Economic Theory of Financial Markets

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Chapter 2:

Utility Theory

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Economic Theory of Financial Markets

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- **Chapter 2: Utility Theory**

Two period model: set of traded positions

- We choose a sufficiently rich probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Let $\mathcal{X}$ be the set of traded positions, and assume

$$\mathcal{X} \ \subset \ L^0(\Omega, \mathcal{F}, \mathbb{P}) = \{X \text{ is a random variable on } (\Omega, \mathcal{F}, \mathbb{P})\}\,.$$

- **Interpretation.** $\mathcal{X}$ is the set of positions that can be purchased at time 0. For a given position $X \in \mathcal{X}$:

  ⋆ $X > 0$ reflects a positive payout (gain) at time 1, and
  ⋆ $X < 0$ reflects a liability (loss) at time 1.

- **Question.** If there is a choice between $X \in \mathcal{X}$ and $Y \in \mathcal{X}$, how should we make a decision between X and Y?

Preference order

A preference order on $\mathcal{X}$ is a relation $\succeq$ with the following two properties:

- completeness: for all $X, Y \in \mathcal{X}$, either $X \succeq Y$ or $Y \succeq X$;
- transitivity: for all $X, Y, Z \in \mathcal{X}$ with $X \succeq Y$ and $Y \succeq Z$, then $X \succeq Z$.

- If $X \succeq Y$, we say we prefer X over Y.
- If $X \succeq Y$ and $Y \succeq X$, we are indifferent between X and Y, write $X \sim Y$.
- If $X \succ Y$, we say that we strictly prefer X over Y.

Numerical representation

A preference order $\succeq$ on $\mathcal{X}$ allows for a numerical representation if there exists a function

$$\mathcal{U} : \mathcal{X} \to \mathbb{R}, \qquad X \mapsto \mathcal{U}(X),$$

such that for all $X, Y \in \mathcal{X}$

$$X \succeq Y \iff \mathcal{U}(X) \geq \mathcal{U}(Y).$$

- A numerical representation is not unique because $X \mapsto \widetilde{\mathcal{U}}(X) = f(\mathcal{U}(X))$ gives the same preference order as numerical representation $\mathcal{U}$ for *any* strictly increasing function $f : \mathbb{R} \to \mathbb{R}$.

- There are necessary and sufficient conditions for the existence of numerical representations of preference orders $\succeq$ on sets $\mathcal{X}$, for details we refer to Theorem 2.6 in Föllmer–Schied (2011).

▷ We will define preference orders through numerical representations.

Expected utility (1/2)

- Assume that $\mathcal{I} \subset \mathbb{R}$ is the minimal interval such that

$$X \in \mathcal{I}, \qquad \mathbb{P}\text{-a.s., for all } X \in \mathcal{X}.$$

- For instance, if $\mathcal{X}$ is a set of non-negative random variables, then $\mathcal{I} \subset \mathbb{R}_+$.

- We construct a numerical representation $\mathcal{U}$ as follows:
  Choose a function $u : \mathcal{I} \to \mathbb{R}$ and define

$$\mathcal{U}(X) = \mathbb{E}[u(X)] = \int_{\mathcal{I}} u(x) dF_X(x), \tag{1}$$

  if F_X denotes the distribution function of $X \in \mathcal{X}$.

- Function (1) generates a preference order on $\mathcal{X}_u = \{X \in \mathcal{X};\ \mathbb{E}[|u(X)|] < \infty\}$.

- In general, we assume that $\mathcal{X}$ and u are such that $\mathcal{X}_u = \mathcal{X}$.

Expected utility (2/2)

- If $u : \mathcal{I} \to \mathbb{R}$ is strictly increasing we call it a utility function.

- A numerical representation $\mathcal{U}(X) = \mathbb{E}[u(X)]$ given by (1) using a utility function u is called Von Neumann–Morgenstern utility or expected utility representation.

- **Interpretation.**

  ⋆ Each financial position $X \in \mathcal{X}$ has a price $\pi(X) \in \mathbb{R}$ at time 0 and a (random) payoff X at time 1.

  ⋆ Each financial agent will be characterized by a utility function $u : \mathcal{I} \to \mathbb{R}$. Moreover, this financial agent will have a budget constraint $B \subset \mathbb{R}$.

  ⋆ This financial agent will then solve the expected utility maximization problem

$$\underset{X \in \mathcal{X} \text{ with } \pi(X) \in B}{\arg\max} \; \mathbb{E}[u(X)].$$

  ⋆ An economic equilibrium will determine the price functional $\pi : \mathcal{X} \to \mathbb{R}$.

Risk aversion

Lemma. Any affine linear transformation $u(\cdot) \mapsto v(\cdot) = a + bu(\cdot)$, $a \in \mathbb{R}$ and $b > 0$, of utility function $u : \mathcal{I} \to \mathbb{R}$ generates the same preference order.

Proof. Use the linearity of expected values to prove this claim. $\square$

General Assumption. $\mathcal{X} \subset L^1(\Omega, \mathcal{F}, \mathbb{P}) = \left\{ X \in L^0(\Omega, \mathcal{F}, \mathbb{P});\ \mathbb{E}[|X|] < \infty \right\}$.

Definition. A financial agent with utility function $u : \mathcal{I} \to \mathbb{R}$ is called

- risk averse if $u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)]$ for all $X \in \mathcal{X}$;
- risk neutral if $u(\mathbb{E}[X]) = \mathbb{E}[u(X)]$ for all $X \in \mathcal{X}$;
- risk seeking if $u(\mathbb{E}[X]) \leq \mathbb{E}[u(X)]$ for all $X \in \mathcal{X}$.

