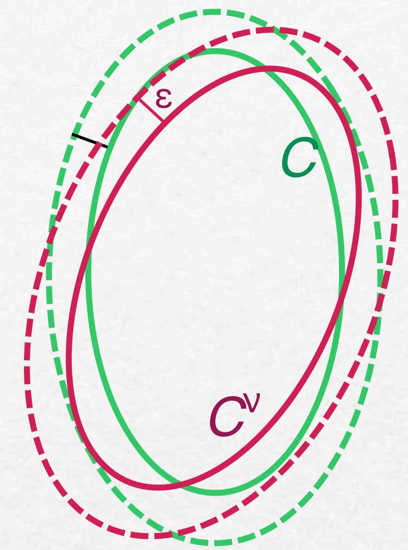
Convergence in probability

Let $\varepsilon^o C = \left\{ x \in \mathbb{R}^m \,\middle|\, d(x,C) < \varepsilon \right\}$, C^ν, C random sets

$\Delta_{\varepsilon,\nu} = \left(C^\nu \setminus \varepsilon^o C\right) \cup \left(C \setminus \varepsilon^o C^\nu\right)$

μ-a.s. convergence: $\mu\left\{ \xi \,\middle|\, C^\nu(\xi) \to C(\xi) \right\} = 1$

in probability: $P\left[\Delta^{-1}_{\varepsilon,\nu}(K) \right] \to 0, \forall \varepsilon > 0, K \in \mathcal{K} =$ cpct-sets



C^ν converges to C in probability

$\Leftrightarrow P(dl(C^\nu, C) > \varepsilon) \to 0$ for all $\varepsilon > 0$

$\Leftrightarrow$ every subsequence of $\{C^\nu\}_{\nu \in \mathbb{N}}$

contains a sub-subsequence converging μ-a.s to C

i.e., in probability $\Rightarrow$ in distribution $\left[\int h(\xi)\, dl(C^\nu(\xi), C(\xi))\, P(d\xi) \to 0 \right]$

Wednesday, May 16, 2012

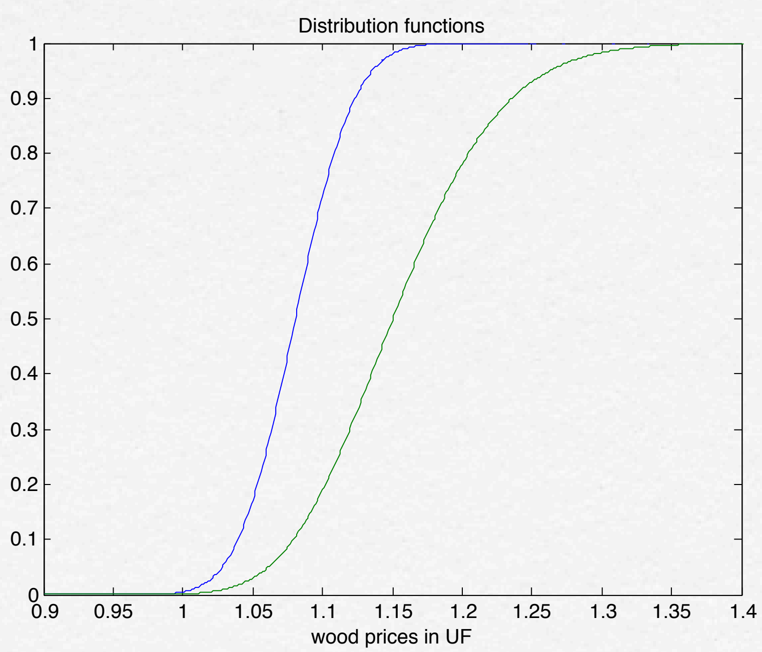
$P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

P^ν, P defined on $(\mathbb{R}, \mathcal{B})$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)\ \forall h$ continuous

$F^\nu(z) = P^\nu\big((-\infty, z)\big), \quad F(z) = P\big((-\infty, z)\big)$, cumulative distributions

$P^\nu \xrightarrow{\mathcal{D}} P \iff F^\nu \xrightarrow{p} F$ on cont $F = \{$ all continuity points of $F\}$



