Random Mappings

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Lecture #3

$$G : E \to \mathbb{R}^d, \quad G^{-1}(0) \text{ soln's of } G(x) = 0, \text{ approximations?}$$

$$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0 \quad \text{"approximated" by } G^\nu(x) = 0$$
$$\xi^1, \dots, \xi^\nu \text{ sample}, \quad G^\nu(x) = \tfrac{1}{\nu} \textstyle\sum_{l=1}^\nu G(\xi^l, x)$$

$$G : \Xi \times D \rightrightarrows E, \text{ set-valued } G(\xi, x) \subset E, \text{ inclusion } \mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$$
$$\xi^1, \dots, \xi^\nu \text{ sample, approximation } \tfrac{1}{\nu} \textstyle\sum_{l=1}^\nu G(\xi^l, x) \ni 0$$

An appendix: more about solution bounds

$$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \textstyle\int_\Xi f(\xi, x)\, P(d\xi)$$
$$\xi^1, \dots \xi^\nu \text{ sample } P^\nu \text{ (random) empirical measure}$$
$$\text{approx.: } \min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \tfrac{1}{\nu} \textstyle\sum_{l=1}^\nu f(\xi^l, x),\ x \in C$$

ε-Solutions Estimates

$$f, g : E \to \overline{\mathbb{R}} \text{ lsc, convex \& } \operatorname{argmin} f \cap \bar\rho\mathbb{B} \neq \emptyset \neq \operatorname{argmin} g \cap \bar\rho\mathbb{B}$$
$$\min f \geq -\bar\rho, \quad \min g \geq -\bar\rho$$

with $\rho > \bar\rho$, $\varepsilon > 0$, $\quad \bar\eta = \hat{d\!l}_\rho(f, g)$:

$$\begin{aligned} \hat{d\!l}_\rho\big(\varepsilon\text{- argmin } f,\ \varepsilon\text{- argmin } g\big) &\leq \bar\eta\Big(1 + \frac{2\rho}{\bar\eta + \varepsilon/2}\Big) \\ &\leq \big(1 + 4\rho\varepsilon^{-1}\big)\hat{d\!l}_\rho(f, g) \end{aligned}$$

Epi-distance alternative

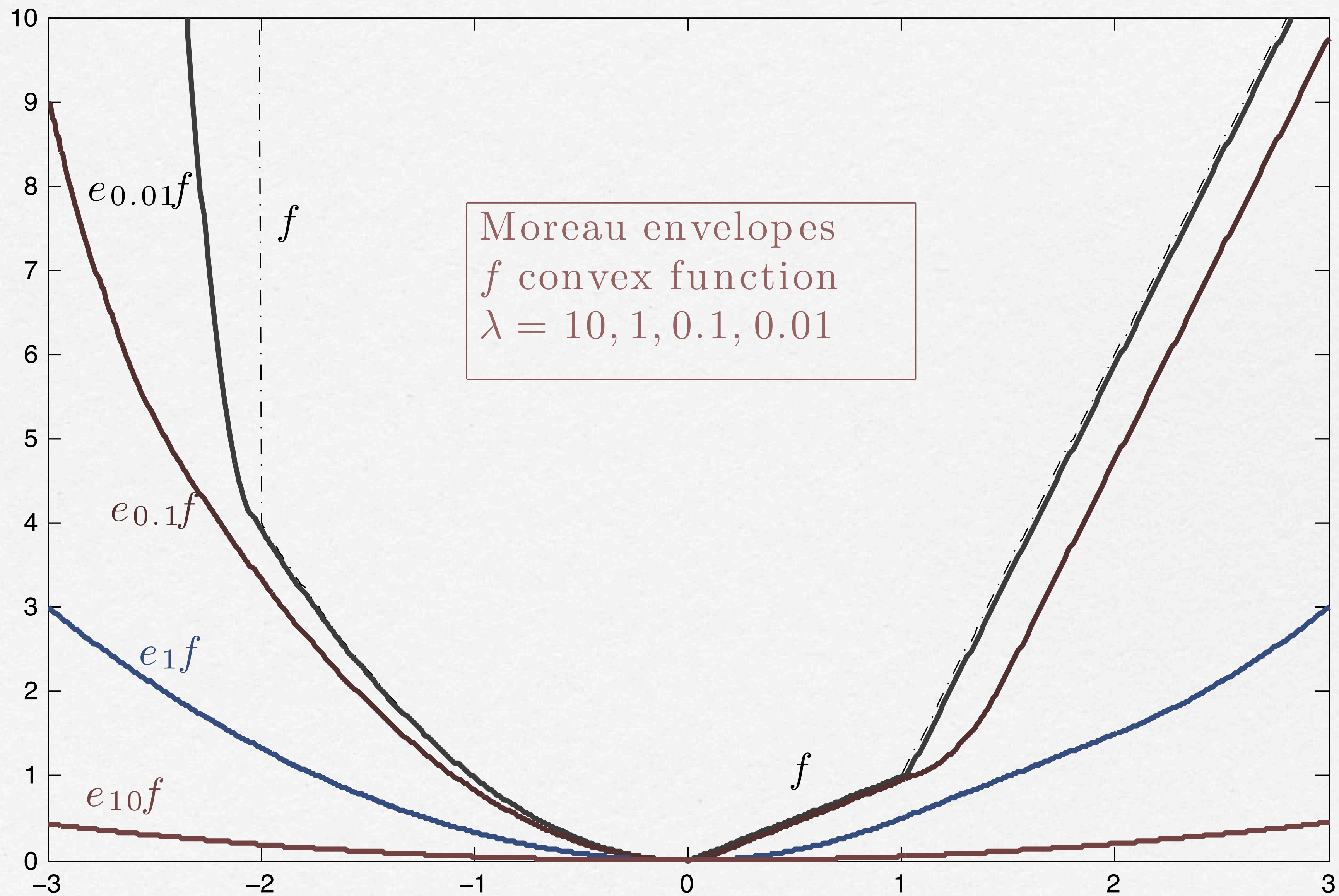
$$\check{d\!l}_{\lambda,\rho}(f,g)$$

same topology: τ_{aw}

Moreau envelopes

epi-sums ~ sum of epigraphs

$$(f \# g)(x) = \inf_u \{f(u) + g(u-x)\}, \quad e_\lambda f(x) \text{ with } g = \frac{1}{2\lambda} |\cdot|^2$$



Alternative epi-distance

$$\check{d\!l}_{\lambda,\rho}(f,g) = \sup\left\{|f_\lambda(x) - g_\lambda(x)| \,\middle|\, x \in \rho\mathbb{B}\right\}$$

$$f, g \quad \text{majorizing} \quad -\alpha_1|\cdot|^p - \alpha_0$$

1. $\forall\, \lambda \geq 0,\ \check{d\!l}_{\lambda,\rho}(f,g) \leq \beta(\lambda,\rho)\hat{d\!l}_{\gamma(\lambda,\rho)}(f,g)$

2. $\hat{d\!l}_{\rho}(f_\lambda, g_\lambda) \leq \check{d\!l}_{\lambda,\rho}(f,g)$
   $\hat{d\!l}_{\rho}(f,g) \leq \check{d\!l}_{\lambda,9\rho}(f,g) + \kappa(\lambda,\alpha_1,\alpha_0,p)$

“Quantitative” LLN-a.s.

E separable Banach space, f random lsc function, $\{\boldsymbol{\xi}, \boldsymbol{\xi}^\nu\}_{\nu \in \mathbb{N}}$ iid

1. $\{f(\xi, \cdot\,), \xi \in \Xi\}$ separable subspace (lsc-fcns$(E), \tau_{aw}$)
2. P-$a.s.$, $\forall\, \theta > 0, \rho \geq 0, \nu$:

   $\check{d\!l}_{\theta,\lambda}\left(\frac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,), \frac{1}{\nu}\sum_{l=1}^{\nu} f_\lambda(\boldsymbol{\xi}^l, \cdot\,)\right) \leq \varepsilon_{\theta,\rho}(\lambda)$

   with $\varepsilon_{\theta,\rho}(\lambda) \to 0$ as $\lambda \searrow 0$
3. $\forall\, \theta > 0, \rho \geq 0, \check{d\!l}_{\theta,\rho}\big(Ef_\lambda, Ef\big) \searrow 0$ as $\lambda \searrow 0$.

Then, $d\!l(E^\nu f, Ef) \to 0 \quad P^\infty$-$a.s.$

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convex

$x\mapsto f(\xi,x)$ convex $\Longrightarrow$ conditions 2 & 3.

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$\hat{d\!l}_\rho\big(\varepsilon\text{- argmin}\, E^\nu f,\, \varepsilon\text{- argmin}\, Ef\big) \leq \big(1 + 4\rho\varepsilon^{-1}\big)\hat{d\!l}_\rho(E^\nu f, Ef)$

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E reflexive, $E^\nu f \xrightarrow[s,w]{} Ef \implies d\!l(E^\nu f, Ef) \to 0\ a.s.$

Approximating Mappings

Why?

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) \ni 0$, approximations?

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$$\xi^1, \dots \xi^\nu \ \text{ sample } P^\nu \text{ (random) empirical measure}$$
$$\text{approx.: } \min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \tfrac{1}{\nu} \textstyle\sum_{l=1}^{\nu} f(\xi^l, x), \ x \in C$$

Examples:

$\min f = f_0 + \iota_C$, optimality: "$0 \in \partial f(\bar{x}) = S(x)$" $\sim 0 = \nabla f(\bar{x})$)

generally, $\partial(f+g) \neq \partial f + \partial g$

$\mathbb{C}.\mathbb{Q}.$ (Constraint Qualification): $-N_C(\bar{x}) \cap \partial^\infty f_0(\bar{x}) = \{0\}$

$v \in \partial^\infty f_0(\bar{x}) =$ horizon subgradient if

$\exists\, x^\nu \to \bar{x}$ with $f(x^\nu) \to f(\bar{x}), v^\nu \in \hat{\partial} f(x^\nu), \lambda_\nu \searrow 0$ & $\lambda_\nu v^\nu \to v$

with $\mathbb{C}.\mathbb{Q}.$ $\bar{x}$ locally optimal $\Rightarrow \partial f_0(\bar{x}) + N_C(\bar{x}) = S(\bar{x}) \ni 0$

f convex ($\Rightarrow$ regular), $\partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0$

$\Rightarrow$ globally optimal (without $\mathbb{C}.\mathbb{Q}$)

When f_0, C are convex: $-\partial f_0(\bar{x}) \in N_C(\bar{x})$,

a functional variational inequality

“Variational” Approximations

(E, d) Polish, in paricular $E = \mathbb{R}^n$

$(\text{cl-sets}(E), d\!\!l)$ complete metric space; Polish if $E = \mathbb{R}^n$

$d\!\!l(C^\nu, C) \to 0 \iff C^\nu \to C$

<u>osc-mappings</u> = closed graph

$(\text{osc-maps}(S), d\!\!l)$ complete, metric space;

Polish if $\text{dom} \subset \mathbb{R}^n$, $\text{rge} \subset \mathbb{R}^m$

Convergence:

$S^\nu \xrightarrow{g} S$ if $d\!\!l(\text{gph}\, S^\nu, \text{gph}\, S) \to 0 \implies (S^\nu)^{-1}(0) \Rightarrow_v S^-1(0)$

Why?

