Economic Theory of Financial Markets

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Chapter 4:

Capital Asset Pricing Model (CAPM)

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Economic Theory of Financial Markets

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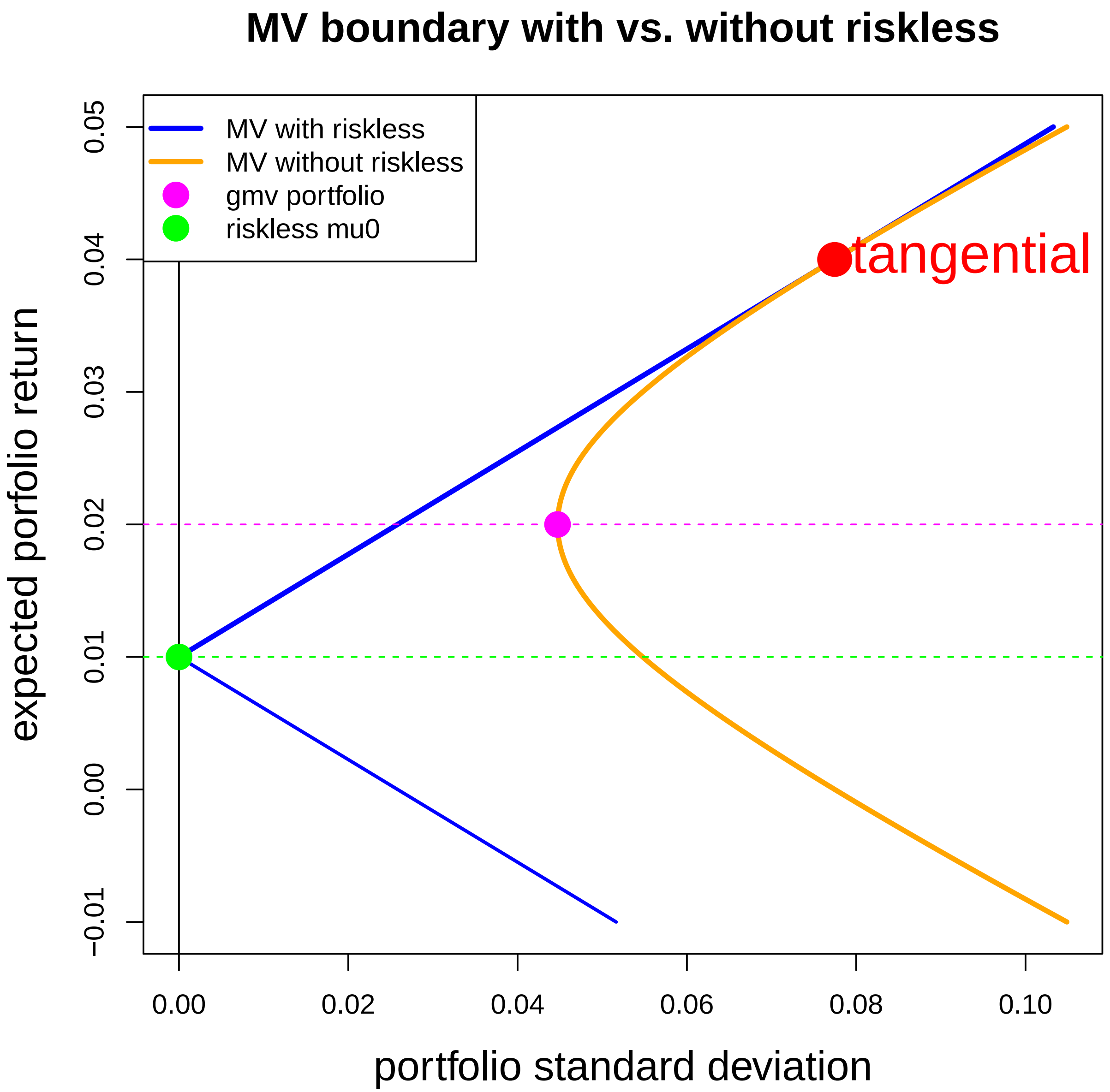
- **Chapter 4: Capital Asset Pricing Model (CAPM)**

- **Recall the Tangential Portfolio**

MV boundary: with vs. without riskless asset



Recall the tangential portfolio

Definition. The mean-variance portfolio $\widetilde{\boldsymbol{x}}_\rho \in \mathbb{R}^{n+1}$ with $\boldsymbol{x}_\rho^\top \boldsymbol{e} = 1$ is called tangential portfolio and its return is denoted by $\rho_{\tan}$.