$\xrightarrow{h}$: hypo-convergence

$P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

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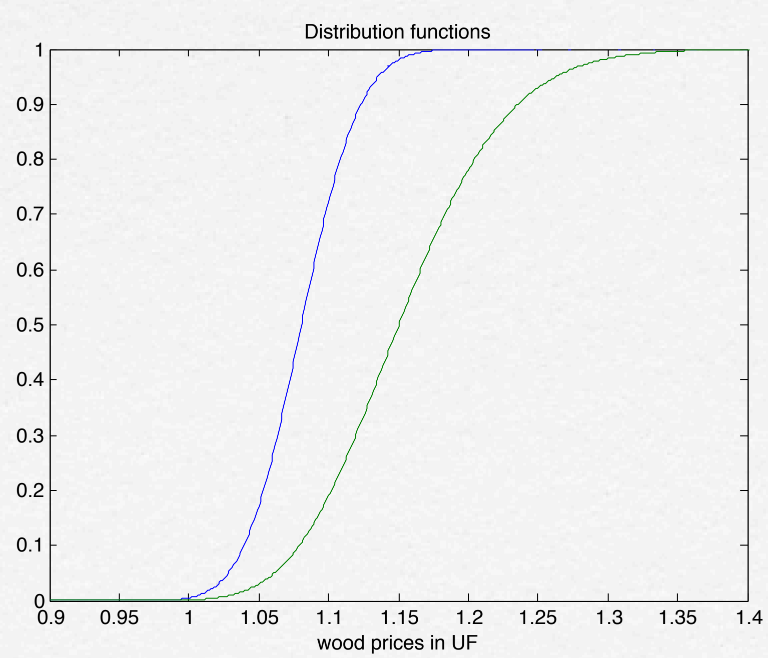


$$P^\nu \xrightarrow{\mathcal{D}} P \iff -F^\nu \xrightarrow{e} -F$$

$(F^\nu \xrightarrow{h} F,\ F$ usc $= -$lsc $)$

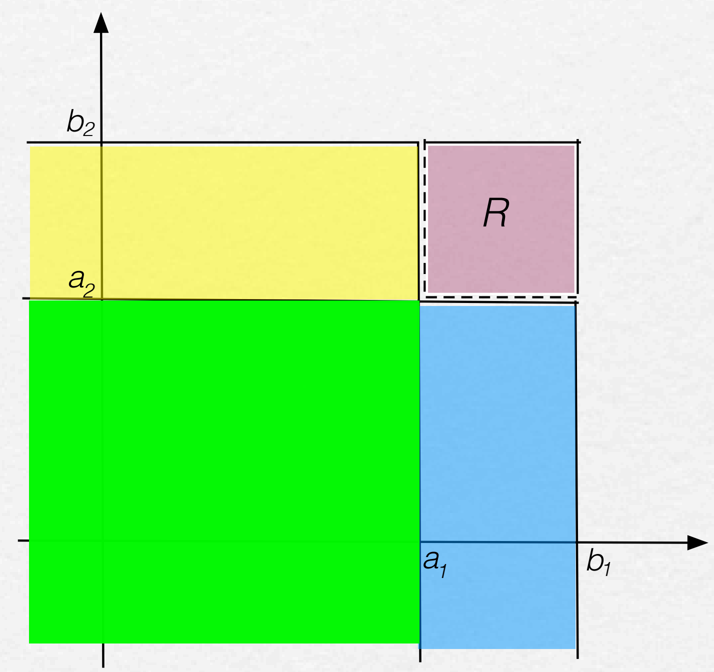
$\xrightarrow{h}$: hypo-convergence

$P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

P^ν, P defined on $(\mathbb{R}^n, \mathcal{B}_n)$ random vectors $\boldsymbol{\xi}^\nu, \boldsymbol{\xi}$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)\;\; \forall h$ continuous

$$F^\nu(z) = P^\nu(\xi_i \le z_i, i = 1, \dots, n), \quad F(z) = P(\xi_i \le z_i, i = 1, \dots, n)$$



1. $z \le \tilde{z} \implies F(z) \le F(\tilde{z})$ "increasing"
2. $\lim_{z \to \infty} F(z) = 1, \;\; \lim_{z_j \to -\infty} F(z) \to 0,$
3. F is usc (upper sc) $\limsup_{z' \to z} F(z') \le F(z),$
4. $R = (a_1, b_1] \times \cdots \times (a_n, b_n], \quad V = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ vertices of R
   $\forall R \subset \mathbb{R}^n, \; P(\boldsymbol{\xi} \in R) = \sum_{v \in V} \operatorname{sgn}(v) F(v), \quad \operatorname{sgn}(v \in V) = (-1)^{\#a \text{ in } v}$

$P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

P^ν, P defined on $(\mathbb{R}^n, \mathcal{B}_n)$ random vectors $\boldsymbol{\xi}^\nu, \boldsymbol{\xi}$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)$ $\forall h$ continuous

$F^\nu(z) = P^\nu(\xi_i \le z_i, i = 1, \dots, n), \quad F(z) = P(\xi_i \le z_i, i = 1, \dots, n)$