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

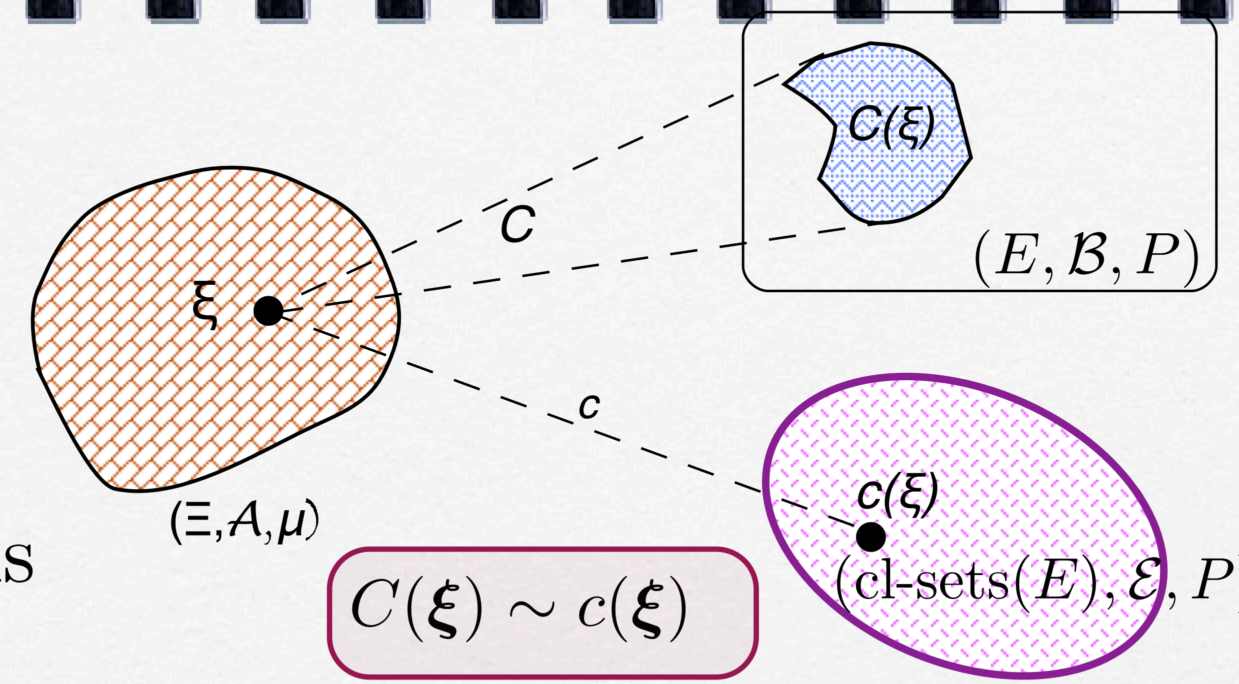
$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \dots \xi^\nu$ sample P^ν (random) empirical measure

approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

Random sets



C 'covered' by countable selections
Castaing represnetation

a.s convergence: $P\{\xi \mid d\!l(C^\nu(\xi), C(\xi)) \to 0\} = 0$

$\Rightarrow$ in probability: $\forall \varepsilon > 0,\ P\{\xi \mid d\!l(C^\nu(\xi), C(\xi)) > \varepsilon\} \to 0$

$\Rightarrow$ in distribution T : cpct-sets$(E) \to [0,1],\ T(\emptyset) = 0,$
(a) $T(K^\nu) \searrow T(K)$ for $K^\nu) \searrow K$, (b) 'rectangle cond'n'
$P^\nu \xrightarrow{\mathcal{D}} P \iff T^\nu \to T$ on cpct-sets$(\mathbb{R}^n)$
or, even, on finite union of closed rational balls.

Random Sets: Expectation

Random set: Expectation

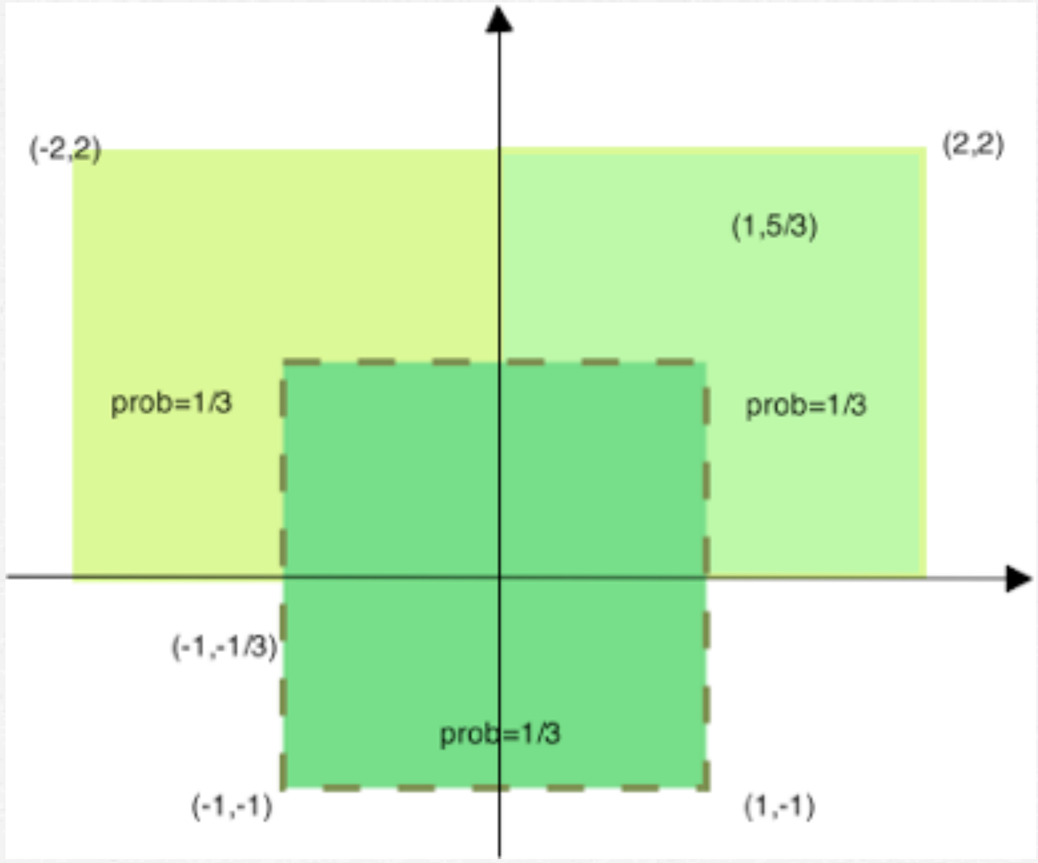
$$EC = \mathbb{E}\{C(\xi)\} = \left\{ \int_\Xi s(\xi)\,P(d\xi) \,\middle\|\, s(\bullet)\ P\text{-summable selection} \right\}$$

..not necessarily closed even when C is closed-valued

Convexity:

C P-atom convex $\Rightarrow EC$ is convex

(certainly when P is atomless).



