Economic Theory of Financial Markets

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Chapter 5:

Arbitrage Pricing Theory (APT)

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Economic Theory of Financial Markets

- Chapter 1: Introduction
- Chapter 2: Utility Theory
- Chapter 3: Mean-Variance Analysis
- Chapter 4: Capital Asset Pricing Model (CAPM)
- Chapter 5: Arbitrage Pricing Theory (APT)
- Chapter 6: Multiperiod Models and Yield Curves

- **Chapter 5: Arbitrage Pricing Theory (APT)**

- **Recall the CAPM Formula**

CAPM formula and a stochastic extension (1/3)

- The CAPM formula for expected return $\mathbb{E}[R_j] = \mu_j$ is given by

$$\mu_j - \mu_0 = \beta_j \left(r^{(M)} - \mu_0 \right),$$

with riskless return μ_0, expected market return $r^{(M)}$ and beta factor β_j.

- Stochastic extension of the CAPM formula: assume we have a risk factor F with $\mathbb{E}[F] = 0$ and $\sigma^2 = \mathrm{Var}(F)$, then we can set for the random return R_j

$$R_j = \mu_j + b_j F = \mu_0 + \beta_j \left(r^{(M)} - \mu_0 \right) + b_j F,$$

for $b_j \neq 0$, $1 \leq j \leq n$.

- This model provides us with $\mathbb{E}[\boldsymbol{R}] = \boldsymbol{\mu}$ and $\Sigma = \mathrm{Cov}(\boldsymbol{R}) = (\sigma^2 b_k b_j)_{1 \leq k,j \leq n}$.

CAPM formula and a stochastic extension (2/3)

- Choose risk factor F with $\mathbb{E}[F] = 0$ and set for $b_j \neq 0$

$$R_j = \mu_0 + \beta_j \left(r^{(M)} - \mu_0 \right) + b_j F.$$

- Assume for the first two risky assets $b_1 \neq b_2$. Then, we can construct an interesting portfolio $\boldsymbol{x} = (x_1, x_2, 0, \ldots, 0)^\top \in \mathbb{R}^n$ with

$$x_1 = \frac{b_2}{b_2 - b_1} \qquad \text{and} \qquad x_2 = \frac{-b_1}{b_2 - b_1}.$$

  This investment strategy gives us full investment in risky assets $\boldsymbol{x}^\top \boldsymbol{e} = 1$.

- The portfolio return of $\boldsymbol{x}$ is given by

$$\boldsymbol{x}^\top \boldsymbol{R} \;=\; \mu_0 + \left[\frac{b_2}{b_2 - b_1} \beta_1 - \frac{b_1}{b_2 - b_1} \beta_2 \right] \left(r^{(M)} - \mu_0 \right).$$

  This portfolio return is riskless!

CAPM formula and a stochastic extension (3/3)

- The portfolio return of $\boldsymbol{x}$ is given by

$$\boldsymbol{x}^\top \boldsymbol{R} \;=\; \mu_0 + \left[\frac{b_2}{b_2 - b_1}\beta_1 - \frac{b_1}{b_2 - b_1}\beta_2\right]\left(r^{(M)} - \mu_0\right).$$

  This portfolio return is riskless!

- Since we can only have one riskless asset (no arbitrage assumption is discussed below) we need to have

$$\frac{b_2}{b_2 - b_1}\beta_1 - \frac{b_1}{b_2 - b_1}\beta_2 = 0, \qquad \text{that is,} \qquad \beta_2 = \frac{b_2}{b_1}\beta_1.$$

- Thus, using a no arbitrage argument, the price system is completely determined by the riskless return μ_0 and the expected return μ_1 of the first risky asset R_1. The latter determines the risk aversion towards risk factor F.

- **APT Without Idiosyncratic Risk**

Financial market model

Assumptions. We have $n+1$ financial assets fulfilling the following assumptions:

- $R_0 = \mu_0$ is the riskless asset.
- Risky assets $\boldsymbol{R} = (R_1, \ldots, R_n)^\top$ fulfill, for $1 \le j \le n$,

$$R_j = \mu_j + \sum_{k=1}^{K} b_{j,k} F_k, \tag{1}$$

  with $b_{j,k} \in \mathbb{R}$ and with centered risk factors F_k, i.e. $\mathbb{E}[F_k] = 0$, for $1 \le k \le K$.
- We have dimension assumption $n > K$.