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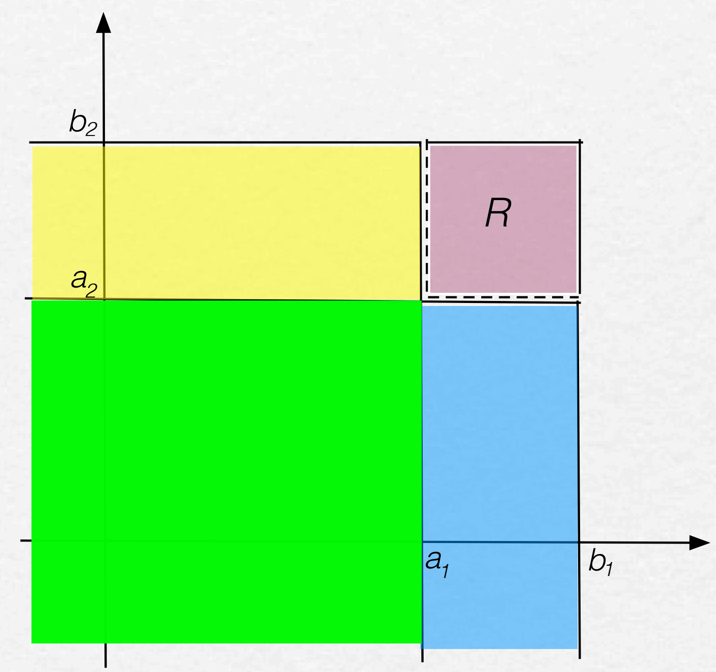

4. $R = (a_1, b_1] \times \cdots \times (a_n, b_n], \quad V = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ vertices of R
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$$P^\nu \xrightarrow{\mathcal{D}} P \iff -F^\nu \xrightarrow{e} -F$$

Distribution of a random set

Borel σ-field: $\mathcal{B} = \sigma\text{-}\{\mathcal{F}^K \mid K \text{ compact}\}$ or $\sigma\text{-}\{\mathcal{F}_O \mid O \text{ open}\}\ldots$

Distribution $(P, \mathcal{B})$ regular, $\mathcal{K}$ compact subsets E

determined by values on $\{\mathcal{F}^K \mid K \in \mathcal{K}\}$ or $\{\mathcal{F}_K \mid K \in \mathcal{K}\}$

Distribution function (Choquet capacity):

$T : \mathcal{K} \to [0,1]$, (2) $T(\varnothing) = 0$ and $\forall \{K^\nu, \nu \in \{0\} \cup \mathbb{N}\} \subset \mathcal{K}$: (1,3)

a) $T(K^\nu) \searrow T(K)$ when $K^\nu \searrow K$ (~ usc on $\mathbb{R}^n$)

b) $\{D_\nu : \mathcal{K} \to [0,1]\}_{\nu \in \mathbb{N}}$ where $D_0(K^0) = 1 - T(K^0)$

(4) $D_1(K^0; K^1) = D_0(K^0) - D_0(K^0 \cup K^1)$ and for $\nu = 2, \ldots$

$D_\nu(K^0; K^1, \ldots, K^\nu) = D_{\nu-1}(K^0; K^1, \ldots, K^{\nu-1}) - D_{\nu-1}(K^0 \cup K^\nu; K^1, \ldots, K^{\nu-1})$

(~ rectangle condition on $\mathbb{R}^n$)

Existence-Uniqueness T

P on $\mathcal{B}$ determines a unique distribution function T on $\mathcal{K}$

$$T(K) = P(\mathcal{F}_K)$$

$$D_\nu(K^0; K^1, \ldots, K^\nu) = P(\mathcal{F}^{K^0} \cap \mathcal{F}_{K^1} \cap \cdots \cap \mathcal{F}_{K^\nu})$$

T on $\mathcal{K}$ determines a unique probability measure P.

Proof. via Choquet Capacity Theorem ***(Matheron)***

(refined) via probabilistic arguments ***(Salinetti-Wets)***

$C : \Xi \rightrightarrows \mathbb{R}^d$ a random closed set

$(P, \mathcal{B})$ induced probability measure:

$$P(\mathcal{F}_G) = P\left[C^{-1}(G)\right] \quad \forall G \in \mathcal{B}, \quad T(K) = P\left[C^{-1}(K)\right] \quad \forall K \in \mathcal{K}$$

Convergence in Distribution

random sets C^ν converge in distribution to C when

induced P^ν narrow-converge to $P : P^\nu \to_n P$ $= P^\nu \overset{\mathcal{D}}{\to} P$

