To show integrality of \( P \), we will show that its corresponding constraint matrix is TU. First observe that \( P \) can be rewritten as

\[
P = \{ x \in \mathbb{R}^A \mid Dx = b, 1 \geq x \geq 0 \},
\]

where, \( 1 = \chi^V \) is the all-ones vector, \( b \in \{-1, 0, 1\}^V \) is defined by

\[
b(v) = \begin{cases} 
1 & \text{if } v = s, \\
-1 & \text{if } v = t, \\
0 & \text{if } v \in V \setminus \{s, t\},
\end{cases}
\]

and \( D \in \{-1, 0, 1\}^{V \times A} \) is the vertex-arc incidence matrix of the directed (loopless) graph \( G \), which is defined as follows: for \( v \in V, a \in A \):

\[
D(v, a) = \begin{cases} 
1 & \text{if } a \in \delta^+(v), \\
-1 & \text{if } a \in \delta^-(v), \\
0 & \text{otherwise}.
\end{cases}
\]

It thus suffices to show that \( D \) is TU to obtain integrality of \( P \).

**Theorem 8.** The vertex-arc incidence matrix \( D \in \{-1, 0, 1\}^{V \times E} \) of any directed (loopless) graph \( G = (V, A) \) is TU.

**Proof.** We apply the Ghouila-Houri characterization to the rows of \( D \). For any subset \( R \subseteq V \) of the rows, we choose the partition \( R_1 = R \) and \( R_2 = \emptyset \). Since each column of \( D \) has only zeros except for precisely one 1 and one \(-1\), summing any subsets of the elements of any column will lead to a total sum of either \(-1, 0, \) or 1. Hence,

\[
\sum_{v \in R_1} D_{v,a} \in \{-1, 0, 1\} \quad \forall a \in A,
\]

as desired. Or more formally, for \((u, v) \in A\), we have

\[
\sum_{v \in R_1} D_{v,a} = \mathbb{1}_{a \in \delta^+(v)} - \mathbb{1}_{a \in \delta^-(v)} \in \{-1, 0, 1\},
\]

where, for \( w \in V \), \( \mathbb{1}_{w \in R_1} = 1 \) if \( w \in R_1 \) and \( \mathbb{1}_{w \in R_1} = 0 \) otherwise. \( \square \)

### 1.5 Spanning tree polytope

Let \( G = (V, E) \) be an undirected graph. For any set \( S \subseteq V \), we denote by \( E[S] \subseteq E \) all edges with both endpoints in \( S \), i.e.,

\[
E[S] = \{ e \in E \mid e \subseteq S \}.
\]

**Theorem 9.** The spanning tree polytope of an undirected graph \( G = (V, E) \) is given by

\[
P = \left\{ x \in \mathbb{R}^E_{\geq 0} \mid \begin{array}{l}
x(E) = |V| - 1, \\
x(E[S]) \leq |S| - 1 & \forall S \subseteq V, |S| \geq 2.
\end{array} \right\}.
\]

Again, one can easily check that \( P \) contains the right integral points, i.e., the \( \{0, 1\} \)-points in \( P \) are precisely the incidence vectors of spanning trees. To check this, the following definition of spanning trees is useful: A set \( T \subseteq E \) is a spanning tree if and only if \( |T| = |V| - 1 \) and \( T \) does no contain any cycle. We will prove integrality of the polytope \( P \) in Section 2 using a powerful technique known as combinatorial uncrossing.

The constraints of the spanning tree polytope are often divided into two groups, namely the nonnegativity constraint \( x \geq 0 \), and all the other constraints which are called spanning tree constraints.
1.6 The $r$-arborescence polytope

**Definition 10** (arborescence, $r$-arborescence). Let $G = (V, A)$ be a directed graph. An arborescence in $G$ is an arc set $T \subseteq A$ such that

(i) $T$ is a spanning tree (when disregarding the arc directions), and

(ii) there is one vertex $r$ from which all edges are directed away, i.e., every vertex $v \in V$ can be reached from $r$ using a directed path in $T$.

The vertex $r$ in condition (ii) is called the root of the arborescence, and $T$ is called an $r$-arborescence.

Figure 4 shows an example of an $r$-arborescence.

![Example of an r-arborescence.](image)

Notice that condition (ii) can equivalently be replaced by

(ii') Every vertex has at most one incoming arc.