Vector notation

- We use vector notation and define matrix $B = (b_{j,k})_{1 \le j \le n; 1 \le k \le K} \in \mathbb{R}^{n \times K}$, that is,
$$B = \begin{pmatrix} b_{1,1} & \cdots & b_{1,K} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,K} \end{pmatrix} \in \mathbb{R}^{n \times K}.$$
Recall $n > K$.

- Set for risk factors $\boldsymbol{F} = (F_1, \ldots, F_K)^\top$.

- We can then rewrite (1) as
$$\boldsymbol{R} = \boldsymbol{\mu} + B\boldsymbol{F}.$$

Arbitrage portfolio

Definition. We call $\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1}$ an arbitrage portfolio if the following three conditions hold:

- $\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{e}} = 0$, i.e., we have a net investment of zero;
- $\mathbb{E}[\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}}] = \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} > 0$, i.e., we have a positive expected return;
- $\mathrm{Var}(\boldsymbol{x}^\top \boldsymbol{R}) = 0$, i.e., no risk is involved.

Interpretation.

- An arbitrage portfolio generates a positive return $\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}} > 0$, $\mathbb{P}$-a.s., from a zero net investment and at zero (downside) risk.
- The positive expected return in the second bullet point can be replaced by a non-zero expected return, note that if $\widetilde{\boldsymbol{x}}$ would generate a negative expected return (with a zero investment and at zero risk), then $-\widetilde{\boldsymbol{x}}$ is an arbitrage portfolio.

No arbitrage theorem

Economic Principle. We exclude the existence of arbitrage portfolios at the financial market.

Theorem. Under the above financial market assumptions and under the exclusion of arbitrage: there exists $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)^\top \in \mathbb{R}^K$ such that

$$\boldsymbol{\mu} = \mu_0 \boldsymbol{e} + B\boldsymbol{\lambda}. \tag{2}$$

Interpretation.

- No arbitrage assigns to every risk factor F_j a pricing factor λ_j, and consistent pricing systems $\boldsymbol{\mu}$ need to be of the form (2), consistent in the sense of no arbitrage.
- For $K = 1$ this gives the CAPM formula, but under a completely different economic principle (no arbitrage versus market clearing).
- The assumption $n > K$ is crucial, as we will see in the proof.

Proof of no arbitrage theorem

Proof. Note that $n > K$ implies that matrix $B \in \mathbb{R}^{n \times K}$ has maximal rank K. Therefore, there exist non-zero vectors $\boldsymbol{x} \in \mathbb{R}^n$ with $B^\top \boldsymbol{x} = 0$, i.e., we can choose $\boldsymbol{x}$ orthogonal to the K column vectors of B.

For this choice $\boldsymbol{x} \in \mathbb{R}^n$ with $B^\top \boldsymbol{x} = 0$, we set $x_0 = -\boldsymbol{x}^\top \boldsymbol{e}$. Then $\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{e}} = 0$ is a zero net investment portfolio. We use $\widetilde{\boldsymbol{x}}$ to construct an arbitrage portfolio. Note

$$\begin{aligned}
\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}} &= x_0 \mu_0 + \boldsymbol{x}^\top \boldsymbol{R} \\
&= x_0 \mu_0 + \boldsymbol{x}^\top (\boldsymbol{\mu} + B\boldsymbol{F}) \\
&= x_0 \mu_0 + \boldsymbol{x}^\top \boldsymbol{\mu} + (B^\top \boldsymbol{x})^\top \boldsymbol{F} \\
&= \boldsymbol{x}^\top (\boldsymbol{\mu} - \mu_0 \boldsymbol{e}) \;=\; \boldsymbol{x}^\top \boldsymbol{\mu}^e.
\end{aligned}$$

As a consequence, $\widetilde{\boldsymbol{x}}$ is a riskless portfolio because it is deterministic, i.e., has zero variance. Exclusion of arbitrage implies that this portfolio has to have a zero return $\boldsymbol{x}^\top \boldsymbol{\mu}^e = 0$.

That is, for *every* non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$ with $B^\top \boldsymbol{x} = 0$ we need to have

$$\boldsymbol{x}^\top \boldsymbol{\mu}^e = 0.$$

This implies that $\boldsymbol{\mu}^e$ is in the span of the column vectors of B and, henceforth, there exists $\boldsymbol{\lambda} \in \mathbb{R}^K$ such that $\boldsymbol{\mu}^e = B\boldsymbol{\lambda}$. This proves the theorem. $\square$

- **APT With Idiosyncratic Risk**

Financial market model

Assumptions. There exists an infinite sequence of risky assets having random returns $R_1, R_2, \ldots$.