$\Leftrightarrow T^\nu \to_p T$ on $\mathcal{K}_{T\text{-cont}}$ (convergence of distribution functions)

$\mathcal{K}_{T\text{-cont}}$?

a) $\forall C^\nu, \nu \in N, \exists$ converging subsequence (pre-compact)

b) $K^\nu \nearrow K =$ cl $\bigcup_\nu K^\nu$ regularly if int $K \subset \bigcup_\nu K^\nu$

c) distribution (fcn) continuity: $\lim_\nu T(K^\nu) = T(\text{cl} \bigcup_\nu K^\nu)$

d) convergence $T^\nu \to_p T$ on C_T continuity set $\Rightarrow P^\nu \to_n P$

e) $P^\nu \to_n P \Leftrightarrow T^\nu \to_p T$ on $C_T^{ub} = C_T \cap \mathcal{K}^{ub}$

$\mathcal{K}^{ub} =$ finite union of rational ball, positive radius

f) $\varepsilon \mapsto T(K + \varepsilon\mathbb{B})$: countable number of discontinuities

a detour about rates

$T^\nu \to_p T$ on $C_T \Leftrightarrow P^\nu \to_n P$ (Polish space: E, d)

P^ν, P defined on $\mathcal{B}$

probability sc-measures on cl-sets(E): λ

(i) $\lambda \geq 0$, (ii) $\lambda \nearrow$ $\lambda(C^1) \leq \lambda(C^2)$ if $C^1 \subset C^2$

(iii) λ is τ_f-usc on cl-sets(E), (iv) $\lambda(\emptyset) = 0, \lambda(E) = 1$

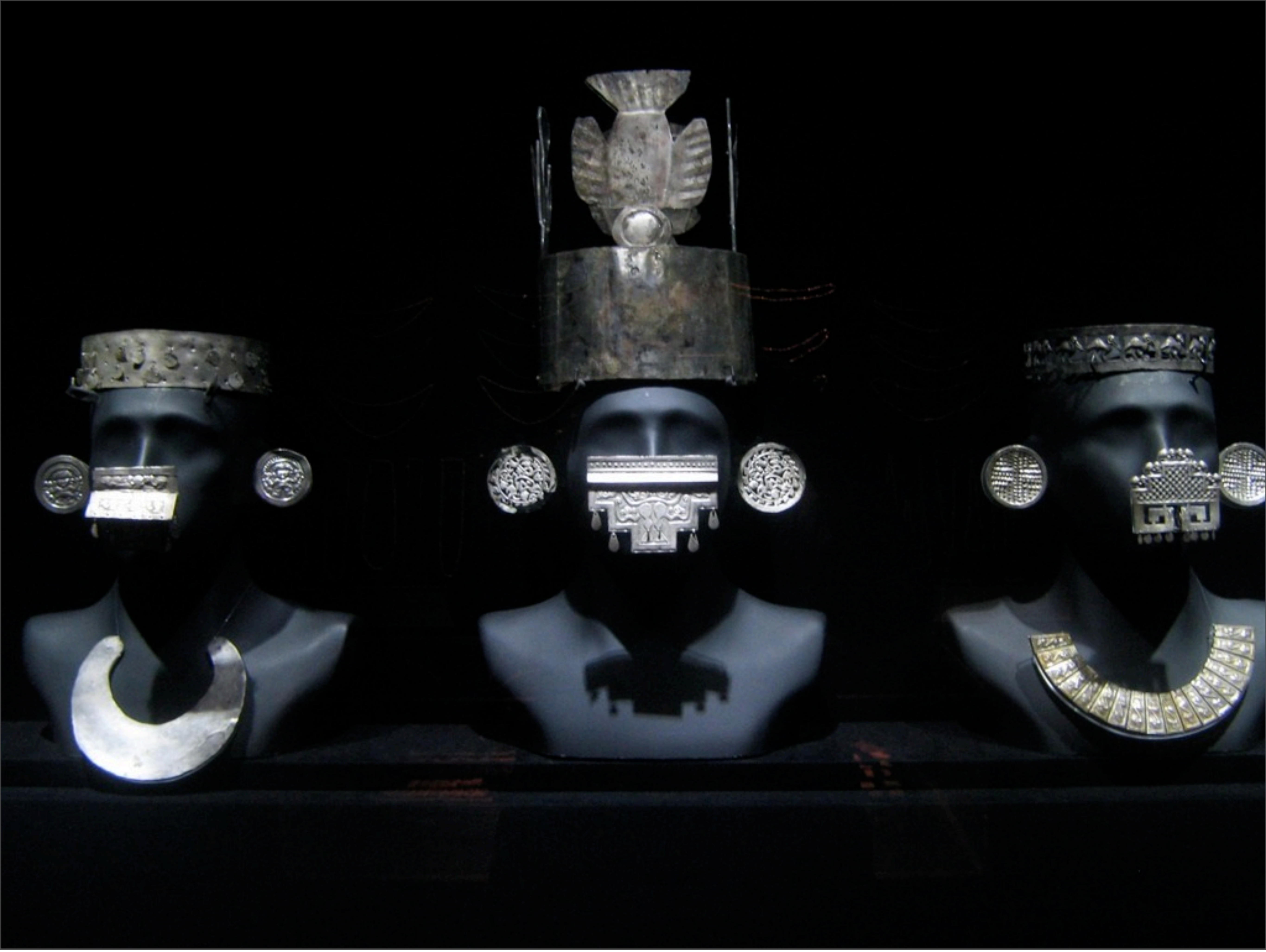
(v) λ modular: $\lambda(C^1) + \lambda(C^2) = \lambda(C^1 \cup C^2) + \lambda(C^1 \cap C^2)$

