Economic Theory of Financial Markets

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Chapter 1:

Introduction

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Literature

- Demange, G., Laroque, G. (2006). *Finance and the Economics of Uncertainty*. Blackwell Publishing.
- Duffie D. (2001). *Dynamic Asset Pricing Theory*. Princeton University Press.
- Föllmer, H., Schied, A. (2011). *Stochastic Finance - An Introduction in Discrete Time*. Walter de Gruyter.
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- **Chapter 1: Introduction**

What is this lecture about?

- Financial decision making
- Consumption of wealth
- Price formation at markets
- Investments, returns and interest
  - ▷ This lecture focuses on the mathematical modeling of these questions.
  - ▷ These questions are also answered in similar lectures in economics.
  - ▷ We focus on the mathematical theory behind these economic questions.
- Prerequisites:
  Good knowledge in Probability Theory, Statistics, Analysis and Linear Algebra.

- **Introductory Example**

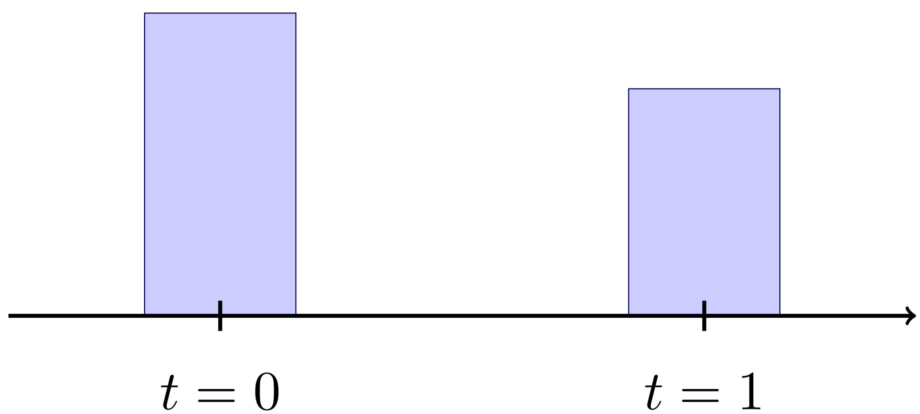
Introductory example

Basic setup: Two period problem under certainty.

- Choose two periods $t = 0, 1$ (today and tomorrow).

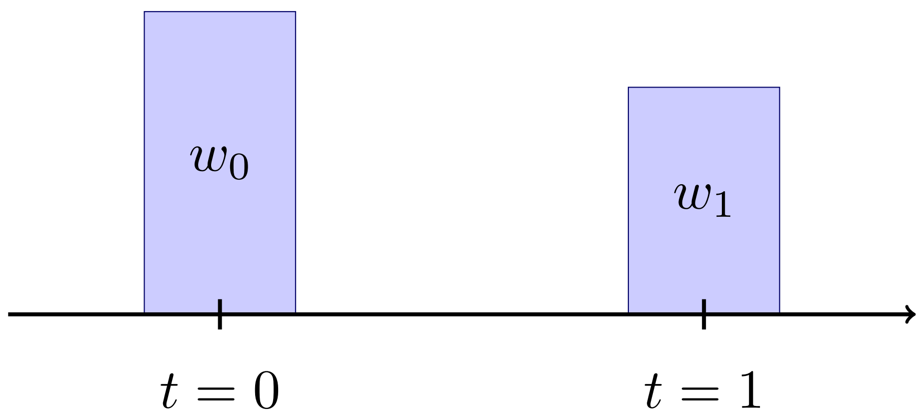
- A single good is traded that cannot be stored, i.e., that needs to be consumed immediately. This good is available at both times $t = 0, 1$.

- Participants in this economy may exchange this good and consume it either today $t = 0$ or tomorrow $t = 1$.



Individual demand for savings (1/2)

- Consider an individual financial agent:
  - ⋆ he/she has an endowment w_0 of this good at time $t = 0$;
  - ⋆ he/she has an endowment w_1 of this good at time $t = 1$.