Remark. Strictly risk averse if $u(\mathbb{E}[X]) > \mathbb{E}[u(X)]$ for all non-deterministic $X \in \mathcal{X}$.

Interpretation of risk aversion

Lemma. Assume $u : \mathcal{I} \to \mathbb{R}$ is a strictly concave utility function, then this financial agent is strictly risk averse.

Proof. Jensen's inequality implies under concavity of u: $u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)]$ for all $X \in \mathcal{X}$. Note that for non-deterministic X the inequality is strict. $\square$

- **Example:**
  - ⋆ Assume u is a strictly risk averse utility function.
  - ⋆ Assume we have a choice between the following to random payouts $X, Y \in \mathcal{X}$

$$X = \begin{cases} -1'000'000 & \text{with probability 1\%}, \\ +10'101 & \text{with probability 99\%}, \end{cases} \qquad \text{and} \qquad Y = 0.$$

  - ⋆ Do you prefer X or Y? Note $\mathbb{E}[X] = \mathbb{E}[Y] = 0$.
  - ⋆ We have $\mathbb{E}[u(X)] < u(\mathbb{E}[X]) = u(0) = \mathbb{E}[u(Y)]$, thus, we have $Y \succ X$.
  - ⋆ Under risk aversion we always prefer mean $\mathbb{E}[X]$ over its random variable X.

Examples: exponential utility function

(Regularity) Assumptions. Assume
- utility functions $u : \mathcal{I} \to \mathbb{R}$ are three times differentiable, and
- strictly risk averse utility functions are strictly concave, i.e. satisfy $u' > 0$ and $u'' < 0$.

▷ This will be assumed in the sequel without explicit further mentioning.

Exponential utility function.
Choose $\mathcal{I} = \mathbb{R}$ and $\alpha > 0$. The exponential utility function is defined by

$$u(x) = -\frac{1}{\alpha} \exp\{-\alpha x\} \qquad \text{for } x \in \mathcal{I}.$$

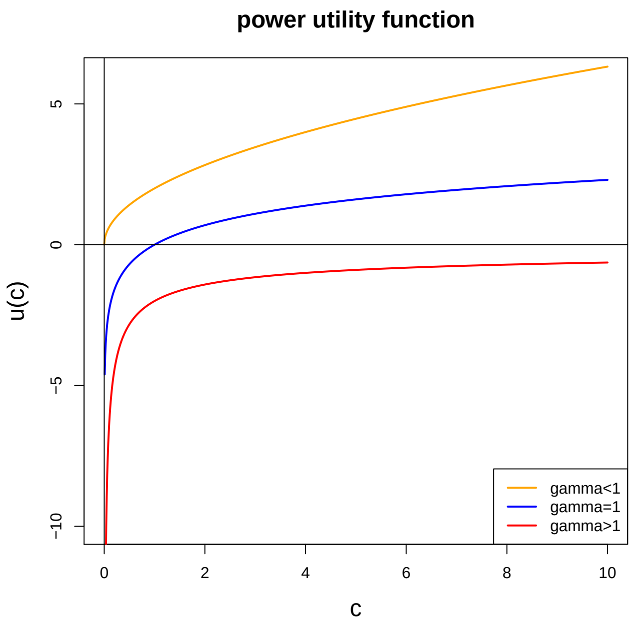
We have $u'(x) = \exp\{-\alpha x\} > 0$ and $u''(x) = -\alpha \exp\{-\alpha x\} < 0$.

Example: power utility function

Power utility function.
Choose $\mathcal{I} = \mathbb{R}_+$ and $\gamma > 0$. The power utility function is defined by

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{for } \gamma > 0 \text{ and } \gamma \neq 1, \\ \log(x) & \text{for } \gamma = 1. \end{cases}$$

We have $u'(x) = x^{-\gamma} > 0$ and $u''(x) = -\gamma x^{-\gamma-1} < 0$.



Absolute and relative risk aversion

- Absolute risk aversion (ARA): $\varrho_{\mathrm{ARA}}(x) = -\frac{u''(x)}{u'(x)}$.

- Relative risk aversion (RRA): $\varrho_{\mathrm{RRA}}(x) = -x\frac{u''(x)}{u'(x)}$.

- **Exponential utility function** (on $\mathcal{I} = \mathbb{R}$).

$$\varrho_{\mathrm{ARA}}(x) = \alpha > 0 \qquad \text{and} \qquad \varrho_{\mathrm{RRA}}(x) = \alpha x.$$

  For this reason the exponential utility function is also called CARA utility function.

- **Power utility function** (on $\mathcal{I} = \mathbb{R}_+$).

$$\varrho_{\mathrm{ARA}}(x) = \gamma x^{-1} > 0 \qquad \text{and} \qquad \varrho_{\mathrm{RRA}}(x) = \gamma > 0.$$

  For this reason the power utility function is also called CRRA utility function.