P and $\lambda = P_{\text{cl-sets}}$ define each other uniquely (E complete $\Rightarrow$ tight)

$\{P^\nu, \nu \in \mathbb{N}\}$ tight: $P^\nu \to_n P \Leftrightarrow \lambda^\nu \to_h \lambda$ ($\sim -\lambda^\nu \to_e -\lambda$) on cl-sets($E$)

tightness ~ equi-usc of $\{\lambda^\nu\}_{\nu \in \mathbb{N}}$ at $\emptyset$

rates: $dl(\lambda^\nu, \lambda) \to 0$ (for $\mathbb{R}$-valued r.v., "~" Skorohod distance)



Random Sets Convergence & Expectation

Artstein-Vitale-Hart-Wets,
Cressis, Hiai, Weyl, ...

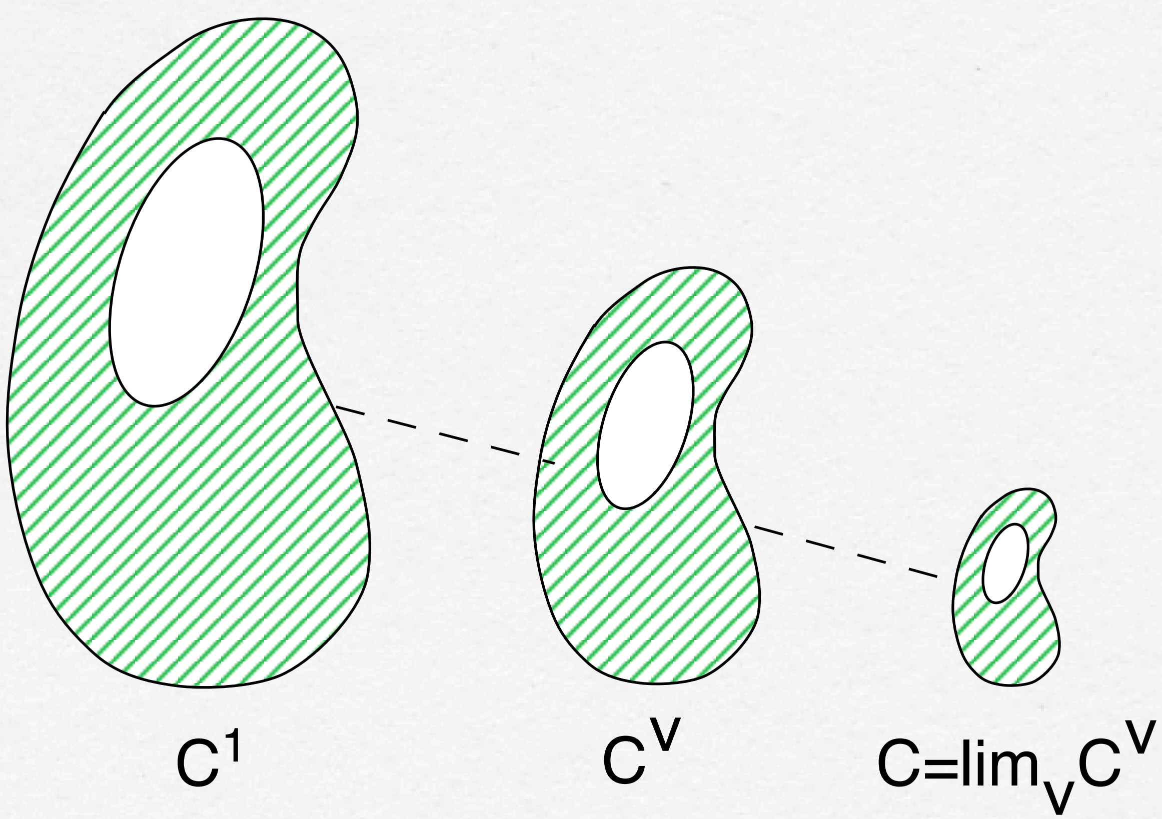
Outer/Inner Limits

outer limit: $\mathrm{Lo}_\nu C^\nu = \left\{ x \in \text{ cluster-points}\{x^\nu\},\, x^\nu \in C^\nu \right\} = \mathrm{Ls}_\nu C^\nu$