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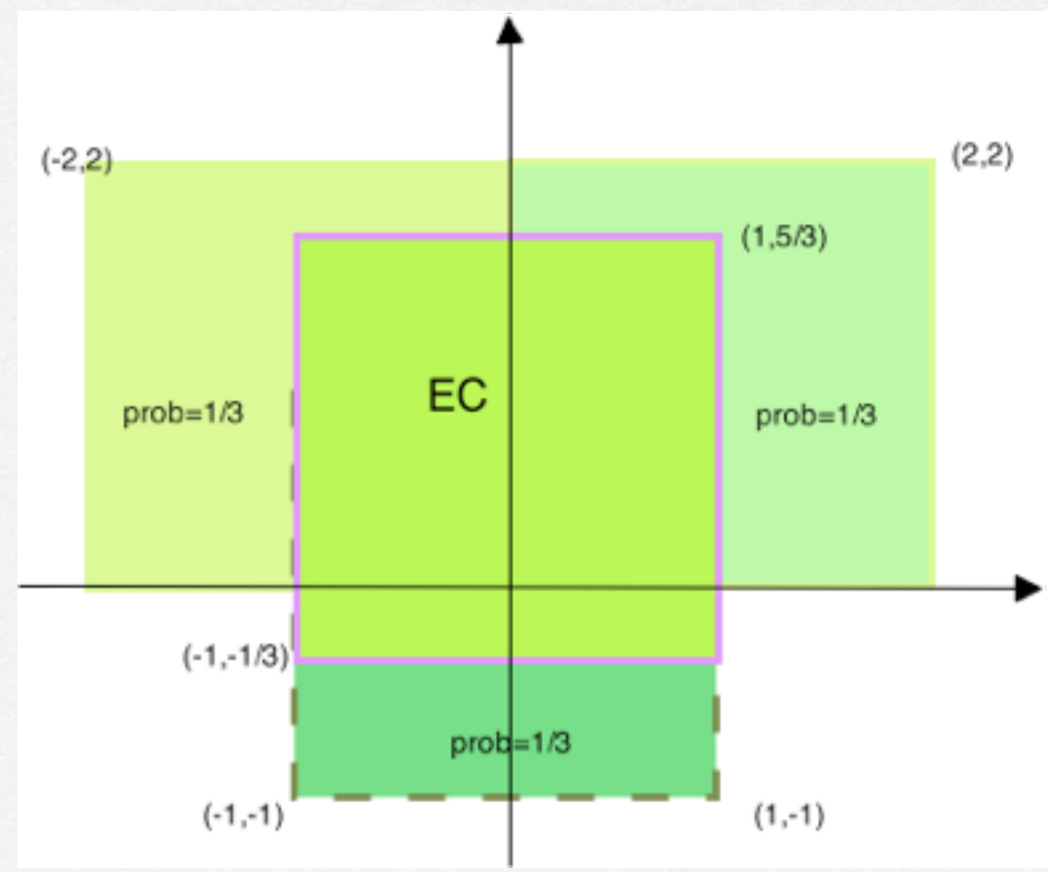
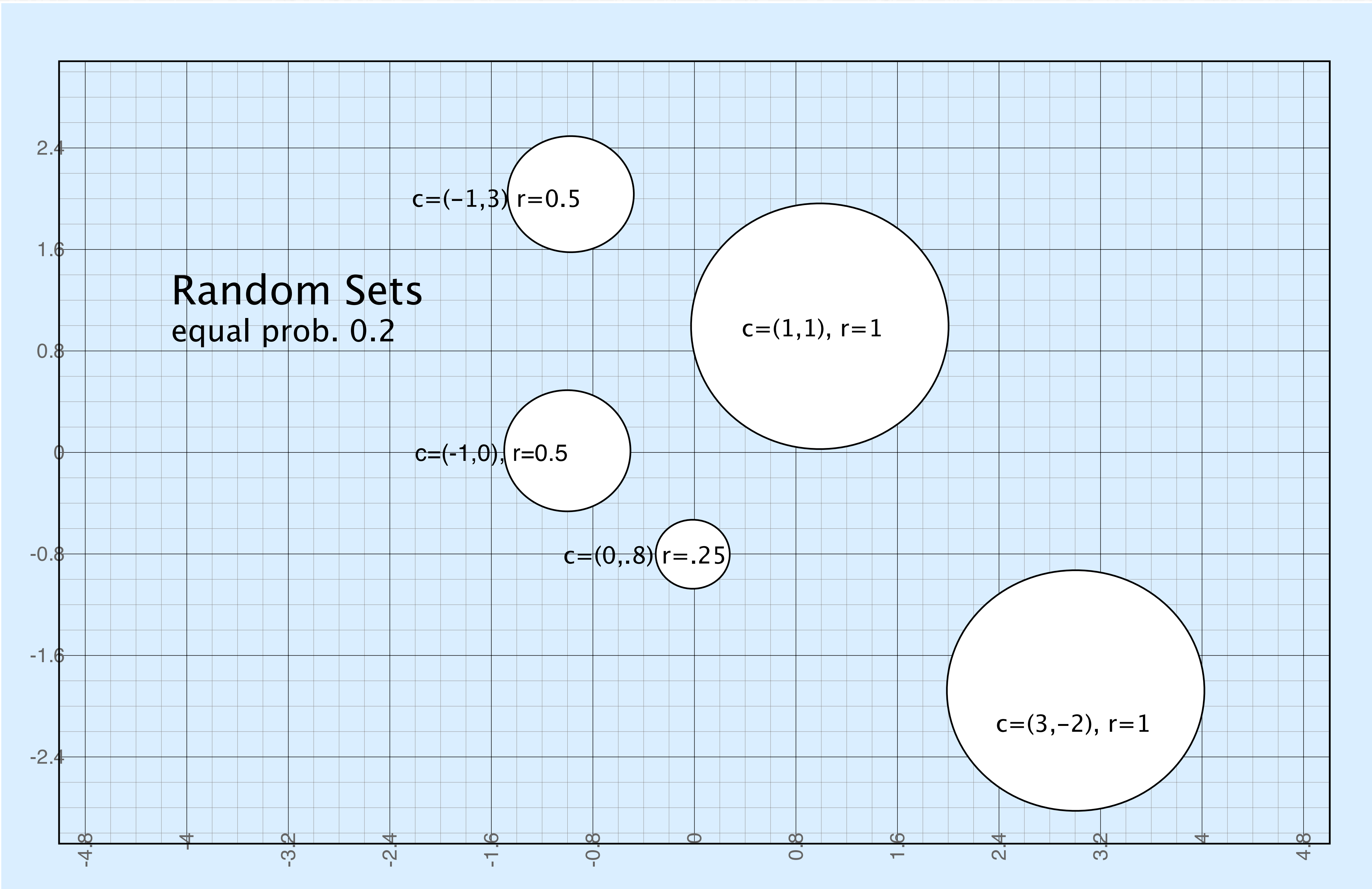
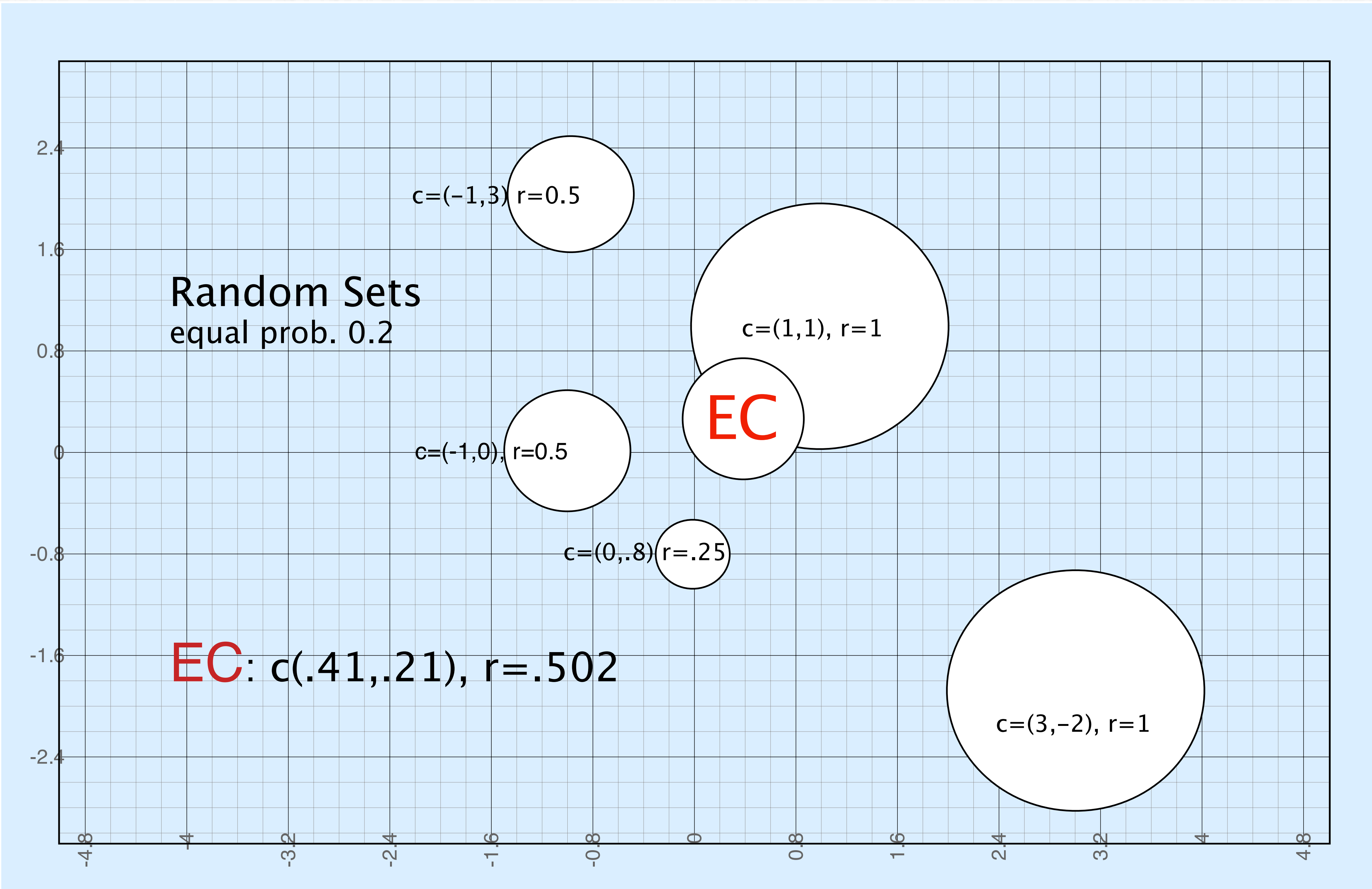