- **Certainty Equivalent**

Certainty equivalent (1/2)

Definition. Assume we have a financial agent with utility function $u : \mathcal{I} \to \mathbb{R}$ and wealth $w \in \mathbb{R}$. The certainty equivalent $x = x(F_X, w, u) \in \mathbb{R}$ of position $X \in \mathcal{X}$ with distribution $X \sim F_X$ is given by the solution of (subject to existence)

$$u(w + x) = \mathbb{E}\left[u(w + X)\right].$$

- **Interpretation and properties.**
  - ⋆ $x = x(F_X, w, u)$ is the deterministic value that makes the agent indifferent $w + x \sim w + X$.
  - ⋆ If the certainty equivalent exists, it is unique. This follows from the strictly increasing property of the utility function u.
  - ⋆ $x = x(F_X, w, u)$ indicates that the certainty equivalent depends on the distribution function F_X of X. If two random variables have the same distribution function they have the same certainty equivalent. This is also called law-invariance.

Certainty equivalent (2/2)

Lemma. Assume that the utility function $u : \mathcal{I} \to \mathbb{R}$ is strictly risk averse (strictly concave) and that the certainty equivalent $x = x(F_X, w, u) \in \mathbb{R}$ exists for given wealth $w \in \mathbb{R}$ and position $X \in \mathcal{X}$ with distribution $X \sim F_X$. We have

$$x = x(F_X, w, u) \;\leq\; \mathbb{E}[X],$$

and the inequality is strict for non-deterministic X.

Proof. We use Jensen's inequality to receive

$$u(w + x) \;=\; \mathbb{E}\left[u(w + X)\right] \;\leq\; u\left(w + \mathbb{E}[X]\right).$$

The claim then follows from the strictly increasing property of u. $\square$

Interpretation. For non-deterministic random variables X we have certainty equivalent $x = x(F_X, w, u) < \mathbb{E}[X]$. Therefore, this agent is willing to exchange X by any deterministic value $y \in (x, \mathbb{E}[X])$ as this implies

$$w + \mathbb{E}[X] \;\succeq\; w + y \;\succeq\; w + x \;\sim\; w + X.$$

More risk averse

Interpretation. For non-deterministic random variables X we have certainty equivalent $x = x(F_X, w, u) < \mathbb{E}[X]$. Therefore, this agent is willing to exchange X by any deterministic value $y \in (x, \mathbb{E}[X])$ as this implies

$$w + \mathbb{E}[X] \ \succeq \ w + y \ \succeq \ w + x \ \sim \ w + X.$$

Definition. Agent 1 with utility function $u_1 : \mathcal{I} \to \mathbb{R}$ is more risk averse than agent 2 with utility function $u_2 : \mathcal{I} \to \mathbb{R}$ if for all position $X \in \mathcal{X}$

$$u_1^{-1}\left(\mathbb{E}[u_1(X)]\right) \ \le \ u_2^{-1}\left(\mathbb{E}[u_2(X)]\right).$$

Assume certainty equivalents $x(F_X, w, u_1)$ and $x(F_X, w, u_2)$ exist, then we have

$$x_1 = x(F_X, w, u_1) \ \le \ x(F_X, w, u_2) = x_2,$$

if agent 1 is more risk averse than agent 2; thus, in this case agent 1 has a bigger interval $(x_1, \mathbb{E}[X])$ if both agents are risk averse. **Proof.** Exercise.

Regularity assumption on the set $\mathcal{X}$

- We have assumed $\mathcal{X} \subset \{X \in L^1(\Omega, \mathcal{F}, \mathbb{P});\ X \in \mathcal{I}, \mathbb{P}\text{-a.s.}\}$.

- Each random variable $X \in \mathcal{X}$ is characterized by a probability measure

$$\mu_X(\cdot) = \mathbb{P}\left[X \in \cdot\ \right] \qquad \text{on interval } \mathcal{I},$$

  and, thus, in the sequel we identify $\mathcal{X}$ with a subset of probability measures $\mathcal{M}_1(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ on $\mathcal{I}$ (note that expected utility is law-invariant).

- By an abuse of notation we use both $\mathcal{X} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{X} \subset \mathcal{M}_1(\mathcal{I}, \mathcal{B}(\mathcal{I}))$.

Assumption. $\mathcal{X} \subset \mathcal{M}_1(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ is convex and contains all point masses δ_x, $x \in \mathcal{I}$.

Convexity of $\mathcal{X}$ implies that if we choose any two measures $\mu_1, \mu_2 \in \mathcal{X}$ we have

$$\alpha\mu_1 + (1-\alpha)\mu_2 \ \in\ \mathcal{X} \qquad \text{for all } \alpha \in [0,1].$$

Main theorem on risk aversion

Theorem. Assume $\mathcal{X} \subset \mathcal{M}_1(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ is convex and contains all point masses δ_x for $x \in \mathcal{I}$. The following statements are equivalent:

(a) agent 1 is more risk averse than agent 2;

(b) $\varrho^{(1)}_{\mathrm{ARA}}(\cdot) \;\geq\; \varrho^{(2)}_{\mathrm{ARA}}(\cdot)$, where these are the ARA of agents 1 and 2, respectively;

(c) $u_1(\cdot) = (v \circ u_2)(\cdot)$ for a strictly increasing and concave function v.

Proof. Under the above assumptions, u_1 and u_2 are strictly increasing and three times differentiable. We start with item (c) and define

$$v(y) = u_1\left(u_2^{-1}(y)\right) \qquad \text{for } y \in u_2(\mathcal{I}).$$

This provides us with

$$(v \circ u_2)(x) = u_1\left(u_2^{-1}(u_2(x))\right) = u_1(x) \qquad \text{for } x \in \mathcal{I},$$

where we have used continuity and strictly increasing a couple of times.