Proposition. Assume $\rho_{\text{gmv}} = b/a \neq \mu_0$. There exists a unique tangential portfolio $\widetilde{\boldsymbol{x}}_{\tan} = \widetilde{\boldsymbol{x}}_{\rho_{\tan}}$ given by

$$\rho_{\tan} = \mu_0 + \frac{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{e}} \qquad \text{and} \qquad \boldsymbol{x}_{\tan} = \frac{\rho_{\tan}^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e.$$

Note that $\rho_{\tan}^e \neq 0$ because $\boldsymbol{x}_{\tan}^\top \boldsymbol{e} = 1$.

Proof. See last chapter. $\square$

Herding effect of mean-variance portfolios

Proposition. Assume $\rho_{\rm gmv} = b/a \neq \mu_0$. Every mean-variance portfolio $\widetilde{\boldsymbol{x}}_\rho$, $\rho \in \mathbb{R}$, is a linear combination of the tangential portfolio $\widetilde{\boldsymbol{x}}_{\rm tan}$ and the riskless portfolio $\widetilde{\boldsymbol{x}}_0 = (1, 0, \dots, 0)^\top \in \mathbb{R}^{n+1}$.

Proof. We decouple a mean-variance portfolio as follows: the risky assets are given by

$$\boldsymbol{x}_\rho = \frac{\rho^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e = \frac{\rho^e}{\rho^e_{\rm tan}} \, \frac{\rho^e_{\rm tan}}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e = \frac{\rho^e}{\rho^e_{\rm tan}} \, \boldsymbol{x}_{\rm tan}.$$

The investment in the riskless asset satisfies

$$(\widetilde{\boldsymbol{x}}_\rho)_0 = 1 - \boldsymbol{x}_\rho^\top \boldsymbol{e} = 1 - \frac{\rho^e}{\rho^e_{\rm tan}} \, \boldsymbol{x}_{\rm tan}^\top \boldsymbol{e} = 1 - \frac{\rho^e}{\rho^e_{\rm tan}}.$$

This implies

$$\widetilde{\boldsymbol{x}}_\rho = \frac{\rho^e}{\rho^e_{\rm tan}} \, \widetilde{\boldsymbol{x}}_{\rm tan} + \left(1 - \frac{\rho^e}{\rho^e_{\rm tan}}\right) \widetilde{\boldsymbol{x}}_0.$$

This proves the claim. $\square$

- **Financial Market Model**

Financial market model and economic assumption

Assumptions.

- **Supply.** We have $n+1$ financial assets fulfilling assumptions (A1)-(A2) from above. Moreover, we assume $\rho_{\text{gmv}} \neq \mu_0$. The total value at time 0 of asset $0 \leq j \leq n$ is given by $M_j > 0$, and the total market capitalization at time 0 of risky assets is given by $M = \sum_{j=1}^{n} M_j$.

- **Demand.** We have N financial agents each holding a mean-variance portfolio $\widetilde{\boldsymbol{x}}^{(i)} \in \mathbb{R}^{n+1}$ with expected return ρ_i, and having initial wealth w_i, $1 \leq i \leq N$.

Economic principle. We assume market clearing, saying Supply=Demand.

Question. What does this imply for the expected returns $\widetilde{\boldsymbol{\mu}}$?

Analysis of supply and demand

- **Supply.** The total market capitalization of risky assets is at time 0 given by

$$(M_1, \ldots, M_n)^\top = M \left(\frac{M_1}{M}, \ldots, \frac{M_n}{M} \right)^\top \stackrel{\text{def.}}{=} M \boldsymbol{x}^{(M)},$$

with weights $\boldsymbol{x}^{(M)} \in \mathbb{R}^n$ satisfying $\sum_{j=1}^n x_j^{(M)} = 1$.