inner limit: $\mathrm{Li}_\nu C^\nu = \left\{ x = \lim_\nu x^\nu, x^\nu \in C^\nu \subset \mathbb{R}^n \right\} \subset \mathrm{Lo}_\nu C^\nu$

limit: $C^\nu \to C$ if $C = \mathrm{Li}_\nu C^\nu = \mathrm{Lo}_\nu C^\nu$ (Painlevé - Kuratowski)

All limit sets are closed



Outer/Inner Limits

outer limit: $\mathrm{Lo}_\nu C^\nu = \{x \in \text{ cluster-points}\{x^\nu\}, x^\nu \in C^\nu\} = \mathrm{Ls}_\nu C^\nu$

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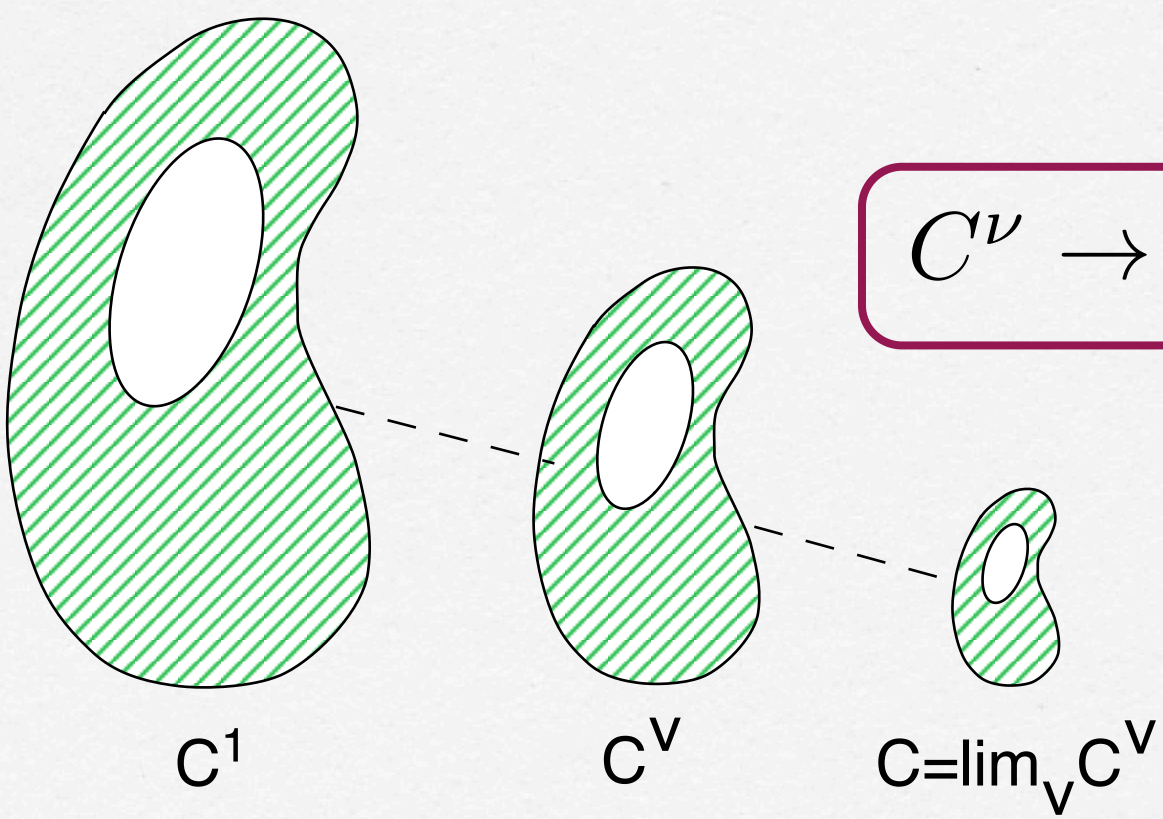

$$C^\nu \to C \iff d\!\!l(C^\nu, C) \to 0$$

Characterizing a.s. convergence

$\Big\{C; C^\nu : \Xi \rightrightarrows \mathbb{R}^n,\ \nu \in \mathbb{N}\Big\}$ random closed sets. Then,