Next we calculate the derivative of v, it is for $y \in u_2(\mathcal{I})$ given by

$$v'(y) = u_1'\left(u_2^{-1}(y)\right) \frac{d}{dy} u_2^{-1}(y) = \frac{u_1'\left(u_2^{-1}(y)\right)}{u_2'\left(u_2^{-1}(y)\right)} \;>\; 0,$$

thus, v is strictly increasing. Next we calculate the second derivative of v

$$\begin{aligned}
v''(y) &= \frac{u_1''\left(u_2^{-1}(y)\right) u_2'\left(u_2^{-1}(y)\right) - u_1'\left(u_2^{-1}(y)\right) u_2''\left(u_2^{-1}(y)\right)}{\left(u_2'\left(u_2^{-1}(y)\right)\right)^2} \frac{d}{dy} u_2^{-1}(y) \\
&= \frac{u_1''\left(u_2^{-1}(y)\right)}{\left(u_2'\left(u_2^{-1}(y)\right)\right)^2} - \frac{u_1'\left(u_2^{-1}(y)\right) u_2''\left(u_2^{-1}(y)\right)}{\left(u_2'\left(u_2^{-1}(y)\right)\right)^3} \\
&= \frac{u_1'\left(u_2^{-1}(y)\right)}{\left(u_2'\left(u_2^{-1}(y)\right)\right)^2} \left[\frac{u_1''\left(u_2^{-1}(y)\right)}{u_1'\left(u_2^{-1}(y)\right)} - \frac{u_2''\left(u_2^{-1}(y)\right)}{u_2'\left(u_2^{-1}(y)\right)}\right] \qquad (2) \\
&= \frac{u_1'\left(u_2^{-1}(y)\right)}{\left(u_2'\left(u_2^{-1}(y)\right)\right)^2} \left[\varrho_{\mathrm{ARA}}^{(2)}\left(u_2^{-1}(y)\right) - \varrho_{\mathrm{ARA}}^{(1)}\left(u_2^{-1}(y)\right)\right].
\end{aligned}$$

Note that the factor in front of the square bracket is strictly positive. Therefore, v is concave if and only if the square bracket is non-positive. This proves the equivalence of (b) and (c).

Next, we reformulate (a). We have

$$
\begin{aligned}
(a) \quad &\iff \quad u_1^{-1}\left(\mathbb{E}[u_1(X)]\right) \;\le\; u_2^{-1}\left(\mathbb{E}[u_2(X)]\right) \qquad \text{for all } X \in \mathcal{X}\\
&\iff \quad \mathbb{E}[u_1(X)] \;\le\; \left(u_1 \circ u_2^{-1}\right)\left(\mathbb{E}[u_2(X)]\right) \qquad \text{for all } X \in \mathcal{X}\\
&\iff \quad \mathbb{E}[u_1(X)] \;\le\; v\left(\mathbb{E}[u_2(X)]\right) \qquad \text{for all } X \in \mathcal{X}.
\end{aligned}
$$

Next we prove that (c) implies (a). Concavity of v implies for all $X \in \mathcal{X}$ (we use Jensen's inequality)

$$
v\left(\mathbb{E}[u_2(X)]\right) \ge \mathbb{E}[(v \circ u_2)(X)] = \mathbb{E}[u_1(X)],
$$

which is equivalent to (a).

Finally, we prove that (a) implies (b). Assume that (b) does not hold, i.e., that there exists $z \in \mathcal{I}$ such that $\varrho_{\mathrm{ARA}}^{(1)}(z) < \varrho_{\mathrm{ARA}}^{(2)}(z)$. Continuity (three times differentiability) implies that there exist an open interval $\mathcal{O} \subset \mathcal{I}$ such that $\varrho_{\mathrm{ARA}}^{(1)}(z) < \varrho_{\mathrm{ARA}}^{(2)}(z)$ for all $z \in \mathcal{O}$. Formula (2) implies that v is strictly convex on $u_2(\mathcal{O})$. Choose a probability measure $\mu \in \mathcal{X} \subset \mathcal{M}_1(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ that is supported in $\mathcal{O}$ and which is not concentrated in a single point (such a measure exists because $\mathcal{O}$ is open and non-empty, and $\mathcal{X}$ is convex and contains all point measures δ_z, $z \in \mathcal{O}$). Choose $X \sim \mu$. Then we have using Jensen's inequality in the second step

$$
\mathbb{E}[u_1(X)] \;=\; \mathbb{E}[(v \circ u_2)(X)] \;>\; v(\mathbb{E}[u_2(X)]) \;=\; u_1\left(u_2^{-1}(\mathbb{E}[u_2(X)])\right).
$$

Thus, agent 1 is *not* more risk averse than agent 2. This finishes the proof of the theorem. $\square$

More risk averse

Corollary. Assume $\mathcal{X} \subset \mathcal{M}_1(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ is convex and contains all point masses δ_x for $x \in \mathcal{I}$. The following statements are equivalent:

(a) $\varrho^{(1)}_{\rm ARA}(x) \;\ge\; \varrho^{(2)}_{\rm ARA}(x)$ for all $x \in \mathcal{I}$;

(b) $x(F_X, w, u_1) \le x(F_X, w, u_2)$ for all $X \in \mathcal{X}$ and $w \in \mathbb{R}$ for which the certainty equivalents exist.

Sketch of proof. (a) is equivalent to $u_1 = v \circ u_2$ for a strictly increasing and concave function v. The latter is equivalent to (b). $\square$

This corollary explains that the pricing interval $(x(F_X, w, u), \mathbb{E}[X])$ widens under increasing risk aversion in u.