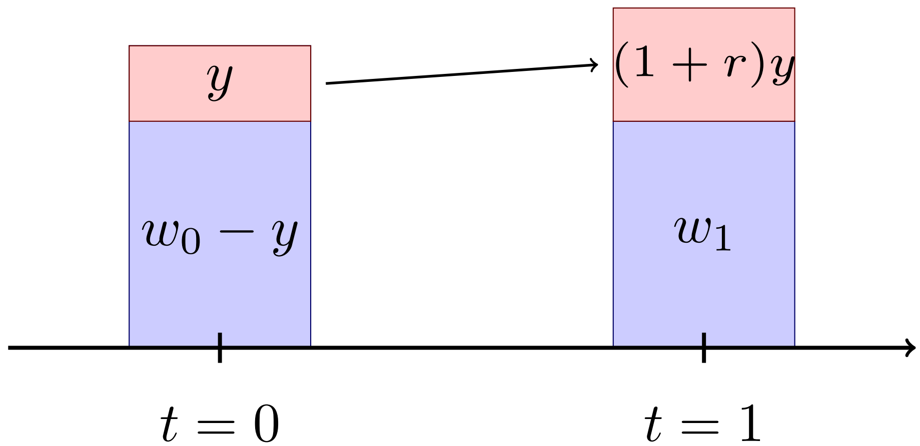
- Since the good cannot be stored it needs to be consumed immediately.

- However, there is a market that allows to lend and borrow this good with other financial agents (market participants).

- Lending and borrowing is done at fixed interest rate $r > -1$.

Individual demand for savings (2/2)

- Consider an individual financial agent with endowment $\boldsymbol{w} = (w_0, w_1)^\top \in \mathbb{R}^2$.



- Assume that the financial agent decides to save a fixed amount of $y \in \mathbb{R}$ at time $t = 0$ of his endowment ($y < 0$ means borrowing).

- This gives consumptions c_t at times $t = 0, 1$ for interest rate $r > -1$

$$
\begin{aligned}
c_0 &= w_0 - y, \\
c_1 &= w_1 + (1+r)y.
\end{aligned}
$$

Optimal consumption stream

- This financial agent has consumption stream $\boldsymbol{c} = (c_0, c_1)^\top \in \mathbb{R}$ with

$$\begin{aligned} c_0 &= w_0 - y, \\ c_1 &= w_1 + (1+r)y. \end{aligned}$$

- **Question:** How much should he/she save at time $t = 0$?
  - ▷ Each financial may have a different answer to this question.
  - ▷ The level of interest rate r is crucial.
  - ▷ How does the market fix the interest rate r?

- Microeconomy considers individual decision making (saving y), and macroeconomy considers the development of the market/economy as a whole (interest rate r).

- This, here, is a problem under certainty because there is no uncertainty involved in endowment $\boldsymbol{w}$ and interest rate r, i.e., for the moment they are perfectly known.

- **Utility Theory**

Budget constraint

- This financial agent has consumption stream $\boldsymbol{c} = (c_0, c_1)^\top \in \mathbb{R}$ with

$$\begin{aligned} c_0 &= w_0 - y, \\ c_1 &= w_1 + (1+r)y. \end{aligned}$$

- Solve the above system for y. This gives

$$y = w_0 - c_0, \qquad \text{and hence} \qquad c_1 = w_1 + (1+r)(w_0 - c_0).$$

- The latter provides us with the intertemporal budget constraint (bc)

$$c_0 + \frac{c_1}{1+r} \;=\; w_0 + \frac{w_1}{1+r}. \tag{1}$$

- The budget constraint gives the (maximal) consumption $\boldsymbol{c}$ in terms of the endowment $\boldsymbol{w}$, respecting time-values (modeled by interest rate r).