1. $C^\nu \to C$ *a.s.*, $d\!\!l(C^\nu, C) \to 0$ *a.s.*, $\mathrm{Lo}_\nu(C^\nu) \subset C \subset \mathrm{Li}_\nu(C^\nu)$ *a.s.*,

2. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$, $d(x, C^\nu(\xi)) \to d(x, C(\xi))$,

3. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$,

$$\lim_{\rho \nearrow \infty} \mathrm{Lo}_\nu \left(C^\nu(\xi) \cap \mathbb{B}(x, \rho)\right) \subset C(\xi) \subset \lim_{\rho \nearrow \infty} \mathrm{Li}_\nu \left(C^\nu(\xi) \cap \mathbb{B}(x, \rho)\right).$$

"Proof 1. ⇔ 2."

$$C^\nu \to C \iff \forall x \in \mathbb{R}^n,\, d(x, C^\nu) \to d(x, C) \text{ provided } E = \mathbb{R}^n.$$

$C^\nu \to C$ if and only if the hit-miss criterion is satisfied

C hits $\mathbb{B}^o(x,\rho)$ then C^ν hits $\mathbb{B}^o(x,\rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \subset \mathrm{Li}_\nu C^\nu \iff d(x,C) \geq \limsup_\nu d(x,C^\nu), \forall x$

C misses $\mathbb{B}(x,\rho)$ then C^ν misses $\mathbb{B}(x,\rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \supset \mathrm{Lo}_\nu C^\nu \iff d(x,C) \geq \liminf_\nu d(x,C^\nu), \forall x$

Building Castaing representations

$C : \Xi \rightrightarrows \mathbb{R}^n$, a random closed set. Let

$$A = \left\{ a_k = (a_k^1, \ldots, a_k^n, a_k^{n+1}) \,\middle|\, a_k^i \in \mathbb{Q}^n \text{ \& aff. independent } \right\}$$

for $\emptyset \neq D = D^0$ closed, define $\operatorname{prj}_D a_k = \operatorname{prj}_{D^n} a_k^{n+1}$
where $D^l = \operatorname{prj}_{D^{l-1}} a_k^l$ for $l = 1, \ldots, n$

$\operatorname{prj}_D a_k$ is a singleton: intersection of n+1 "aff. independent" spheres.

Moreover, $\left\{ \operatorname{prj}_D a_k, a_k \in A \right\}$ also dense in D

$s_k : \Xi \to \mathbb{R}^n$ with $s_k(\xi) = \operatorname{prj}_{C(\xi)} a_k$ is a measurable selection of C

$\square$ When D is a random closed set, so is $\xi \mapsto \operatorname{prj}_{D(\xi)} a$, $a \in \mathbb{R}^n$
repeat the argument $n + 1$ times to obtain s_k measurable. $\square$

Converging Castaing representations

$C^\nu : \Xi \rightrightarrows \mathbb{R}^n$ random closed sets converging P-$a.s.$ to C, $\operatorname{dom} C^\nu = \operatorname{dom} C$.
Then, $\exists \{ s_k^\nu, k \in \mathbb{N} \}$ Castaing representations of C^ν converging for each k
to a Castaing representation $\{ s_k, k \in \mathbb{N} \}$ of C.

$\square$ All Castaing representations are built via our earlier "projections".
Then, $\forall \xi \in \Xi_1, s_k^\nu(\xi) \to s_k(\xi)$, $P(\Xi_1) = 1$ the set of $a.s.$-convergence.
Since, P-$a.s.$ convergence of $C^\nu \to C \Longrightarrow$ (rely on 2. earlier)

$$d(a_k^1, s_k^\nu(\xi)) = d(a_k^1, C^\nu(\xi)) \to d(a_k^1, C(\xi)) = d(a_k^1, s_k(\xi)), \forall \xi \in \Xi_1. \quad \square$$

(a) Convergence of Castaing representations $\not\Rightarrow$ convergence of random sets!
(b) v meas-selection of $C \Rightarrow \exists v^\nu$ meas-selection of C^ν converging $a.s.$ to v.

“Simple” random sets

$C : \Xi \rightrightarrows \mathbb{R}^n$ is a *simple* random set if $\operatorname{rge} C$ is finite.
C is a closed random set $\iff$ $C = P\text{-}a.s.$ limit of simple random sets.