- **Utility Indifference Price**

Utility indifference price

Definition. Assume we have a financial agent with utility function $u : \mathcal{I} \to \mathbb{R}$ and wealth $w \in \mathbb{R}$. The utility indifference price $\pi = \pi(F_X, w, u) \in \mathbb{R}$ of risky position $X \in \mathcal{X}$ with distribution $X \sim F_X$ is given by the solution of (subject to existence)

$$u(w) = \mathbb{E}\left[u(w - X + \pi)\right].$$

$\pi_r = \pi - \mathbb{E}[X]$ is called premium risk loading.

- **Interpretation.**
  - ⋆ Because of the sign switch, a positive X has now the interpretation of an insurance risk or an insurance claim.
  - ⋆ $\pi = \pi(F_X, w, u)$ is the deterministic value that makes the agent indifferent in accepting the risk X at price π or not insuring risk X, i.e. $w \sim w - X + \pi$.
  - ⋆ We have law-invariance of $\pi(F_X, w, u)$.
  - ⋆ If the utility indifference price exists it is unique. **Proof.** Exercise.
  - ⋆ 0 is the certainty equivalent of $\pi - X$, i.e., π is such that $x(F_{\pi - X}, w, u) = 0$.

Premium risk loading

Corollary. Assume the utility function $u : \mathcal{I} \to \mathbb{R}$ is strictly risk averse (strictly concave) and that the utility indifference price $\pi = \pi(F_X, w, u) \in \mathbb{R}$ exists for given wealth $w \in \mathbb{R}$ and risky position $X \in \mathcal{X}$ with distribution $X \sim F_X$. We have

$$\pi_r = \pi - \mathbb{E}[X] \geq 0,$$

and the inequality is strict for non-deterministic X.

Proof. This is an easy consequence of the lemma on slide 15 by noting that the utility indifference price π satisfies $x(F_{\pi - X}, w, u) = 0$.

$\square$

Exponential utility function indifference price

Theorem. Assume that u is a risk averse utility function.
The following are equivalent:

(a) $\pi = \pi(F_X, w, u)$ does not depend on w for all $X \in \mathcal{X}$;

(b) $u(x) = -a \exp\{-\alpha x\} + b$ for some $a > 0$, $\alpha > 0$ and $b \in \mathbb{R}$.

Proof. We use our standing assumptions here: u is three times differentiable, strictly increasing and strictly concave on $\mathcal{I}$, as well as that $\mathcal{X}$ is convex and containing all point measures δ_x of $x \in \mathcal{I}$, where $\mathcal{I}$ is an interval with non-empty interior.

We first prove that (b) implies (a). Using a lemma from above we know that we can drop a and b because any affine linear transformation of a utility function generates the same preference order. In particular, statement (b) just considers the exponential utility function. Therefore, we receive the utility indifference price as the solution π of

$$\exp\{-\alpha w\} = \mathbb{E}[\exp\{-\alpha(w - X + \pi)\}] = \exp\{-\alpha(w + \pi)\}\mathbb{E}[\exp\{\alpha X\}].$$

This implies utility indifference price which does not depend on w

$$\pi = \frac{1}{\alpha} \log\left(\mathbb{E}[\exp\{\alpha X\}]\right).$$

We prove that (a) implies (b). Since $\mathcal{X}$ is a convex set containing all point measures δ_x, $x \in \mathcal{I}$, we can choose any Bernoulli random variable X that takes values x_1 and x_2 in $\mathcal{I}$ with probabilities p and $1-p$. Choice of Bernoulli random variables will imply that all subsequent expected values and derivatives are well-defined.

Assume that $\pi(F_X, w, u)$ is the utility indifference price of such a Bernoulli random variable and for a given w in the interior of $\mathcal{I}$. Then it satisfies

$$u(w) = \mathbb{E}\left[u(w - X + \pi(F_X, w, u))\right].$$

Calculating the derivative of the above w.r.t. w implies (we use in the 2nd step that the utility indifference price does not depend on w)

$$\begin{aligned} u'(w) &= \mathbb{E}\left[u'(w - X + \pi(F_X, w, u))\left(1 + \frac{d}{dw}\pi(F_X, w, u)\right)\right] \\ &= \mathbb{E}\left[u'(w - X + \pi(F_X, w, u))\right]. \end{aligned}$$

Define $v = -u'$. Since u is risk averse, we have $u'' < 0$ and, henceforth, $v' > 0$, which implies that v is a twice differentiable utility function. This implies

$$v(w) = \mathbb{E}\left[v(w - X + \pi(F_X, w, u))\right],$$

and, henceforth, π is also the utility indifference price of v. That is, we have for all such Bernoulli random variables X

$$\pi(F_X, w, u) = \pi(F_X, w, v) = \pi.$$

This implies for all such Bernoulli random variables X

$$v^{-1}\left(\mathbb{E}\left[v(w-X+\pi)\right]\right)=w=u^{-1}\left(\mathbb{E}\left[u(w-X+\pi)\right]\right).$$

This implies that agent u and agent v have the same risk aversion, and using the main theorem on risk aversion from above we find that $\varrho^u_{\mathrm{ARA}}(\cdot)\equiv\varrho^v_{\mathrm{ARA}}(\cdot)$ on $\mathcal{I}$. The latter is equivalent to

$$\frac{u''(x)}{u'(x)}=\frac{v''(x)}{v'(x)}=\frac{u'''(x)}{u''(x)}\qquad \text{for all } x\in\mathcal{I}.$$