Preferences through utility functions

- **Main Question:** How do individuals take financial decisions?
- We model their preferences (happiness) about decisions by utility functions.
- A utility function is a map

$$\begin{aligned} \mathcal{U} : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ \boldsymbol{c} &\mapsto \mathcal{U}(\boldsymbol{c}) = u_0(c_0) + \frac{1}{1+\delta}\, u_1(c_1), \end{aligned}$$

  with

  ⋆ $u_0(c_0)$ happiness contribution of consumption c_0 at time $t = 0$,
  ⋆ $u_1(c_1)$ happiness contribution of consumption c_1 at time $t = 1$,
  ⋆ $(1+\delta)^{-1}$ impatience factor for waiting for a later consumption.

Optimal consumption

- Assume that the financial agent is characterized by utility function

$$\begin{aligned} \mathcal{U}:\mathbb{R}^2 &\to \mathbb{R} \\ \boldsymbol{c} &\mapsto \mathcal{U}(\boldsymbol{c}) = u_0(c_0) + \frac{1}{1+\delta}\, u_1(c_1). \end{aligned}$$

- His/her optimal consumption $\boldsymbol{c}^*$ is determined by maximizing the utility

$$\boldsymbol{c}^* = \underset{\boldsymbol{c}\in\mathbb{R}^2}{\arg\max}\, \mathcal{U}(\boldsymbol{c}) \qquad \text{subject to budget constraint (1).}$$

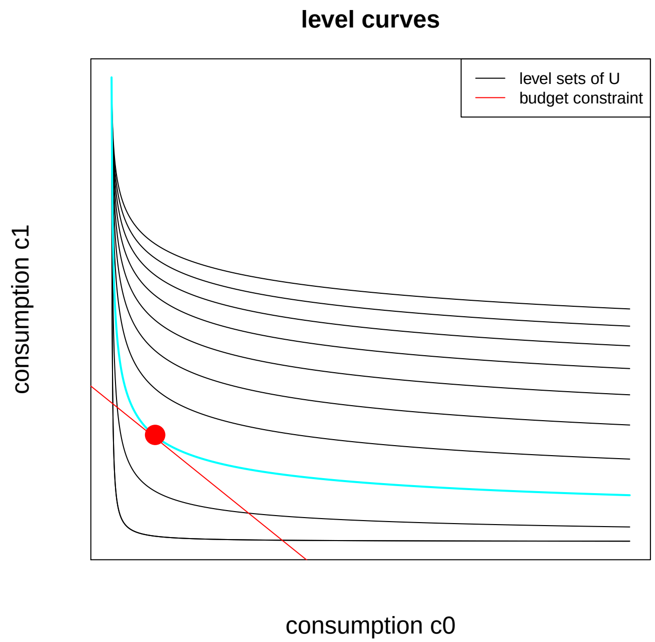
- Existence and uniqueness of optimal consumption $\boldsymbol{c}^*$ remains to be checked; in fact, this is one of the main questions, here.

Graphical solution of optimal consumption

- The optimal consumption $\boldsymbol{c}^*$ is given by

$$\boldsymbol{c}^* = \arg\max_{\boldsymbol{c} \in \mathbb{R}^2} \mathcal{U}(\boldsymbol{c}) \qquad \text{subject to budget constraint (1).}$$

- Plot level curves $\{\boldsymbol{c} \in \mathbb{R}^2;\ \mathcal{U}(\boldsymbol{c}) = \text{const}\}$, this gives the black lines:



- $\boldsymbol{c}^*$ is determined by the level curve that is tangential to budget constraint (1).

Mathematical solution of optimal consumption (1/2)

- Graphical solution is illustrative but it is not possible to derive mathematical properties of this solution.

- Mathematical solution: rewrite the utility function by using budget constraint (1)

$$\begin{aligned}\mathcal{U}(\boldsymbol{c}) &= \mathcal{U}\left((c_0, c_1)^\top\right) \stackrel{(1)}{=} \mathcal{U}\left((c_0, c_1(c_0))^\top\right) \\ &= u_0(c_0) + \frac{1}{1+\delta}\, u_1(c_1(c_0)) \stackrel{\text{def.}}{=} U(c_0),\end{aligned}$$

  using budget constraint $c_1 = c_1(c_0) = w_1 + (1+r)(w_0 - c_0)$.