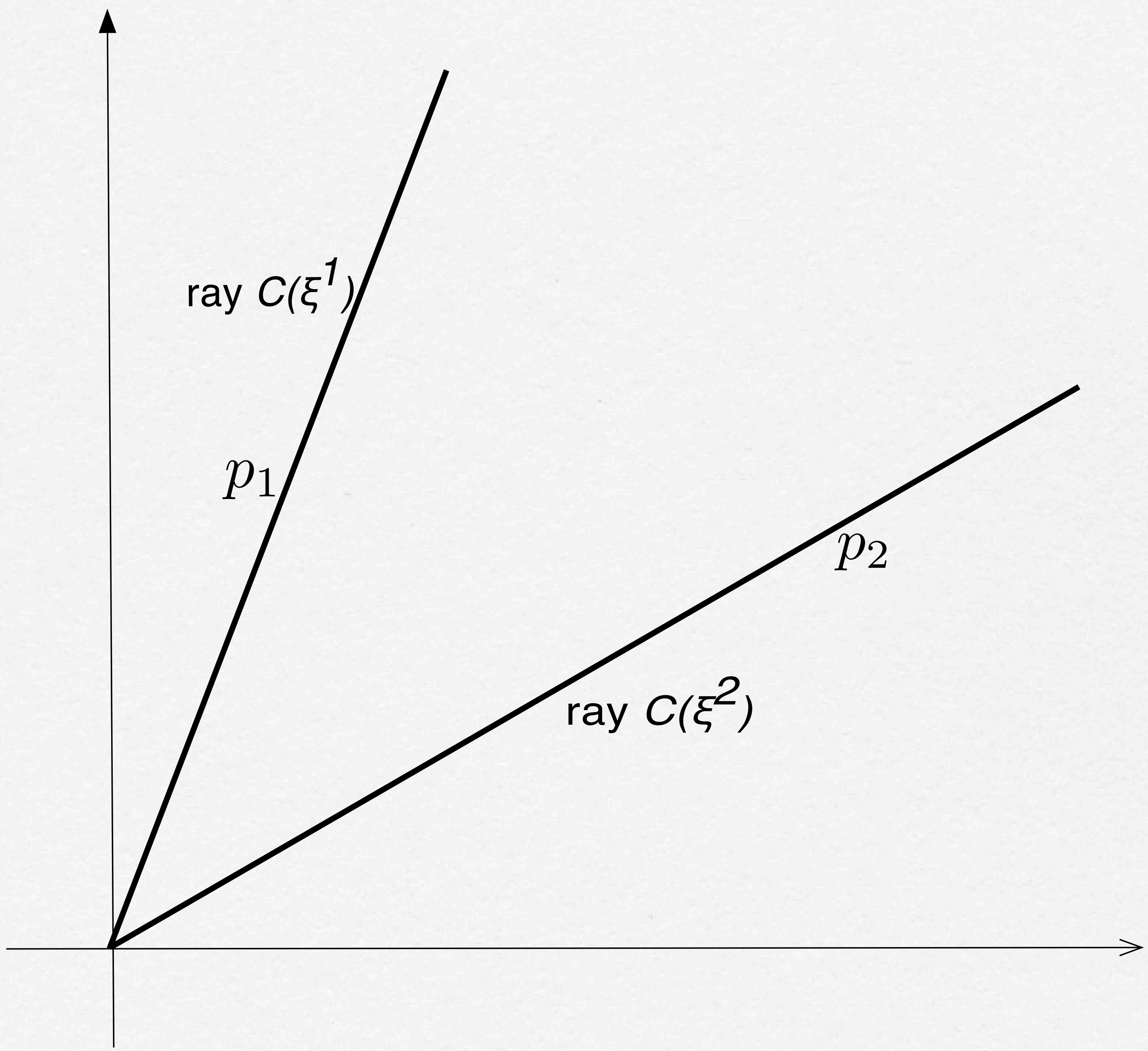
Bounded random set



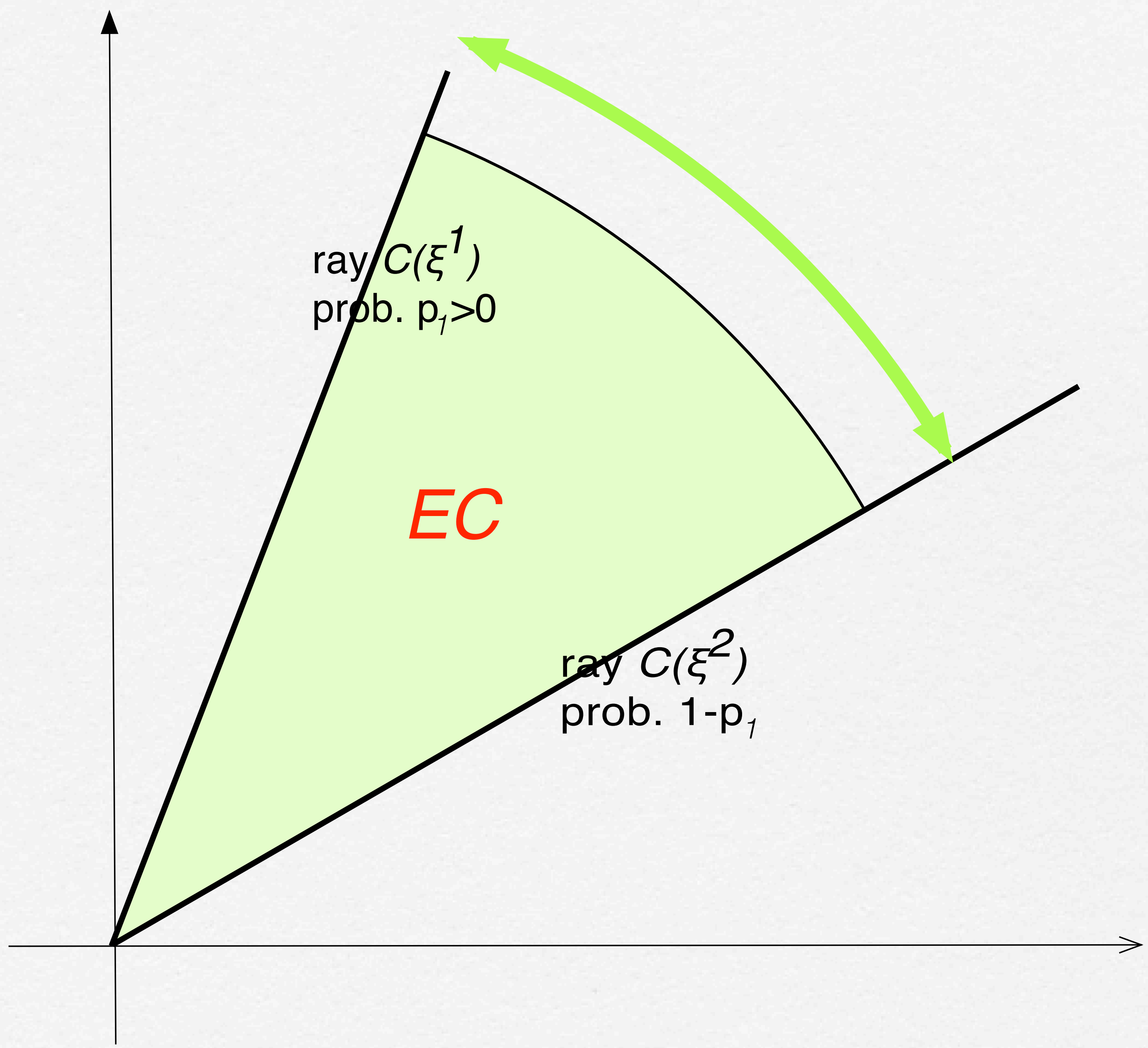
Expectation: Bounded r. set



Expectation: Unbounded r. sets



Expectation: Unbounded r. sets



Some properties: $\mathbb{E}\{C(\boldsymbol{\xi})\}$

- measure P atomless, then $EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ is convex (Richter, Lyapounov,...)
- P is P-atom convex $\Longrightarrow$ EC is convex; [an atom contains no (measurable) subset of positive probability]
- C a random set, $\emptyset \neq EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ contains no line, then

$$\mathrm{con}\, EC = \mathbb{E}\{\mathrm{con}\, X(\boldsymbol{\xi})\}$$

  this essentially requires that $C(\xi) \subset$ a pointed cone

- in general, the expectation of a (closed-valued) random set is *not* closed
- if $|C| = \mathbb{E}\big\{\sup\left[\,|s(\xi)|\,\big|\, s(\xi) \in C(\xi)\,\right]\big\} < \infty$ then EC is closed; C is then *integrably bounded.*

Strong law of large numbers
for random sets *(Artstein-Hart)*

$C:\Xi \rightrightarrows E$ measurable, $\{\xi^\nu, \nu \in \mathbb{N}\}$ iid Ξ-valued random variables

$C(\xi^\nu)$ iid random sets (i.e. induced P^ν independent and identical)

$EC = \mathbb{E}\{C(\bullet)\} = \left\{ \int_\Xi s(\xi) P(d\xi) \,\middle|\, s: P\text{-summable } C(\xi)\text{-selection} \right\}$

independence $\Rightarrow$ all (measurable) selections are independent

$\{C(\xi^\nu): \Xi \rightrightarrows \mathbb{R}^m \; \nu \in \mathbb{N}\}$ iid with $EC \neq \emptyset$. Then, with

$$C^\nu(\xi^\infty) = \nu^{-1}\left(\sum_{k=1}^{\nu} C(\xi^k)\right) \to \bar{C} = \text{cl con } EC \;\; P^\infty\text{-a.s.}$$

$\text{Lo}_\nu C^\nu(\xi^\infty) \subset \bar{C} \;\Leftrightarrow\; \limsup_\nu \sigma_{C^\nu} \leq \sigma_{\bar{C}}$ support functions

$\text{Li}_\nu C^\nu(\xi^\infty) \supset \bar{C}$ relies on LLN for (vector-valued) selections

Proof: time allowing

Random mappings

$S : \Xi \times E \rightrightarrows \mathbb{R}^m, \quad E \subset \mathbb{R}^n$

$\mathcal{A} \otimes \mathcal{B}^n$-jointly measurable: $S^{-1}(O) \in \mathcal{A} \otimes \mathcal{B}^n$, O open

$\Rightarrow \forall\, x : \xi \mapsto S(\xi, x)$ a random set

random closed set when S is closed-valued

$ES : E \rightrightarrows \mathbb{R}^m$ with $ES(x) = \mathbb{E}\{S(\xi, x)\}$ expected mapping

ES convex-valued when $\xi \mapsto S(\xi, \cdot)$ P-atom convex

Law of Large Numbers for random sets

applies pointwise

Sample Average Approximation (SAA)

stochastic variational problem: $\bar{S}(x) = \mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

$S : \Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ random set-valued mapping

$\boldsymbol{\xi}$ random vector with values $\xi \in \Xi \subset \mathbb{R}^N$

solution (a 'stationary point') $\bar{x} \in \bar{S}^{-1}(0)$

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sample $\overset{\rightarrow \nu}{\xi} = (\xi^1, \ldots, \xi^\nu)$ of $\boldsymbol{\xi}$

$\frac{1}{\nu}\left(\sum_{k=1}^{\nu} S(\xi^k, x)\right) = S^\nu(\overset{\rightarrow \nu}{\xi}, x) \ni 0$, approximating system?

i.e., $\left(S^\nu\right)^{-1}(0) \Rightarrow_\nu \bar{S}^{-1}(0)$ a.s. ???

So far ...

➪ *generalized equations*

$S : \Xi \times D \rightrightarrows E$, set-valued $S(\xi, x) \subset E$, inclusion $\mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

iid-sample $\vec{\xi}^{\,\nu} = \xi^1, \dots, \xi^\nu$ and $x \mapsto S(\xi, x)$ osc

SAA-mapping $S^\nu : \Xi^\infty \times D \rightrightarrows E$, random osc mappings

$$S^\nu(\xi, x) = \frac{1}{\nu} \sum_{k=1}^{\nu} S(\xi^k, x) \approx S^\nu(\vec{\xi}^{\,\nu}, x), \ \forall \xi \in \Xi^\infty$$

So far ...