We calculate the first derivative of the ARA of u

$$\frac{d}{dx}\varrho^u_{\mathrm{ARA}}=-\frac{d}{dx}\frac{u''(x)}{u'(x)}=-\frac{u'''(x)u'(x)-(u''(x))}{(u'(x))^2}=-\frac{u''(x)}{u'(x)}\left[\frac{u'''(x)}{u''(x)}-\frac{u''(x)}{u'(x)}\right]=0.$$

This implies that $\varrho^u_{\mathrm{ARA}}\equiv\alpha>0$, we use risk averse here. This provides differential equation

$$u''(x)+\alpha u'(x)=0,$$

and solving this differential equation provides the exponential utility function. $\square$

Interpretation of exponential utility function

Assume u is a risk averse utility function. The following are equivalent:

(a) $\pi = \pi(F_X, w, u)$ does not depend on w for all $X \in \mathcal{X}$;

(b) $u(x) = -a \exp\{-\alpha x\} + b$ for some $a > 0$, $\alpha > 0$ and $b \in \mathbb{R}$.

- The utility indifference price under the exponential utility function does not depend on the size w of the insurance company. This is not a reasonable model property because bigger insurance companies expect to be able to better diversify claims. Henceforth, they should charge a smaller premium risk loading $\pi_r = \pi - \mathbb{E}[X]$ for bigger w. Therefore, the exponential utility function should not be used!

- Actuaries like the exponential utility function because it has nice analytical properties, i.e., the utility indifference price

$$\pi = \frac{1}{\alpha} \log \left(\mathbb{E}[\exp\{\alpha X\}] \right) = \frac{1}{\alpha} \log M_X(\alpha),$$

relies on the moment generating function M_X of X evaluated in $\alpha \in \mathbb{R}_+$.

Decreasing absolute risk aversion

Theorem. Assume that u is a risk averse utility function.
The following are equivalent:

(a) $\pi = \pi(F_X, w, u)$ is decreasing in w for all $X \in \mathcal{X}$;

(b) $\varrho^u_{\mathrm{ARA}}(x) = -\frac{u''(x)}{u'(x)}$ is decreasing in $x \in \mathcal{I}$.

Sketch of proof. The utility indifference price of w and random variable X fulfills

$$u(w) = \mathbb{E}\left[u(w - X + \pi(F_X, w, u))\right].$$

Calculating the derivative of the above w.r.t. w implies (this needs a bit of work)

$$u'(w) \quad = \quad \mathbb{E}\left[u'(w - X + \pi(F_X, w, u))\left(1 + \frac{d}{dw}\pi(F_X, w, u)\right)\right].$$

The implicit function theorem provides differentiability of π w.r.t. w and this implies that (a) is equivalent to $\frac{d}{dw}\pi(F_X, w, u) \le 0$ for all $X \in \mathcal{X}$. The latter is equivalent to

$$u'(w) \quad \le \quad \mathbb{E}\left[u'(w - X + \pi(F_X, w, u))\right] \qquad \text{for all } X \in \mathcal{X}.$$

Define $v = -u'$. This is a twice differentiable utility function due to the risk aversion of u. This implies that (a) is equivalent to

$$v(w) \quad \geq \quad \mathbb{E}\left[v(w - X + \pi(F_X, w, u))\right] \qquad \text{for all } X \in \mathcal{X}.$$

In turn this implies that (a) is equivalent to

$$\pi(F_X, w, u) \leq \pi(F_X, w, v) \qquad \text{for all } X \in \mathcal{X}.$$

This statement is equivalent to saying that agent v is more risk averse than agent u and, thus, (a) is equivalent to

$$-\frac{u''(x)}{u'(x)} \leq -\frac{v''(x)}{v'(x)} = -\frac{u'''(x)}{u''(x)} \qquad \text{for all } x \in \mathcal{I}.$$

Using the first derivative of the ARA of u this is equivalent to

$$\frac{d}{dx}\varrho^u_{\mathrm{ARA}} = -\frac{u''(x)}{u'(x)}\left[\frac{u'''(x)}{u''(x)} - \frac{u''(x)}{u'(x)}\right] \leq 0 \qquad \text{for all } x \in \mathcal{I},$$

i.e., the ARA is a decreasing function.

$\square$

Interpretation of decreasing absolute risk aversion

The following are equivalent:

(a) $\pi = \pi(F_X, w, u)$ is decreasing in w for all $X \in \mathcal{X}$;

(b) $\varrho^u_{\mathrm{ARA}}(x)$ is decreasing in $x \in \mathcal{I}$.

- The ARA should be a decreasing function in order to reflect that bigger insurance companies should charge smaller premium risk loadings π_r.

- The power utility (CRRA utility) function has this property

$$\varrho_{\mathrm{ARA}}(x) = \gamma x^{-1} \qquad \text{and} \qquad \varrho_{\mathrm{RRA}}(x) = \gamma.$$

  The CRRA utility function is only defined on $\mathcal{I} = \mathbb{R}_+$, therefore, it only allows to consider bounded risks X for utility indifference pricing. That is, we need to assume $X \leq M$, $\mathbb{P}$-a.s., for some fixed constant M.