**Theorem 11.** The arborescence polytope of a directed graph $G = (V, A)$ is given by

$$P = \left\{ x \in \mathbb{R}_+^A \mid \begin{array}{l} x(A) = |V| - 1, \\
 x(A[S]) \leq |S| - 1 \quad \forall S \subseteq V, |S| \geq 2, \\
 x(\delta^+(v)) \leq 1 \quad \forall v \in V. \end{array} \right\},$$

where $A[S] \subseteq A$ for $S \subseteq V$ denotes all arcs with both endpoints in $S$.

A polyhedron that is closely related to the arborescence polytope and has a very elegant description, is the so-called **dominant of the arborescence polytope**. The dominant of the arborescence polytope will also provide an excellent example to show how integrality of a polyhedron can be proven using a technique called **combinatorial uncrossing**. The dominant can be defined for any polyhedron.

**Definition 12** (dominant of a polyhedron). The dominant $\text{dom}(P)$ of a polyhedron $P \subseteq \mathbb{R}^n$ is the polyhedron defined by

$$\text{dom}(P) = P + \mathbb{R}^n_+ = \{ x + y \mid x \in P, y \in \mathbb{R}^n_+ \}.$$  

Notice that the dominant of any nonempty polyhedron is an unbounded set, and therefore not a polytope, which is bounded by definition.

Apart from sometimes having a simpler description, the dominant of a polytope can also often be used for optimization. For example, consider the problem of finding a minimum weight $r$-arborescence with respect to some positive arc weights $w \in \mathbb{Z}_+^A$. Let $P$ be the $r$-arborescence polytope. Then this problem corresponds to minimizing $w^T x$ over all $x \in P$. However, this is equivalent to minimizing $w^T x$ over $x \in \text{dom}(P)$. Indeed, any $x \in \text{dom}(P)$ can be written as $x = y + z$, where $y \in P$ and $z \in \mathbb{R}^A_+$. Therefore for $x \in \text{dom}(P)$ to be a minimizer of $w^T x$, we must have $z = 0$; for otherwise, $w^Ty < w^Tx$ and $y \in P \subseteq \text{dom}(P)$, violating that $x \in \text{dom}(P)$ minimizes $w^T x$.

**Theorem 13.** The dominant of the $r$-arborescence polytope is given by

$$P = \left\{ x \in \mathbb{R}^A_+ \mid x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \right\}.$$  

We will prove integrality of this polytope later, when talking about combinatorial uncrossing.
1.7 Non-bipartite matchings

We will start by introducing the perfect matching polytope and then derive therefrom the description of the matching polytope.

**Theorem 14.** The perfect matching polytope of $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \mid \begin{array}{l} x(\delta(v)) = 1 \quad \forall v \in V, \\ x(\delta(S)) \geq 1 \quad \forall S \subseteq V, |S| \text{ odd.} \end{array} \right\}.$$ 

**Proof.** It is easy to check that $P$ contains the right integral points. Thus, it remains to show integrality of $P$.

By sake of contradiction assume that there are graphs $G = (V, E)$ for which $P$ is not integral. Among all such graphs let $G = (V, E)$ be a one that minimizes $|V| + |E|$, i.e., we look at a smallest bad example, and let $P$ be the corresponding polytope as defined in Theorem 14. Notice that we must have that $|V|$ is even. For otherwise the polytope $P$ is indeed the perfect matching polytope because it is empty, which follows from the constraint $x(\delta(V)) \geq 1$, which is impossible to satisfy since $\delta(V) = \emptyset$.

Let $y \in P$ be a vertex of $P$ that is fractional. We start by observing some basic properties that follow from the fact that we chose a smallest bad example. First, there is no 0-edge, i.e., an edge $e \in E$ such that $y(e) = 0$, because then one could delete this edge and obtain a smaller bad example. Similarly, there is no 1-edge $e = \{u, v\} \in E$, because in this case one can delete $u$ and $v$ together with all edges incident to them. One can observe that through this operation a smaller graph is obtained, namely $G' = G[V \setminus \{u, v\}]$, such that the restriction of $y$ to the edges of $G'$ is a vertex of the polytope $P'$ as defined by Theorem 14 that corresponds to $G'$. This violates the minimality of our bad example. Hence, $y$ is fractional on all edges, i.e., $y(e) \in (0, 1) \forall e \in E$. Additionally, we can assume that $G$ is connected. For otherwise, one gets a smaller bad example by only considering one of its connected components.

... Proof to be finished next time.