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$\forall x \in D$, $S(\cdot, x)$, closed random set,

let $\bar{S} = \mathrm{cl}\,\mathrm{con}\, ES$, $\quad ES(x) = \mathbb{E}\{S(x, \boldsymbol{\xi})\}$

Artstein-Hart LLN applies: $S^\nu \xrightarrow{p} \bar{S}$ $a.s.$ when $E = \mathbb{R}^m$

but $\xrightarrow{p} \not\Rightarrow (S^\nu)^{-1}(0) \rightrightarrows \bar{S}^{-1}(0)$. Needed $S^\nu \xrightarrow{g} \bar{S}$

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but $\xrightarrow{p}\ \not\Rightarrow (S^\nu)^{-1}(0) \rightrightarrows \bar{S}^{-1}(0)$. Needed $S^\nu \xrightarrow{g} \bar{S}$

recall: $\bar{S}(x) = \mathrm{cl}\, ES(x)$ when P-atom convex, $ES(x)$ closed if $\xi \mapsto S(\xi, x)$ is integrably bounded and compact if $\mathrm{rge}\, S(\cdot, x)$ is bounded.

Consistent approximations?

$$S^\nu(\xi,\cdot) \to_p \bar{S} \;\; P^\infty\text{-a.s.} \Rightarrow ? \;\; S^\nu(\xi,\cdot)^{-1}(0) \Rightarrow_\nu \bar{S}^{-1}(0)$$

sometimes!

graphical rather than pointwise convergence is required

$$S^\nu(\xi,\cdot) \underset{\text{gph}}{\longrightarrow} \bar{S} \;\; P^\infty\text{-a.s. is needed}$$

relationship between graphical and pointwise convergence?

Some Examples

Stochastic VI, Variational Inequality

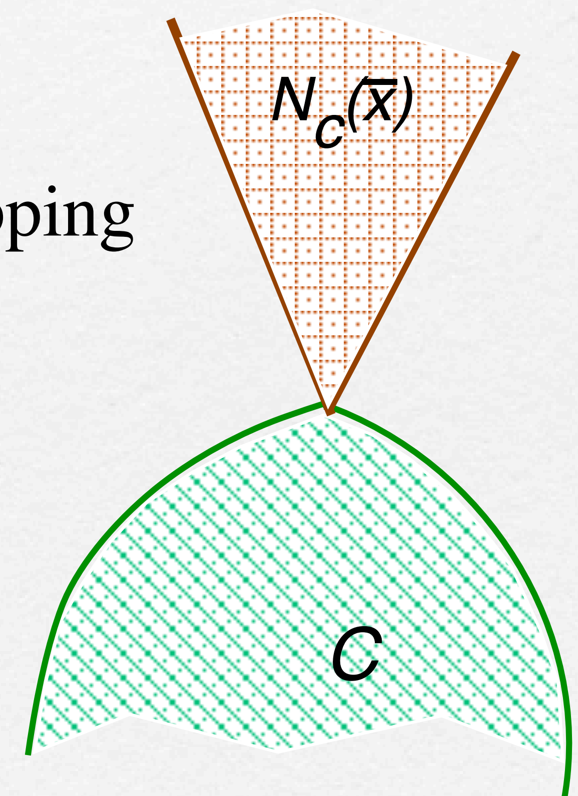
Network flow equilibrium with stochastic demand and link capacities
Economic equilibrium in a stochastic environment

$\xi = (\xi^1, \xi^2, \ldots)$, $G^\nu(\cdot, x)$ σ-$(\xi^1, \ldots \xi^\nu)$ measurable

$-G^\nu(\xi, x) \in N_C(x)$, C compact, convex

$N_{C(x)} + G^\nu(\xi, x) = S^\nu(x) \ni 0$, S^ν closed set-valued mapping

$$G^\nu(\xi, \cdot) \to^? G(\xi, \cdot)$$



$x^\nu(\xi)$ solution of $-G^\nu(\xi, x) \in N_C(x)$ for sample $\xi \approx \vec{\xi}^{\,\nu}$

does $x^\nu(\xi) \to$ a solution of $-G(\xi, x) \in N_C(x)$? a.s.

what if C depends on (ξ, ν): sequence of random sets $C^\nu(\xi)$?

"static" Walras Equilibrium

agent's problem: $a \in \mathcal{A}$, $|\mathcal{A}|$ finite, possibly "large"

$\bar{x}_a \in \arg\max u_a(x_a)$ so that $\langle p, x_a \rangle \le \langle p, e_a \rangle$, $x_a \in X_a$

e_a: endowment of agent a, $e_a \in \operatorname{int} X_a$

u_a: utility of agent a, concave, usc

$u_a : X_a \to \mathbb{R}$, $X_a \subset \mathbb{R}^n$ (survival set) convex

market clearing: $s(p) = \sum_{a \in \mathcal{A}} (e_a - \bar{x}_a)$ excess supply

equilibirum price: $\bar{p} \in \Delta$ such that $s(\bar{p}) \ge 0$, Δ unit simplex

Walras: a Variational Inequality

$$c_a = \arg\max_x u_a(x) \text{ so that } \langle p,x\rangle \le \langle p,e\rangle,\ x \in C_a$$

$$\sum_a (e_a - c_a) = s(p) \ge 0.$$

$$\Updownarrow$$

$$N_D(\bar{z}) = \left\{ v \middle| \langle v, z-\bar{z}\rangle \le 0, \forall z \in D \right\}$$

$$G(p,(x_a),(\lambda_a)) = \left[\sum_a (e_a - x_a); (\lambda_a p - \nabla u_a(x_a)); \langle p, e_a - x_a\rangle \right]$$

$$D = \Delta \times \left(\prod_a C_a \right) \times \left(\prod_a \mathbb{R}_+ \right)$$

$$-G(\bar{p},(\bar{x}_a),(\bar{\lambda}_a)) \in N_D(\bar{p},(\bar{x}_a),(\bar{\lambda}_a))$$

D unbounded $\to \hat{D}$ bounded

Equilibrium: stochastic environment

$$(c_a^1, y_a, c_{a,\xi}^2) = \arg\max_{x^1, y \in \mathbb{R}^L, x^2_\bullet \in \mathcal{M}} u_a^1(x^1) + \mathbb{E}^a \left\{ u_a^2(\boldsymbol{\xi}, x^2(\boldsymbol{\xi})) \right\}$$

$$\text{such that } \left\langle p^1, x_a^1 + T_a^1 y \right\rangle \leq \left\langle p^1, e_a^1 \right\rangle$$

$$\left\langle p_\xi^2, x_{a,\xi}^2 \right\rangle \leq \left\langle p_\xi^2, e_{a,\xi}^2 + T_{a,\xi}^2 y \right\rangle, \ \forall \xi \in \Xi$$

$$x_a^1 \in X_a^1, \ x_{a,\xi}^2 \in X_{a,\xi}^2, \quad \forall \xi \in \Xi$$

$\mathbb{E}^a \left\{ \bullet \right\}$ expecttion with respect to a-beliefs, Ξ finite support

2-stage stochastic programs with recourse

solution procedures & approximation theory "well-estblished"

$T_a^1, T_{a,\xi}^2$: input-output matrices (production, investments)

$e_a^1 \in \text{int}\, X_a^1, \ e_{a,\xi}^2 \in \text{int}\, X_{a,\xi}^2$ for all ξ

Market Clearing ~Equilibrium

excess supply: agent-a: $\left(c_a^1, y_a^1, \left\{ c_{a,\xi}^2 \right\}_{\xi\in\Xi} \right)$

$$\sum_{a\in\mathcal{A}} \left(e_a^1 - (c_a^1 + T_a^1 y_a) \right) = s^1 \left(p^1, \{p_\xi^2\}_{\xi\in\Xi} \right) \geq 0$$

$$\forall \xi,\ \sum_{a\in\mathcal{A}} \left((e_{a,\xi}^2 + T_{a,\xi}^2) - c_{a,\xi}^2 \right) = s_\xi^2 \left(p^1, \{p_\xi^2\}_{\xi\in\Xi} \right) \geq 0$$

Variational inequality: $-G(p,(x_a),(\lambda_a)) \in N_D(p,(x_a),(\lambda_a))$,

$$p = \left(p^1, \{p_\xi^2\}_{\xi\in\Xi} \right),\ x = \left(x^1, \{x_\xi^2\}_{\xi\in\Xi} \right),\ \lambda = \left(\lambda^1, \{\lambda_\xi^2\}_{\xi\in\Xi} \right)$$

$$S(\xi,(p,x,\lambda)) = G(\xi,(x,p,\lambda)) + N_{D(\xi)}(p,x,\lambda)),$$

$$\mathbb{E}\left\{ S(\xi,(p,x,\lambda)) \right\} \ni 0$$

a.s. Congergence of SAA-mappings

Graphical vs Pointwise convergence

$S, S^\nu : X \rightrightarrows \mathbb{R}^m$. Then, $S^\nu \underset{\text{point}}{\longrightarrow} S$ and $S^\nu \underset{\text{gph}}{\longrightarrow} S$ (at x)

$\Leftrightarrow \{C^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically) (at x)

~ Arzela-Ascoli Theorem for set-valued mappings

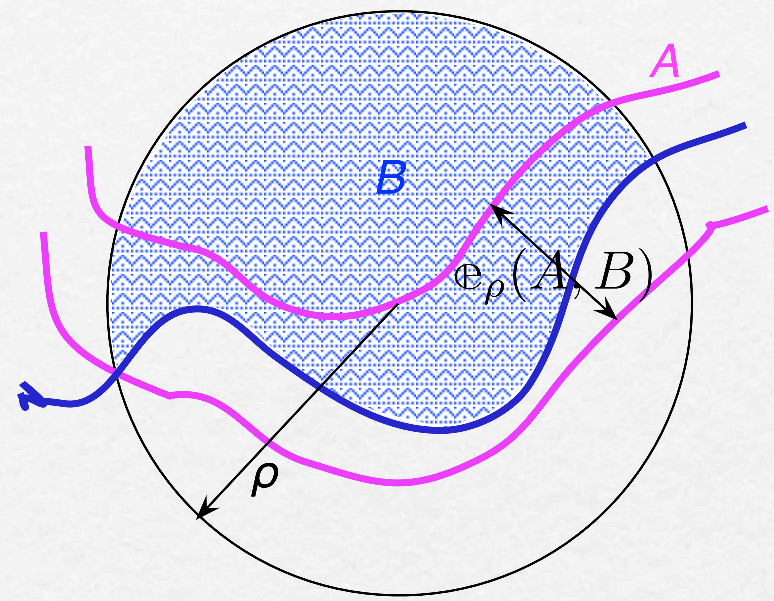
S random mapping, P^∞-a.s., $S^\nu(\xi, \cdot) \underset{\text{point}}{\longrightarrow} \operatorname{cl} \operatorname{con} ES = \bar{S}$

then $S^\nu \underset{\text{gph}}{\longrightarrow} \bar{S} \Leftrightarrow \{S^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically)