- **Risk Exchange Economy**

Introduction to risk exchange economy

- So far, we have discussed properties of certainty equivalents.
- To consider expected utility maximization

$$\underset{X \in \mathcal{X} \text{ with } X \in B}{\arg\max} \; \mathbb{E}[u(X)],$$

  we need a budget constraint $B \subset \mathcal{X}$ as, in general, we cannot freely attain any position $X \in \mathcal{X}$.

- The budget constraint will be determined by prices $\pi(X) \in \mathbb{R}$ at time 0 of positions $X \in \mathcal{X}$, if X reflects the (random) payout at time 1.
- These prices will be calculated from a market equilibrium that describes demand and supply if financial agents are allowed to exchange positions $X \in \mathcal{X}$ at time 0.

Assumptions

- To keep things simple in this section, we choose a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $|\Omega| < \infty$ and $\mathcal{F}$ being the resulting power set on Ω.

- Assume that $\mathcal{X}$ (only) contains all strictly positive random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e.,
$$X(\omega) > 0 \qquad \text{for all } \omega \in \Omega \text{ and } X \in \mathcal{X}.$$

- Assume we have $N \geq 2$ financial agents each holding a given position at time 0 that provides payoff $X_i \in \mathcal{X}$ at time 1, $1 \leq i \leq N$.

- The total market capitalization at time 1 is given by
$$Z = \sum_{i=1}^{N} X_i \in \mathcal{X}.$$

State price deflator

- We assume that each agent i can trade at time 0 his initial holding $X_i \in \mathcal{X}$ against any other position $Y_i \in \mathcal{X}$ as long as a certain budget constraint is fulfilled.

- For this we need the notion of a price $\pi(X) \in \mathbb{R}$ of all $X \in \mathcal{X}$ at time 0.

- We introduce a financial pricing kernel φ which itself is a random variable in $\mathcal{X}$ with normalization $\mathbb{E}[\varphi] = 1$.[1]

- This financial pricing kernel allows us to define prices at time 0 for all $X \in \mathcal{X}$

$$\pi_\varphi(X) = \mathbb{E}[\varphi X] \ \in (0, \infty).$$

- Strict positivity $\varphi > 0$, $\mathbb{P}$-a.s., is crucial (and trivial in our toy example). Such financial pricing kernels are called state price deflators or stochastic discount factors. A simple example is $\varphi \equiv 1$.

[1] Normalization $\mathbb{E}[\varphi] = 1$ assumes that the interest rate is 0, we can easily generalize this to positive interest rates $r > 0$ by assuming $\mathbb{E}[\varphi] = (1 + r)^{-1}$.

Toy model for risk exchange economy

Assumptions. We have a state price deflator $\varphi \in \mathcal{X}$ and $N \geq 2$ financial agents $1 \leq i \leq N$ with:

- each agent holds an initial position $X_i \in \mathcal{X}$ at time 1;

- each agent may trade his initial position against any other position $Y_i \in \mathcal{X}$ subject to his budget constraint

$$B_i^{\varphi} = \{X \in \mathcal{X};\ \pi_{\varphi}(X) = \pi_{\varphi}(X_i)\} \quad \subset \quad \mathcal{X};$$

- each agent is described by strictly risk averse utility function u_i on $\mathbb{R}_+$.

- Each agent tries to achieve by trading

$$X_i^* \;=\; \arg\max_{X \in B_i^{\varphi}} \mathbb{E}[u_i(X)]. \qquad (3)$$

First order conditions

Theorem. The optimal position X_i^* of (3) fulfills the first order conditions

$$u_i'(X_i^*) = \lambda_i \varphi \qquad \mathbb{P}\text{-a.s.},$$

for some $\lambda_i > 0$.

Proof. Working on finite probability spaces allows us to directly apply the method of Lagrange. The Lagrange function is given by

$$\mathcal{L} = \mathbb{E}[u_i(X)] - \lambda_i \left(\pi_\varphi(X) - \pi_\varphi(X_i) \right),$$

with Lagrange multiplier $\lambda_i \in \mathbb{R}$. The optimal position X_i^* is found by maximizing the Lagrange function $\mathcal{L}$; note that u_i is a concave function and all side constraints are linear. This optimization is most easily solved by considering directional derivatives, i.e. we perturb X_i^* by a position $\widetilde{X} \in \mathcal{X}$ for small $\varepsilon \in \mathbb{R}$ such that $X = X_i^* + \varepsilon \widetilde{X} \in \mathcal{X}$. This gives us Lagrangian

$$\mathcal{L}(\varepsilon; \widetilde{X}) = \mathbb{E}\left[u_i(X_i^* + \varepsilon \widetilde{X}) \right] - \lambda_i \left(\pi_\varphi(X_i^* + \varepsilon \widetilde{X}) - \pi_\varphi(X_i) \right).$$

The optimal position X_i^* needs to provide a critical point of $\mathcal{L}(\varepsilon; \widetilde{X})$ in $\varepsilon = 0$ for all $\widetilde{X} \in \mathcal{X}$.

Therefore, we consider score equations

$$0 = \frac{d}{d\varepsilon}\mathcal{L}(\varepsilon; \widetilde{X})\bigg|_{\varepsilon=0} = \mathbb{E}\left[u_i'(X_i^*)\widetilde{X}\right] - \lambda_i \mathbb{E}[\varphi \widetilde{X}].$$

Thus, we obtain requirement

$$\mathbb{E}\left[u_i'(X_i^*)\widetilde{X}\right] = \lambda_i \mathbb{E}[\varphi \widetilde{X}] \qquad \text{for all } \widetilde{X} \in \mathcal{X}.$$

But this implies the claim (use e.g. definition of conditional expectation w.r.t. information $\mathcal{F}$), and positivity of $\lambda_i > 0$ is received because both u_i' and φ are strictly positive. $\square$

Herding behavior in our toy model

Corollary. The optimal positions

$$X_i^* = (u_i')^{-1}(\lambda_i \varphi)$$

are comonotone for $1 \le i \le N$.