- This provides optimization problem

$$c_0^* = \arg\max_{c_0 \in \mathbb{R}} U(c_0),$$

  subject to existence and uniqueness, and $\boldsymbol{c}^* = (c_0^*, c_1(c_0^*))^\top$.

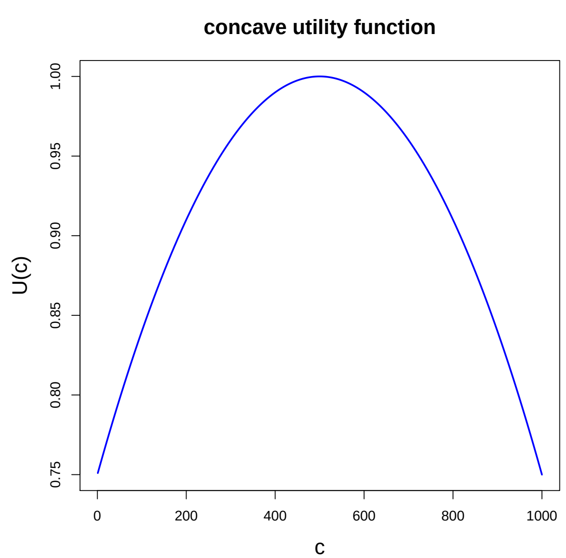
Mathematical solution of optimal consumption (2/2)

- Consider optimization problem

$$c_0^* \;=\; \arg\max_{c_0 \in \mathbb{R}} U(c_0).$$

- Under suitable regularity conditions the (global) maximum c_0^* of U is found by solving $U'(c) = 0$ and $U''(c) < 0$.

- Typically, we assume U is concave. This implies existence of at most 1 maximum.

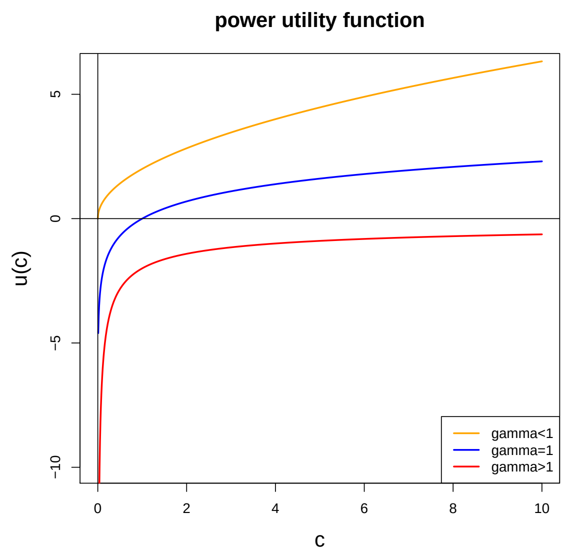


Power utility example

- Assume only positive consumptions are allowed $\boldsymbol{c} \in \mathbb{R}_+^2$.
- Define power utility function

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{for } \gamma > 0 \text{ and } \gamma \neq 1, \\ \log(c) & \text{for } \gamma = 1. \end{cases}$$

- u is concave on $\mathbb{R}_+$ with $u'(c) = c^{-\gamma} > 0$ and $u''(c) = -\gamma c^{-\gamma-1} < 0$.



Solution to the optimal consumption problem

Choosing power utility for u_0 and u_1, we have (we choose $\gamma \neq 1$)

$$\begin{aligned} U(c) &= u_0(c) + \frac{1}{1+\delta}\, u_1(c_1(c)) \\ &= \frac{c^{1-\gamma}}{1-\gamma} + \frac{1}{1+\delta}\, \frac{(w_1 + (1+r)(w_0 - c))^{1-\gamma}}{1-\gamma}. \end{aligned}$$

We calculate the score equation

$$\frac{\partial U(c)}{\partial c} = c^{-\gamma} - \frac{1+r}{1+\delta}\, (w_1 + (1+r)(w_0 - c))^{-\gamma} \stackrel{!}{=} 0.$$