Semicontinuity: osc/isc

$S : D \rightrightarrows \mathbb{R}^m$ continuous at $\bar{x}$ if $\lim_{x^\nu \to \bar{x}} d\!l\big(S(x^\nu), S(\bar{x})\big) \to 0$

$$d\!l(S(x^\nu), S(\bar{x})) \to 0 \iff d\!l_\rho(S(x^\nu), S(\bar{x})) \to 0$$
$$\iff \hat{d\!l}_\rho(S(x^\nu), S(\bar{x})) \to 0. \forall\, \rho > \bar{\rho} \geq 0$$



Semicontinuity: osc/isc

$$S : D \rightrightarrows \mathbb{R}^m \text{ continuous at } \bar{x} \text{ if } \lim_{x^\nu \to \bar{x}} d\!l\big(S(x^\nu), S(\bar{x})\big) \to 0$$

$$d\!l(S(x^\nu), S(\bar{x})) \to 0 \iff d\!l_\rho(S(x^\nu), S(\bar{x})) \to 0$$
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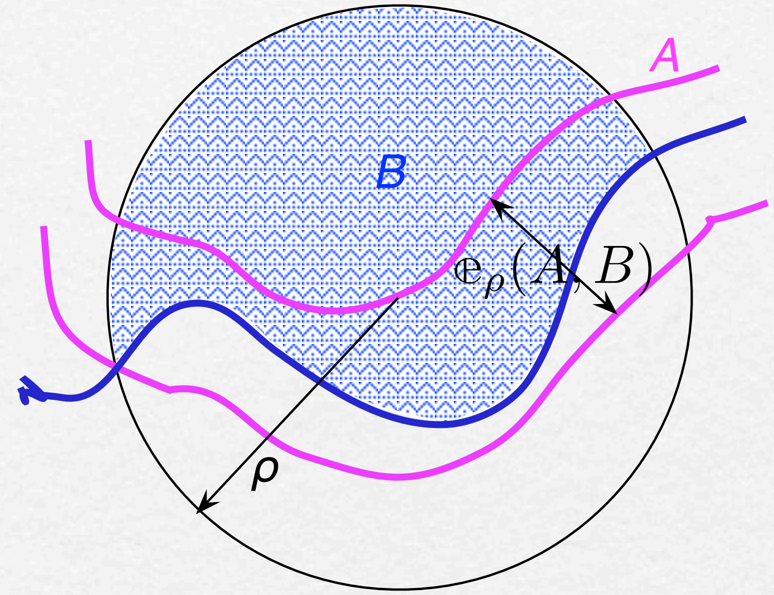

$$\hat{d\!l}_\rho\big(S(x^\nu), S(\bar{x})\big) = \max\left[\, e_\rho\big(S(x^\nu), S(\bar{x})\big), e_\rho\big(S(\bar{x}), S(x^\nu)\big)\,\right]$$

Semicontinuity: osc/isc

$S : D \rightrightarrows \mathbb{R}^m$ continuous at $\bar{x}$ if $\lim_{x^\nu \to \bar{x}} d\!l\big(S(x^\nu), S(\bar{x})\big) \to 0$

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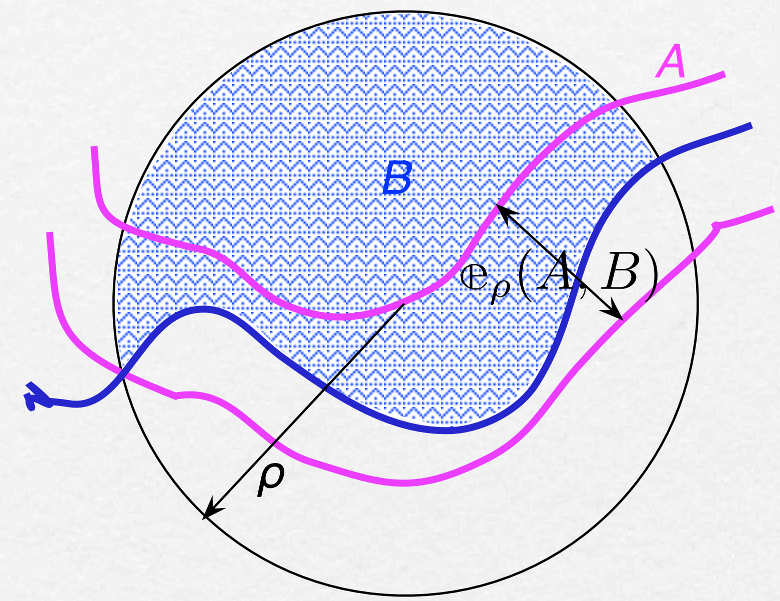

$$\hat{d\!l}_\rho\big(S(x^\nu), S(\bar{x})\big) = \max\big[\, e_\rho\big(S(x^\nu), S(\bar{x})\big), e_\rho\big(S(\bar{x}), S(x^\nu)\big)\,\big]$$

S is osc (outer semicontinuous) at $\bar{x}$ if $e_\rho\big(S(x^\nu), S(\bar{x})\big) \to 0$ as $x^\nu \to \bar{x}$

S is isc (inner semicontinuous) at $\bar{x}$ if $e_\rho\big(S(\bar{x}), S(x^\nu)\big) \to 0$ as $x^\nu \to \bar{x}$

Equi-osc mappings

$S : D \rightrightarrows \mathbb{R}^m,\ D \subset \mathbb{R}^n$ is osc if gph S is closed

osc at $\bar{x}$: given any $\rho > 0, \epsilon > 0$

$$\exists V \in \mathcal{N}(\bar{x}) : d\!l_\rho\big(S(x), S(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$\{S^\nu : D \rightrightarrows \mathbb{R}^m\}$ are equi-osc at $\bar{x}$

given any $\rho > 0, \epsilon > 0$

$$\exists V \in \mathcal{N}(\bar{x}) : d\!l_\rho\big(S^\nu(x), S^\nu(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$V = V(\rho, \epsilon)$ doesn't depend on ν.

G-convergence of SAA-mappings

$S: \Xi \times X \rightrightarrows \mathbb{R}^m$ random mapping, $(\Xi, \mathcal{A}, P)$

P^∞-a.s.: $S^\nu(\boldsymbol{\xi}, \cdot) \underset{\text{gph}}{\longrightarrow} \bar{S}$ at $\bar{x} \Leftrightarrow$ SAA-mappings $\{S^\nu(\boldsymbol{\xi}, \cdot)\}$ equi-osc at $\bar{x}$

$\Rightarrow$ sol'ns of $S^\nu(\xi, \cdot) \ni 0 \Rightarrow_\nu$ sol'ns of $\bar{S}(\cdot) \ni 0$

Sufficient condition: P^∞-a.s.

$S(\boldsymbol{\xi}, \cdot)$ stably osc & steady under averaging $\Rightarrow$ $\{S^\nu(\boldsymbol{\xi}, \cdot)\}$ equi-osc

G-convergence of SAA-mappings

$S: \Xi \times X \rightrightarrows \mathbb{R}^m$ random mapping, $(\Xi, \mathcal{A}, P)$

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$\Rightarrow$ sol'ns of $S^\nu(\xi,\cdot) \ni 0 \Rightarrow_\nu$ sol'ns of $\bar{S}(\cdot) \ni 0$

Sufficient condition: P^∞-a.s.

$S(\boldsymbol{\xi},\cdot)$ stably osc & steady under averaging $\Rightarrow$ $\{S^\nu(\boldsymbol{\xi},\cdot)\}$ equi-osc

Law of large Numbers for Random Mappings

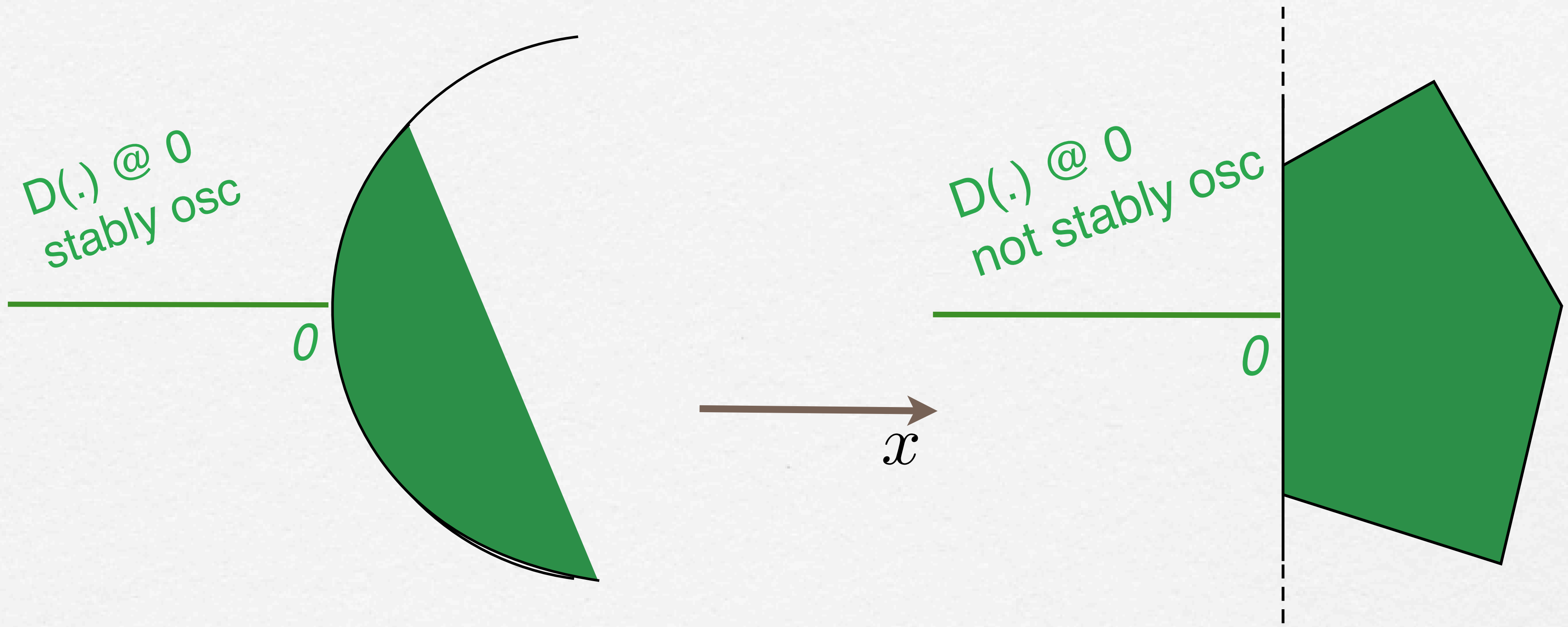
S random osc mapping: $\Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$

stably osc & steady under averaging

$\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \ldots,$ iid random variables (values in Ξ), distribution P