For all $n \geq 1$, set $\boldsymbol{R}^{(n)} = (R_1, \ldots, R_n)^\top$ and assume the following structural form

$$\boldsymbol{R}^{(n)} = \boldsymbol{\mu}^{(n)} + B^{(n)}\boldsymbol{F} + \boldsymbol{\varepsilon}^{(n)},$$

where

- $\boldsymbol{\mu}^{(n)} \in \mathbb{R}^n$ are the first n elements of an infinite sequence $\mu_1, \mu_2, \ldots$,
- $B^{(n)} \in \mathbb{R}^{n \times K}$ are the first n rows of an $\infty \times K$ matrix B,
- $\boldsymbol{F}$ is a K-dimensional risk factor with $\mathbb{E}[\boldsymbol{F}] = 0$.
- $\boldsymbol{\varepsilon}^{(n)}$ are the first n elements of an infinite sequence $\varepsilon_1, \varepsilon_2, \ldots$ with
  - ⋆ $\mathbb{E}[\varepsilon_j] = 0$ for all $j \geq 1$,
  - ⋆ $\Phi^{(n)} = \mathrm{Cov}(\boldsymbol{\varepsilon}^{(n)})$ has uniformly (in n) bounded eigenvalues with bound $\bar{\lambda} < \infty$.

Interpretation of financial market model

- Set $\boldsymbol{R}^{(n)} = (R_1, \ldots, R_n)^\top$ and assume the following structural form

$$\boldsymbol{R}^{(n)} = \boldsymbol{\mu}^{(n)} + B^{(n)}\boldsymbol{F} + \boldsymbol{\varepsilon}^{(n)},$$

  where $\boldsymbol{\varepsilon}^{(n)}$ is idiosyncratic risk, i.e., this is the part that cannot be fully described by the common K-dimensional risk factors $\boldsymbol{F}$.

- Of course, idiosyncratic risk can be reduced by increasing K. There will be a trade-off between applicability and accuracy that will determine a suitable K which, of course, should be finite.

- The crucial assumption is: $\Phi^{(n)} = \mathrm{Cov}(\boldsymbol{\varepsilon}^{(n)})$ has uniformly (in n) bounded eigenvalues with bound $\bar{\lambda} < \infty$. This will imply an asymptotically perfect knowledge about the common risk factors $\boldsymbol{F}$, diversifying idiosyncratic risk. For this to happen, we need to assume boundedness of the noisy idiosyncratic part.

  Basically, this assumption plays the role of $n > K$ in the previous section.

Asymptotic arbitrage portfolios

Definition. We call the sequence $\boldsymbol{x}^{(n)} \in \mathbb{R}^n$, $n \geq 1$, an asymptotic arbitrage opportunity if the following three conditions hold:

- $(\boldsymbol{x}^{(n)})^\top \boldsymbol{e}^{(n)} = 0$, with $\boldsymbol{e}^{(n)} = (1, \dots, 1)^\top \in \mathbb{R}^n$, net investment of zero;
- $\limsup_{n \to \infty} \mathbb{E}[(\boldsymbol{x}^{(n)})^\top \boldsymbol{R}^{(n)}] \geq \delta > 0$;
- $\lim_{n \to \infty} \mathrm{Var}((\boldsymbol{x}^{(n)})^\top \boldsymbol{R}^{(n)}) = 0$, i.e., asymptotically vanishing risk.

Remark that different (stochastic) market models typically need slightly different arbitrage assumptions.

No arbitrage theorem

Economic Principle. We exclude asymptotic arbitrage opportunities.