This provides

$$c = \left(\frac{1+r}{1+\delta}\right)^{-1/\gamma} (w_1 + (1+r)(w_0 - c))\,.$$

Bringing all c's to the same side

$$\left[\left(\frac{1+r}{1+\delta}\right)^{1/\gamma} + (1+r)\right] c = w_1 + (1+r) w_0.$$

Finally, solving this for c provides us with optimal consumption $\boldsymbol{c}^*$ for endowment $w_0 + w_1/(1+r) > 0$

$$\begin{aligned} c_0^* &= \alpha(r;\delta,\gamma) \left[w_0 + \frac{w_1}{1+r} \right] > 0, \\ c_1^* &= (1+r)\left(1-\alpha(r;\delta,\gamma)\right) \left[w_0 + \frac{w_1}{1+r} \right] > 0, \end{aligned}$$

with

$$\alpha(r;\delta,\gamma) = \left[1 + \frac{1}{1+r} \left(\frac{1+r}{1+\delta} \right)^{1/\gamma} \right]^{-1} \in (0,1).$$

This mathematical solution has many advantages over the graphical solution because we can study the properties of the solution.

Properties of the optimal consumption

- The optimal consumption $\boldsymbol{c}$ is

$$\begin{aligned} c_0^* &= \alpha(r;\delta,\gamma)\left[w_0 + \frac{w_1}{1+r}\right], \\ &\quad \text{with} \quad \alpha(r;\delta,\gamma) = \frac{(1+r)(1+\delta)^{1/\gamma}}{(1+r)(1+\delta)^{1/\gamma} + (1+r)^{1/\gamma}}, \\ c_1^* &= (1+r)\left(1-\alpha(r;\delta,\gamma)\right)\left[w_0 + \frac{w_1}{1+r}\right], \\ &\quad \text{with} \quad (1+r)\left(1-\alpha(r;\delta,\gamma)\right) = \frac{(1+r)(1+r)^{1/\gamma}}{(1+r)(1+\delta)^{1/\gamma} + (1+r)^{1/\gamma}}. \end{aligned}$$

- This, for instance, immediately implies that c_0^* is increasing in δ and

$$\delta > r \quad \Longleftrightarrow \quad c_0^* > c_1^*.$$

- **Market Equilibrium**

Interest rate and markets

- Where does the interest rate $r > -1$ come from?

- Assume we have N financial agents with

  ⋆ each agent $1 \le i \le N$ has an endowment $\boldsymbol{w}^{(i)} = (w_0^{(i)}, w_1^{(i)})^\top \in \mathbb{R}^2$;

  ⋆ each agent $1 \le i \le N$ takes an optimal consumption $\boldsymbol{c}^{(i)}$ w.r.t. his/her utility function

$$\boldsymbol{c}^{(i)} \;=\; \underset{\boldsymbol{c}\in\mathbb{R}^2}{\arg\max}\, \mathcal{U}^{(i)}(\boldsymbol{c}) \;=\; \underset{\boldsymbol{c}\in\mathbb{R}^2}{\arg\max} \left\{ u_0^{(i)}(c_0) + \frac{1}{1+\delta^{(i)}}\, u_1^{(i)}(c_1) \right\},$$

  subject to the agent's budget constraint

$$c_0 + \frac{c_1}{1+r} \;=\; w_0^{(i)} + \frac{w_1^{(i)}}{1+r}. \tag{2}$$

  ⋆ For simplicity, assume existence of unique solutions for all interest rates $r > -1$.

Arreu–Debreu equilibrium (special case)

An equilibrium is given by an interest rate $r > -1$ and consumption streams $\boldsymbol{c}^{(i)}$ of each agent $1 \le i \le N$ if the following two conditions hold:

1. $\boldsymbol{c}^{(i)}$ maximizes the utility $\mathcal{U}^{(i)}$ of each agent $1 \le i \le N$ subject to his budget constraint (2) for interest rate r;

2. we have market clearing at times $t = 0, 1$

$$\sum_{i=1}^{N} c_t^{(i)} = \sum_{i=1}^{N} w_t^{(i)}.$$

- Remarks:
  - ⋆ If such a market equilibrium exists, the resulting interest rate $r^* > -1$ is called equilibrium rate.
  - ⋆ We may have multiple solutions or no solution to the equilibrium problem.
  - ⋆ Equilibrium is an economic principle that explains price formation.