$\square$ $\Leftarrow$: the limit of a sequence of random sets is a random set
$\Rightarrow$: let $C^\nu = C \cap \nu\mathbb{B}$, unif. bounded closed random set, $C = \operatorname{Lm}_\nu C^\nu$
build (via "prj") Castaing representations $\{r_k^\nu\}_{k\in\mathbb{N}}$ of the C^ν
let $\{s_k^\nu\}_{k\in\mathbb{N}'} = \bigcup_{\upsilon\leq\nu} \{r_k^\upsilon\}_{\upsilon\in\mathbb{N}}$, also Castaing for C^ν
$D_k^\nu = \bigcup_{j\leq k} s_j^\nu$ $d\!l$-converge uniformly to C^ν as $k \to \infty$
since each $s_k^\nu = \lim_{l\to\infty} s_{kl}^\nu$ uniformly, s_{kl}^ν simple random variables
$\Delta_{kl}^\nu = \bigcup_{j\leq k} s_{jl}^\nu$ is a simple random set, $C(\xi) = \operatorname{Lm}_\nu \operatorname{Lm}_k \operatorname{Lm}_l \Delta_{kl}^\nu(\xi)$

$\Delta_{kl}^\nu \xrightarrow{u} D_k^\nu \xrightarrow{u} C^\nu$ allows diagonalization to find $\Delta_{k^\nu l^\nu}^\nu \to C$. $\square$

Sierpiński-Lyapunov Theorems

$(\Xi, \mathcal{A})$ a measure space

Sierpiński (1922). Suppose P is an atomless probability measure.
Given $A_0, A_1 \in \mathcal{A}$ with $0 \leq P(A_0) \leq P(A_1) \leq 1$, then
$\forall \lambda \in [0,1]$, $\exists\ A_\lambda \in \mathcal{A}$ such that $P(A_\lambda) = (1-\lambda)P(A_0) + \lambda P(A_1)$.
In particular, it implies $\forall\, \lambda \in [0,1]$, $\exists\ A \in \mathcal{A}$ such that $P(A) = \lambda$;
choose $A_0 = \emptyset$ and $A_1 = \Xi$.

Lyapunov (1940) $\mu : \mathcal{A} \to \mathbb{R}^n$ atomless, σ-additive measure.
For $A \in \mathcal{A}$, define $\operatorname{rge}\mu(A) = \left\{\mu(B) \,\middle|\, B \subset A \cap \mathcal{A}\right\}$. Then,
$\operatorname{rge}\mu(\Xi) \subset \mathbb{R}^n$ is convex and if μ is also bounded, it's compact.

Expectation: simple random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a simple random set, i.e., $\operatorname{rge} C = \left\{ z^k \in \mathbb{R}^n \,\middle|\, k \in K, |K| \text{ finite } \right\}$
Given $\bar{r}, \bar{s} \in EC = \mathbb{E}\{C(\boldsymbol{\xi})\} \implies$

$\exists$ simple selections $r, s : \Xi \to \mathbb{R}^n$ with $\mathbb{E}\{r(\boldsymbol{\xi})\} = \bar{r}, \mathbb{E}\{s(\boldsymbol{\xi})\} = \bar{s}$.

Let $\lambda \in [0, 1]$. Define $v : \Xi \to \mathbb{R}^n$ as follows:

1. partition Ξ into subsets $A_=$ and $\mathcal{A}_{\neq}$
2. $A_= = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = s(\xi) \right\} \in \mathcal{A}$
3. $A = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = z_k, s(\xi) = z_l, k \neq l \right\} \in \mathcal{A}_{\neq}$, a finite collection
4. split each $A \in \mathcal{A}_{\neq}$, $P(A_r) = \lambda P(A)$ & $A_s = A \setminus A_r$ (Sierpiński)

$$\text{set } v(\xi) = \begin{cases} r(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_r \;\cup\; A_= \\ s(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_s \end{cases}$$

then $\bar{v} = \mathbb{E}\{v(\xi)\} = \lambda \bar{r} + (1 - \lambda)\bar{s} \implies EC$ convex.
Clearly EC is bounded and it's easy to show it's also closed $\implies$ compact.

Expectation of random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a closed random set

$\Longrightarrow C = P\text{-}a.s.$ limit of simple random sets,

say $C^\nu \underset{a.s.}{\to} C$ with $C^\nu \nearrow$ w.l.o.g

$EC^\nu = \mathbb{E}\{C^\nu(\boldsymbol{\xi})\} \nearrow$ are convex, compact $\Rightarrow$

$EC = \mathbb{E}\{C(\boldsymbol{\xi})\} = \bigcup_\nu EC^\nu$

$\Longrightarrow EC$ convex

$\Longrightarrow EC$ closed if C is integrably bounded

$\Longrightarrow$ compact if $\operatorname{rge} C$ is bounded