Then, $\nu^{-1}\sum_{k=1}^{\nu} S(\xi^k,\cdot) \to_{\text{gph}} \bar{S} = \operatorname{cl}\operatorname{con} E\{S(\boldsymbol{\xi}^0,\cdot)\}$ P^∞-a.s.

Stably osc mapping

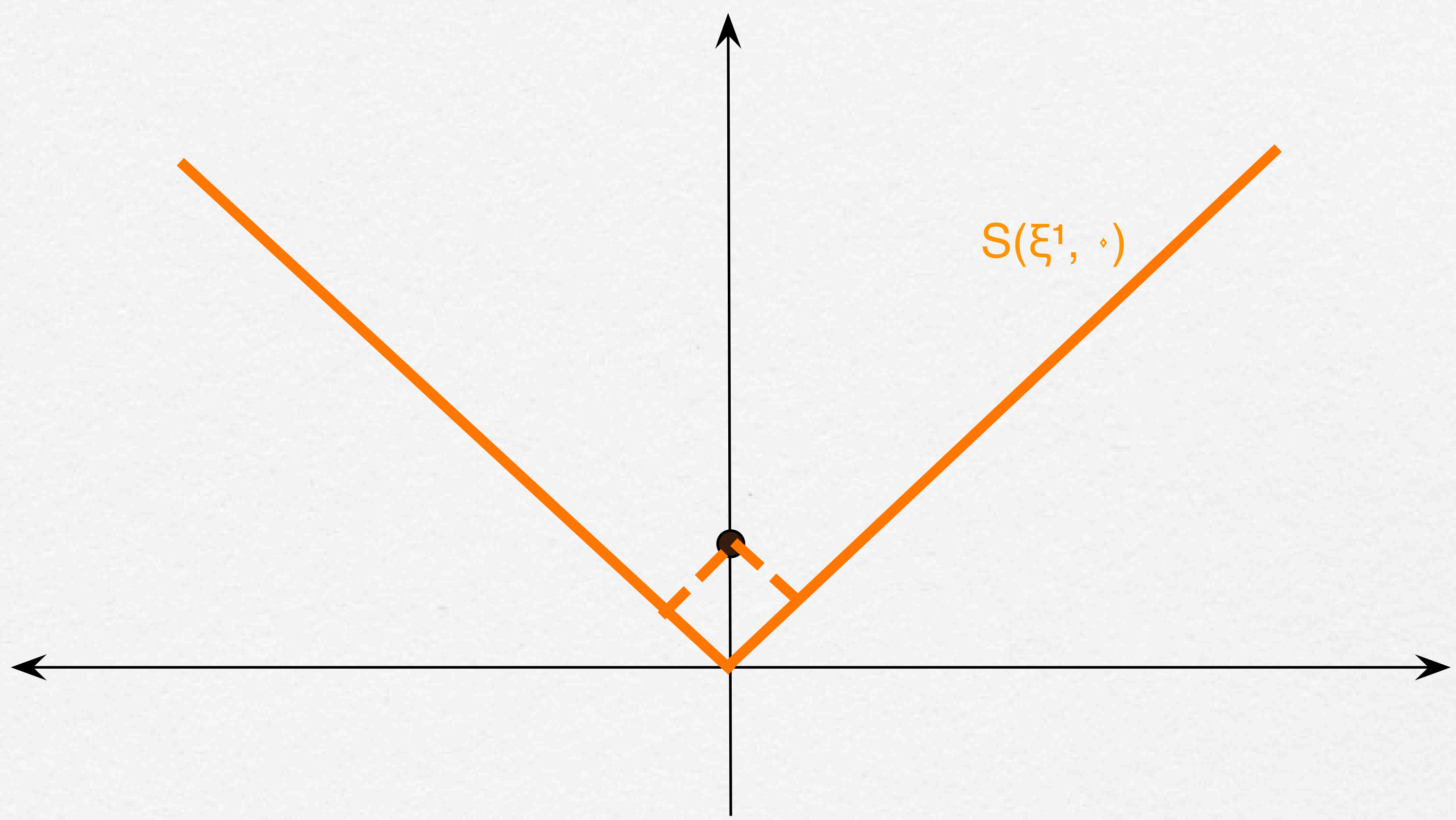


S stably osc near $\bar{x}$ if P-a.s.,

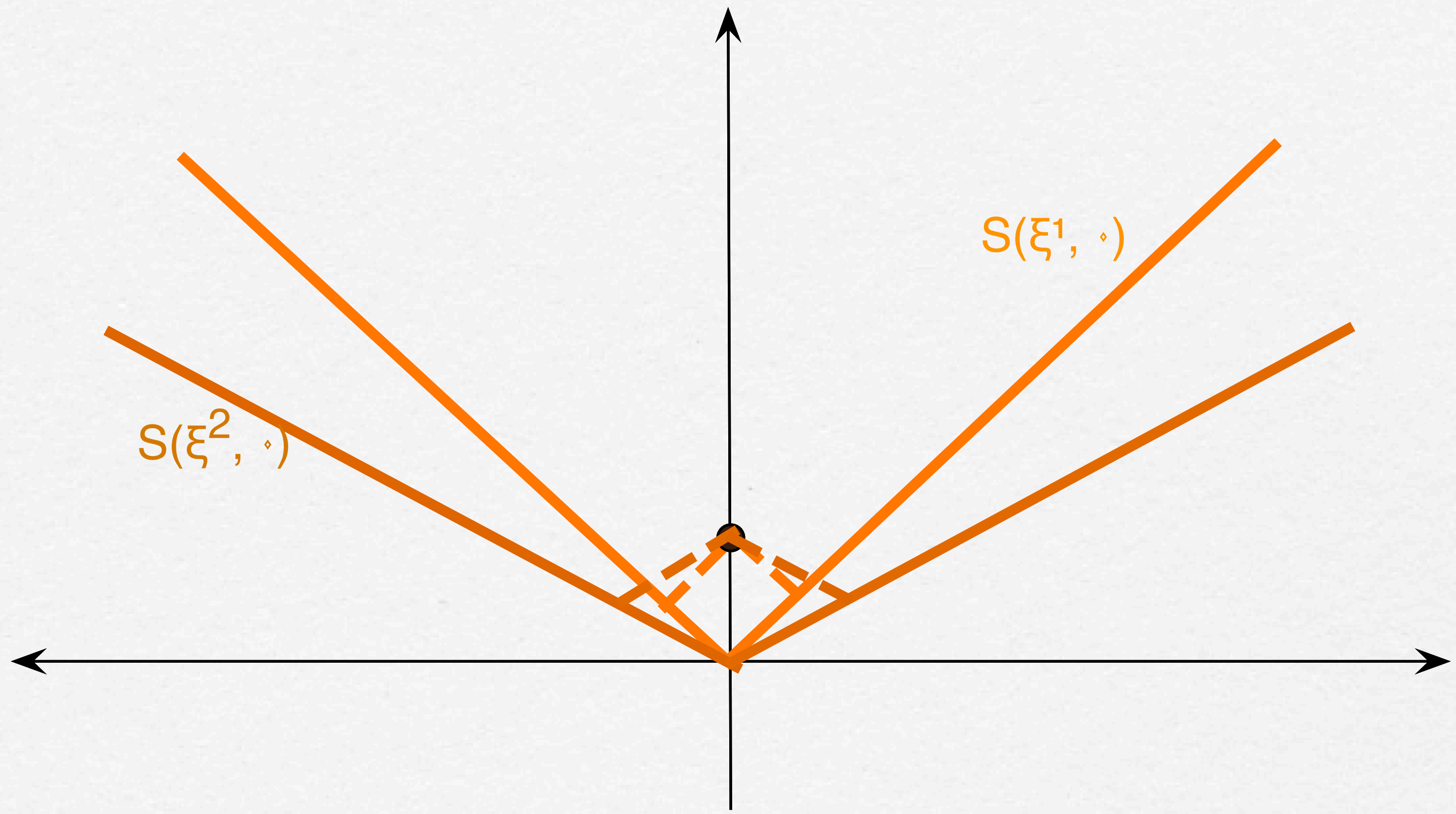
$$\forall \rho > 0, \varepsilon > 0,\ \exists W \in \mathcal{N}(\bar{x})\ \&\ \eta\mathbb{B}\ (\eta > 0):$$

$$\mathrm{e}_\rho\big(S(\xi, x'), S(\xi, x)\big) < \varepsilon,\ \forall\, x' \in x + \eta\mathbb{B},\ x \in W$$