Theorem. Under the above financial market assumptions and under the exclusion of asymptotic arbitrage opportunities: there exists an infinite sequence $(\widetilde{\boldsymbol{\lambda}}^{(n)})_n$ with $\widetilde{\boldsymbol{\lambda}}^{(n)} = (\lambda_0^{(n)}, \lambda_1^{(n)}, \ldots, \lambda_K^{(n)})^\top \in \mathbb{R}^{K+1}$ such that

$$\boldsymbol{\mu}^{(n)} = \lambda_0^{(n)} \boldsymbol{e}^{(n)} + B^{(n)} \boldsymbol{\lambda}^{(n)} + \boldsymbol{v}^{(n)}, \tag{3}$$

with correction/error terms $\boldsymbol{v}^{(n)} \in \mathbb{R}^n$ satisfying

$$\lim_{n \to \infty} \frac{1}{n} \|\boldsymbol{v}^{(n)}\|_2^2 \; = \; 0.$$

In contrast to the case without idiosyncratic risk, we only get an asymptotic result with idiosyncratic risk, saying that the correction terms $\|\boldsymbol{v}^{(n)}\|_2^2/n$ vanish, the more assets we consider for determining the common risk factors $\boldsymbol{F}$.

Proof of no arbitrage theorem

Proof. Consider (3)

$$\boldsymbol{\mu}^{(n)} = \lambda_0^{(n)} \boldsymbol{e}^{(n)} + B^{(n)} \boldsymbol{\lambda}^{(n)} + \boldsymbol{v}^{(n)},$$

this is a linear regression equation with responses μ_j, covariates $\boldsymbol{b}_j = (b_{j,1}, \ldots, b_{j,K})^\top \in \mathbb{R}^K$ (the j-th row of B), regression parameter $\widetilde{\boldsymbol{\lambda}}^{(n)}$ and errors $v_j^{(n)}$, i.e.,

$$\mu_j = \lambda_0^{(n)} + \boldsymbol{b}_j^\top \boldsymbol{\lambda}^{(n)} + v_j^{(n)},$$

for all $1 \leq j \leq n$. Parameter estimation in linear regression is received by minimizing the squared error loss objective function. Thus, we consider the objective function

$$\widetilde{\boldsymbol{\lambda}}^{(n)} \;\mapsto\; \mathcal{L}_{\boldsymbol{\mu}^{(n)}}(\widetilde{\boldsymbol{\lambda}}^{(n)}) = \frac{1}{2} \left\| \boldsymbol{\mu}^{(n)} - \lambda_0^{(n)} \boldsymbol{e}^{(n)} - B^{(n)} \boldsymbol{\lambda}^{(n)} \right\|_2^2 = \frac{1}{2} \left\| \boldsymbol{v}^{(n)} \right\|_2^2 .$$

We minimize this in $\widetilde{\boldsymbol{\lambda}}^{(n)}$ for $n > K$. We start with the intercept, this gives us score equation

$$\frac{\partial \mathcal{L}_{\boldsymbol{\mu}^{(n)}}(\widetilde{\boldsymbol{\lambda}}^{(n)})}{\partial \lambda_0^{(n)}} = \left\langle -\boldsymbol{e}^{(n)}, \boldsymbol{\mu}^{(n)} - \lambda_0^{(n)} \boldsymbol{e}^{(n)} - B^{(n)} \boldsymbol{\lambda}^{(n)} \right\rangle = -\langle \boldsymbol{e}^{(n)}, \boldsymbol{v}^{(n)} \rangle \;\overset{!}{=}\; 0, \qquad (4)$$

i.e. the resulting optimal correction terms $\boldsymbol{v}^{(n)}$ are centered (for all $n \geq K$).

The score equations for the regression parameters in $\boldsymbol{\lambda}^{(n)}$ are

$$\frac{\partial \mathcal{L}_{\boldsymbol{\mu}^{(n)}}(\widetilde{\boldsymbol{\lambda}}^{(n)})}{\partial \lambda_k^{(n)}} = \left\langle - \begin{pmatrix} b_{1,k} \\ \vdots \\ b_{n,k} \end{pmatrix}, \boldsymbol{\mu}^{(n)} - \lambda_0^{(n)} \boldsymbol{e}^{(n)} - B^{(n)} \boldsymbol{\lambda}^{(n)} \right\rangle = - \left\langle \begin{pmatrix} b_{1,k} \\ \vdots \\ b_{n,k} \end{pmatrix}, \boldsymbol{v}^{(n)} \right\rangle \stackrel{!}{=} 0, \tag{5}$$

i.e.,the resulting optimal correction terms $\boldsymbol{v}^{(n)}$ are orthogonal to the columns of $B^{(n)}$.

These (optimal) correction terms $\boldsymbol{v}^{(n)}$ are now used to construct asymptotic arbitrage opportunities. We define portfolios

$$\boldsymbol{x}^{(n)} = \frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} \boldsymbol{v}^{(n)} \ \in \ \mathbb{R}^n.$$

From the first score equation (4) we know that this portfolio gives us a zero net investment.

