3 Ellipsoid Method

The ellipsoid method is a procedure that can be used to solve in particular linear, and more generally, convex optimization problems. We will focus on linear optimization problems with a bounded feasible region, i.e., a polytope, since this is the most important use of the ellipsoid algorithm in the context of combinatorial optimization. The problem setting that is considered by the ellipsoid algorithm is the following.

$$\text{Given a polytope } P \subseteq \mathbb{R}^n \text{, find a point } x \in P \text{, or decide that } P = \emptyset. \tag{17}$$

To simplify the exposition, we assume that P is a full-dimensional polytope. As we will discuss later, a linear programming problem over some (full-dimensional) polytope Q, i.e.,

$$\begin{aligned} \max \quad & w^T x \\ & x \in Q \end{aligned} \tag{18}$$

can be reduced to finding a point $x \in P = Q \cap \{w^T x \geq b\}$.

A crucial advantage of the ellipsoid method compared to other methods for linear programming, like the simplex algorithm or interior point methods, is that one does not need an explicit facet description of the facets of P. More precisely, one only needs to be able to solve an arguably simpler subproblem, known as the *separation problem*.

Separation problem
Given a point $y \in \mathbb{R}^n$: • Decide whether $y \in P$, or if this is not the case, • find a non-zero vector $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n \mid c^T x \leq c^T y\}$.

A procedure that solves the separation problem is often called a *separation oracle*. Furthermore, a hyperplane $H = \{x \in \mathbb{R}^n \mid c^T x = \alpha\}$ with $c^T y \geq \alpha$ and $P \subseteq \{x \in \mathbb{R}^n \mid c^T x \leq \alpha\}$ is called a *separating hyperplane*, or more precisely, a *y-separating hyperplane*. Notice that the separation problem as we stated it above asks to find a separating hyperplane that goes through y. However, any separating hyperplane $H = \{x \in \mathbb{R}^n \mid c^T x = \alpha\}$ can easily be transformed into one that goes through y by considering $H' = \{x \in \mathbb{R}^n \mid c^T x = c^T y\}$.

The fact that the ellipsoid algorithm is based on a separation oracle and does not need a facet description allows it to solve in polynomial time many linear programs with an exponential number of constraints. For example, one can use the ellipsoid method to optimize any linear function over the matching polytope, and thus, in particular, one can find a maximum weight matching via the ellipsoid method. The reason for this is that even though the matching polytope has an exponential number of constraints, one can construct a polynomial time separation oracle.