- **Demand.** Each financial agent $1 \le i \le N$ is a mean-variance optimizer and, henceforth, holds assets

$$w_i \widetilde{\boldsymbol{x}}^{(i)} = w_i \left(\frac{\rho_i^e}{\rho_{\tan}^e} \, \widetilde{\boldsymbol{x}}_{\tan} + \left(1 - \frac{\rho_i^e}{\rho_{\tan}^e} \right) \widetilde{\boldsymbol{x}}_0 \right).$$

Market clearing

- Supply equal to demand implies for the risky assets

$$M\boldsymbol{x}^{(M)} \;=\; \sum_{i=1}^{N} w_i \frac{\rho_i^e}{\rho_{\tan}^e}\, \boldsymbol{x}_{\tan} \;=\; \left(\sum_{i=1}^{N} w_i \frac{\rho_i^e}{\rho_{\tan}^e}\right) \boldsymbol{x}_{\tan}.$$

- As an immediate consequence of market clearing we see

$$\boldsymbol{x}^{(M)} = \boldsymbol{x}_{\tan} \qquad \text{and} \qquad M = \sum_{i=1}^{N} \frac{w_i(\rho_i - \mu_0)}{\rho_{\tan}^e},$$

in particular, the tangential portfolio is equal to the market portfolio of risky assets.

- **CAPM Formula**

CAPM formula

Theorem. (Sharpe-Lintner-Mossin (1964-1966)). Under the above assumptions and market clearing we receive for all assets $1 \leq j \leq n$

$$\mu_j - \mu_0 = \beta_j \left(r^{(M)} - \mu_0 \right),$$

with expect market return of risky assets $r^{(M)} = \mathbb{E}[(\boldsymbol{x}^{(M)})^\top \boldsymbol{R}]$ and beta's

$$\beta_j = \frac{\mathrm{Cov}(R_j, (\boldsymbol{x}^{(M)})^\top \boldsymbol{R})}{\mathrm{Var}((\boldsymbol{x}^{(M)})^\top \boldsymbol{R})}.$$

Proof. Since $\boldsymbol{x}^{(M)} = \boldsymbol{x}_{\tan}$, the market portfolio is a mean-variance portfolio with $r^{(M)} = \rho_{\tan}$. Choose unit vector $\boldsymbol{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^n$ and consider

$$\begin{aligned} \mathrm{Cov}(R_j, (\boldsymbol{x}^{(M)})^\top \boldsymbol{R}) &= \boldsymbol{e}_j^\top \Sigma \boldsymbol{x}^{(M)} = \frac{\rho^e_{\tan}}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \boldsymbol{e}_j^\top \Sigma \Sigma^{-1} \boldsymbol{\mu}^e \\ &= \frac{\rho^e_{\tan}}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \mu^e_j = \ldots = \frac{\mathrm{Var}((\boldsymbol{x}^{(M)})^\top \boldsymbol{R})}{\rho^e_{\tan}} \left(\mu_j - \mu_0 \right). \end{aligned}$$

□

Interpretation of CAPM formula

We have expected returns

$$\mu_j = \mu_0 + \beta_j \left(r^{(M)} - \mu_0 \right),$$

with beta's

$$\beta_j = \frac{\mathrm{Cov}(R_j, (\boldsymbol{x}^{(M)})^\top \boldsymbol{R})}{\mathrm{Var}((\boldsymbol{x}^{(M)})^\top \boldsymbol{R})}.$$

- The expected returns are determined by the riskless return μ_0, the expected market return $r^{(M)}$ and the β_j's, E.g. $\beta_j = 1$ means that asset j is expected to perform as the market, often smaller firms have smaller β_j's.

- The β_j's are estimated with regression from time series.

- Small correlation of return R_j with the market return gives a small β_j and, henceforth, a small expected return μ_j. I.e. assets that have low correlation with the market have higher prices (because investors prefer them for diversification).

Interpretation of CAPM formula

We have expected returns

$$\mu_j = \mu_0 + \beta_j \left(r^{(M)} - \mu_0 \right),$$

with beta's

$$\beta_j = \frac{\mathrm{Cov}(R_j, (\boldsymbol{x}^{(M)})^\top \boldsymbol{R})}{\mathrm{Var}((\boldsymbol{x}^{(M)})^\top \boldsymbol{R})}.$$

- Underlying assumptions of the CAPM formula that are often criticized:
  - ⋆ All financial agents are mean-variance optimizers.
  - ⋆ All financial agents work with the same mean $\boldsymbol{\mu}$ and variance Σ (estimates).
  - ⋆ We have a closed market and only one currency.
  - ⋆ Individual financial agents cannot influence prices (everyone is price taker).

- Further points that lead to discussions:
  - ⋆ The model does not clearly separate endogenous from exogenous factors.
  - ⋆ CAPM is a one-factor formula, multifactor extensions are considered, see e.g. Fama-French (1993) 3-factor model.