Interpretation of equilibrium

- There is a supply and a demand for savings, i.e., carry forward part of endowments.
- Size of savings is steered by the size of interest rate r.
- In an equilibrium everyone is happy w.r.t. the given interest rate r, because there is an equality between supply and demand resulting in a full consumption of the endowments in each period $t = 0, 1$

$$\sum_{i=1}^{N} c_t^{(i)} = \sum_{i=1}^{N} w_t^{(i)}.$$

- Remark that market clearing at $t = 0$ is sufficient (due to budget constraints (2))

$$\sum_{i=1}^{N} c_1^{(i)} \overset{(2)}{=} \sum_{i=1}^{N} w_1^{(i)} + (1+r)\left(w_0^{(i)} - c_0^{(i)}\right) \overset{\text{clearing } t=0}{=} \sum_{i=1}^{N} w_1^{(i)}.$$

Coming back to the power utility example (1/2)

- Assume homogeneity among financial agents w.r.t. utility function $\mathcal{U}^{(i)} \equiv \mathcal{U}$.
- Assume power utility function

$$\mathcal{U}(\boldsymbol{c}) = \frac{c_0^{1-\gamma}}{1-\gamma} + \frac{1}{1+\delta} \frac{c_1^{1-\gamma}}{1-\gamma},$$

with optimal consumptions of agents $1 \le i \le N$

$$c_0^{(i)} = \alpha(r;\delta,\gamma) \left[w_0^{(i)} + \frac{w_1^{(i)}}{1+r} \right],$$

$$c_1^{(i)} = (1+r)\left(1-\alpha(r;\delta,\gamma)\right) \left[w_0^{(i)} + \frac{w_1^{(i)}}{1+r} \right],$$

with $\quad \alpha(r;\delta,\gamma) = \dfrac{(1+r)(1+\delta)^{1/\gamma}}{(1+r)(1+\delta)^{1/\gamma} + (1+r)^{1/\gamma}}.$

Coming back to the power utility example (2/2)

- The market clearing condition implies for $t = 0$

$$\sum_{i=1}^{N} w_0^{(i)} = \sum_{i=1}^{N} c_0^{(i)} = \alpha(r;\delta,\gamma) \sum_{i=1}^{N} \left[w_0^{(i)} + \frac{w_1^{(i)}}{1+r} \right].$$

- We isolate all terms involving the interest rate

$$(1+r)\,\frac{1-\alpha(r;\delta,\gamma)}{\alpha(r;\delta,\gamma)} \;=\; \frac{\sum_{i=1}^{N} w_1^{(i)}}{\sum_{i=1}^{N} w_0^{(i)}} \;\stackrel{\text{def.}}{=}\; 1+g,$$

where g is the growth rate of the aggregate endowment (supply).

- We receive a unique equilibrium rate r^* in this example given by

$$r^* \;=\; (1+\delta)\,(1+g)^{\gamma} - 1 \;>\; -1,$$

for impatience rate $\delta > -1$, growth rate $g > -1$ and $\gamma > 0$.

Summary of this introduction

- In the toy example of homogeneous power utilities, a single non-storable good and under certainty we receive a unique equilibrium rate.

- What about the general case:

  ⋆ heterogeneity between different financial agents in terms of utility functions;
  ⋆ other utility functions;
  ⋆ uncertainty in the second endowments $w_1^{(i)}$, $1 \leq i \leq N$, i.e., if we need to model these with random variables;
  ⋆ multi-period models $t \geq 2$;
  ⋆ multiple goods and investment vehicles;
  ⋆ exogenous factors that may change markets over time (physical capital, human capital, technological progress, regulation, etc.);
  ⋆ other economic principles besides market clearing.