Next we calculate the return of this portfolio, we use score equations (5) in the third step,

$$\begin{aligned}
(\boldsymbol{x}^{(n)})^\top \boldsymbol{R}^{(n)} &= \frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} (\boldsymbol{v}^{(n)})^\top \left(\boldsymbol{\mu}^{(n)} + B^{(n)} \boldsymbol{F} + \boldsymbol{\varepsilon}^{(n)} \right) \\
&= \frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} \left((\boldsymbol{v}^{(n)})^\top \boldsymbol{\mu}^{(n)} + (\boldsymbol{v}^{(n)})^\top B^{(n)} \boldsymbol{F} + (\boldsymbol{v}^{(n)})^\top \boldsymbol{\varepsilon}^{(n)} \right) \\
&= \frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} \left((\boldsymbol{v}^{(n)})^\top \boldsymbol{\mu}^{(n)} + (\boldsymbol{v}^{(n)})^\top \boldsymbol{\varepsilon}^{(n)} \right).
\end{aligned}$$

We calculate expected return and variance of this portfolio

$$\begin{aligned}
\mathbb{E}[(\boldsymbol{x}^{(n)})^\top \boldsymbol{R}^{(n)}] &= \frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} \left((\boldsymbol{v}^{(n)})^\top \boldsymbol{\mu}^{(n)} + (\boldsymbol{v}^{(n)})^\top \mathbb{E}[\boldsymbol{\varepsilon}^{(n)}] \right) \\
&= \frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} (\boldsymbol{v}^{(n)})^\top \boldsymbol{\mu}^{(n)} \\
&= \frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} (\boldsymbol{v}^{(n)})^\top \left(\lambda_0^{(n)} \boldsymbol{e}^{(n)} + B^{(n)} \boldsymbol{\lambda}^{(n)} + \boldsymbol{v}^{(n)} \right) \\
&= \frac{\|\boldsymbol{v}^{(n)}\|_2}{\sqrt{n}}, \qquad (6)
\end{aligned}$$

where we have used score equations (4)-(5). The variance of this portfolio is

$$\begin{aligned}
\mathrm{Var}((\boldsymbol{x}^{(n)})^\top \boldsymbol{R}^{(n)}) &= \mathrm{Var}\left(\frac{1}{\sqrt{n}} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2} \left((\boldsymbol{v}^{(n)})^\top \boldsymbol{\mu}^{(n)} + (\boldsymbol{v}^{(n)})^\top \boldsymbol{\varepsilon}^{(n)} \right) \right) \\
&= \frac{1}{n} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2^2} \mathrm{Var}\left((\boldsymbol{v}^{(n)})^\top \boldsymbol{\varepsilon}^{(n)} \right) \\
&= \frac{1}{n} \frac{1}{\|\boldsymbol{v}^{(n)}\|_2^2} (\boldsymbol{v}^{(n)})^\top \Phi^{(n)} \boldsymbol{v}^{(n)} \;\le\; \frac{\bar{\lambda}}{n}.
\end{aligned}$$

The latter converges to zero for $n \to \infty$. Exclusion of asymptotic arbitrage implies that also (6) needs to converge to zero. This proves the claim because $(\widetilde{\boldsymbol{\lambda}}^{(n)})_n$ has the necessary properties. $\square$

No arbitrage theorem

Economic Principle. We exclude asymptotic arbitrage opportunities.

Theorem. Under the above financial market assumptions and under the exclusion of asymptotic arbitrage opportunities: there exists an infinite sequence $(\widetilde{\boldsymbol{\lambda}}^{(n)})_n$ with $\widetilde{\boldsymbol{\lambda}}^{(n)} = (\lambda_0^{(n)}, \lambda_1^{(n)}, \ldots, \lambda_K^{(n)})^\top \in \mathbb{R}^{K+1}$ such that

$$\boldsymbol{\mu}^{(n)} = \lambda_0^{(n)} \boldsymbol{e}^{(n)} + B^{(n)} \boldsymbol{\lambda}^{(n)} + \boldsymbol{v}^{(n)},$$

with correction/error terms $\boldsymbol{v}^{(n)} \in \mathbb{R}^n$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \|\boldsymbol{v}^{(n)}\|_2^2 \;=\; 0.$$