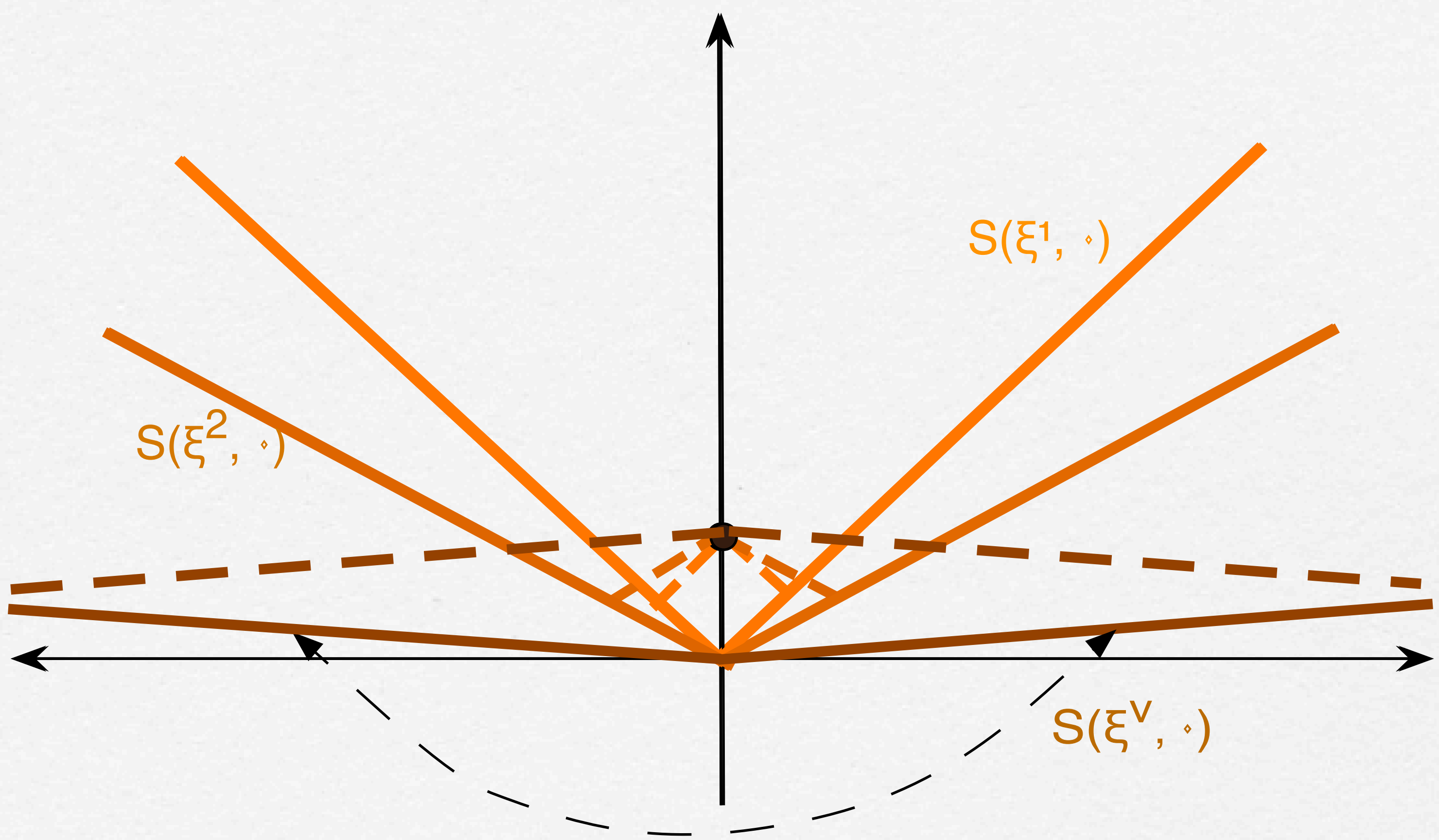
Steady under averaging



Steady under averaging

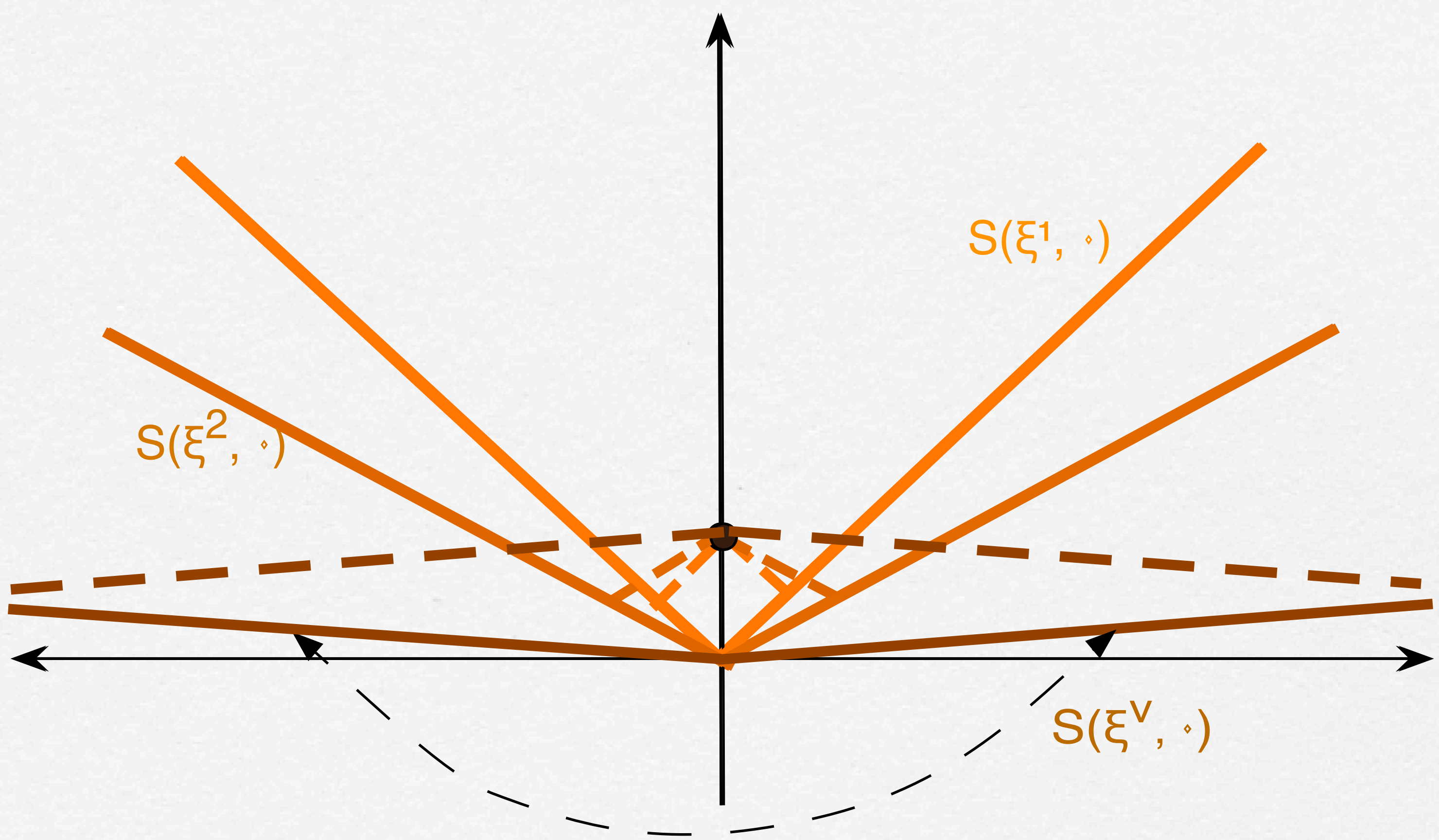


Steady under averaging



Sunday, May 27, 2012

Steady under averaging



$$u \in S^{\nu}(\overset{\rightarrow\nu}{\xi}, x) \cap \rho\mathbb{B} \Rightarrow \exists\, \hat{\rho} \geq \rho,\ u^k \in S(\xi^k, x) \cap \hat{\rho}\mathbb{B} \text{ such that}$$

$$u = \nu^{-1}(u^1 + \cdots + u^{\nu});\ S^{\nu}(\overset{\rightarrow\nu}{\xi}, x) \cap \rho\mathbb{B} \subset \frac{1}{\nu}\left[\sum_{k=1}^{\nu} S(\xi^k, x) \cap \hat{\rho}\mathbb{B}\right]$$

Steady u. averaging & stably osc

rge $S \subset B$ bounded $\Rightarrow$ steady under averaging

S cone-valued and rge $S \subset$ pointed cone K. Then,

$\bar{S} = ES$ and $\Rightarrow$ steady under averaging.

S, R steady under averaging $\Rightarrow$ so is $S + R$

$R(\xi, x) = R(x) \Rightarrow R$ steady under averaging

rge S bounded + R constant $\Rightarrow$ steady under averaging

$G(\xi, x) + N_C(x) \Rightarrow$ steady under averaging (V.I.)

provided $G : \Xi \times X \to \mathbb{R}^n$ is bounded

S, R stably osc $\Rightarrow S + R$ stably osc

although D^1, D^2 osc $\not\Rightarrow D^1 + D^2$ osc

$\mathbb{B}$ closed, convex $\quad x \mapsto N_{\mathbb{B}}(x)$ osc

but not stably osc ($x^\nu \in$ int $\mathbb{B} \to \bar{x} \in$ bdry $\mathbb{B}$)

Implementing SAA ** locally

$$EG(x) = \mathbb{E}\{G(\xi,x)\} \in R(x)$$

(V.I.: $S = N_C$, applied to option pricing, ...)

$G^\nu(\vec{\xi}^\nu,\cdot) = \nu^{-1}\sum_{k=1}^{\nu} G(\xi^k,x)$. Assume $G^\nu(\vec{\xi}^\nu,\cdot)$, $EG \in C^1(\mathbb{R}^n;\mathbb{R}^n)$,

$\bar{x}$ strongly regular solution [Robinson] of $EG(x) \in R(x)$,

$\exists V \in \mathcal{N}(\bar{x}), \rho > 0$ such that $\forall z \in \rho\mathbb{B}$:

$$z + EG(\bar{x}) + \nabla EG(\bar{x})\big(x - \bar{x}\big) \in S(x)$$

has a unique solution $\bar{x}(z) \in V$, Lipschitz continuous on $\rho\mathbb{B}$, and

$\left\| G^\nu(\vec{\xi}^\nu,\cdot) - EG \right\| \to 0$ P-a.s. Then, for ν sufficiently large

on a neighborhood of $\bar{x}$, $G^\nu(\vec{\xi}^\nu,\cdot) \in R(x)$ has a unique solution

$$\bar{x}(\vec{\xi}^\nu) \to \bar{x} \quad P\text{-a.s.}$$

Ciao!