Proof. Comonotonicity means that all X_i^* can be described by strictly decreasing transformation of a common latent risk factor. This is the case here, we have common latent risk factor φ and $z \mapsto (u_i')^{-1}(\lambda z)$ are strictly decreasing functions due to strict concavity of u_i. $\square$

- This can be interpreted as herding behavior because in this toy model all agents have the "same" optimal strategy.
- Remaining question: where does state price deflator φ come from?

Market clearing and state price deflator

Assumption. We require market clearing in our risk exchange economy saying that the total market capitalization is shared in an optimal equilibrium

$$Z = \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} X_i^*.$$

Theorem. Under market clearing the optimal asset allocations X_i^* are comonotonic to the market capitalization Z, and $Z = v(\varphi)$ for a strictly decreasing function v.

Proof. Market clearing provides

$$Z = \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} X_i^* = \sum_{i=1}^{N} (u_i')^{-1} (\lambda_i \varphi) = v(\varphi),$$

where the latter defines function v.

$\square$

Interpretation of theorem

Theorem. Under market clearing the optimal asset allocations X_i^* are comonotonic to the market capitalization Z, and $Z = v(\varphi)$ for a strictly decreasing function v.

- In this toy example market clearing provides a pricing function π_φ that can be calculated from the market capitalization. Thus, prices are given endogenously under market clearing.
- The agents diversify all idiosyncratic risks and are only left by systematic risk reflected by Z which can be interpreted as the overall growth of the economy.
- This toy example generalizes the savings example of the introduction where we have assumed that there is no uncertainty at time 1. In that example, the equilibrium rate r^* was (also) determined from the growth rate g of the economy.
- By the Radon–Nikdym derivative $d\mathbb{P}^*/d\mathbb{P} = \varphi$ we can interpret the state price deflator φ in terms of a pricing measure $\mathbb{P}^*$ because $\pi_\varphi(X) = \mathbb{E}[\varphi X] = \mathbb{E}^*[X]$.

Example: exponential utility function

- We assume that all financial agents have an exponential utility function

$$u_i(x) = -\frac{1}{\alpha_i}\exp\{-\alpha_i x\},$$

where we allow for heterogeneity $\alpha_i > 0$ between different financial agents i.

- We have

$$(u_i')^{-1}(y) = -\frac{1}{\alpha_i}\log y.$$

- This provides us with optimal positions

$$X_i^* = -\frac{1}{\alpha_i}\log(\lambda_i \varphi) = -\frac{1}{\alpha_i}\log(\lambda_i) - \frac{1}{\alpha_i}\log(\varphi).$$

- Market clearing requires

$$Z = \sum_{i=1}^N X_i = \sum_{i=1}^N X_i^* = -\sum_{i=1}^N \frac{1}{\alpha_i}\log(\lambda_i) - \log(\varphi)\sum_{i=1}^N \frac{1}{\alpha_i}.$$

- This motivates definition of the aggregate market risk aversion

$$\alpha^* = \left(\sum_{i=1}^N \frac{1}{\alpha_i} \right)^{-1} \;>\; 0,$$

which implies

$$Z \;=\; -\sum_{i=1}^N \frac{1}{\alpha_i} \log(\lambda_i) - \frac{\log(\varphi)}{\alpha^*}.$$

- This gives us equilibrium state price deflator

$$\varphi = \exp\left\{ -\alpha^* Z - \alpha^* \sum_{i=1}^N \frac{1}{\alpha_i} \log(\lambda_i) \right\}.$$

- Normalization $\mathbb{E}[\varphi] = 1$ gives us

$$\varphi = \frac{\exp\{-\alpha^* Z\}}{\mathbb{E}[\exp\{-\alpha^* Z\}]}. \tag{4}$$

- Thus, budget constraints under the equilibrium state price deflator are given by

$$\pi_\varphi(X_i) = \mathbb{E}[\varphi X_i] = \frac{\mathbb{E}[\exp\{-\alpha^* Z\} X_i]}{\mathbb{E}[\exp\{-\alpha^* Z\}]}.$$

  Note that $\pi_\varphi(X_i) < \mathbb{E}[X_i]$ if $\exp\{-\alpha^* Z\}$ and X_i are negatively correlated, i.e., in that case we want a positive expected return.

- The individual Lagrange multipliers λ_i are determined from the budget constraint

$$\pi_\varphi(X_i^*) = \pi_\varphi\left(-\frac{1}{\alpha_i}\log(\lambda_i) - \frac{1}{\alpha_i}\log(\varphi)\right) = \pi_\varphi(X_i),$$

  using (4). This gives us optimal position

$$X_i^* \;=\; \pi_\varphi(X_i) + \frac{\alpha^*}{\alpha_i}(Z - \pi_\varphi(Z)) \;=\; \left(\pi_\varphi(X_i) - \frac{\alpha^*}{\alpha_i}\pi_\varphi(Z)\right) + \frac{\alpha^*}{\alpha_i}Z.$$

- The expected wealth at time 1 is $\mathbb{E}[X_i^*] > \pi_\varphi(X_i)$ due to negative correlation.