Economic Theory of Financial Markets

Mario Wüthrich

Chapter 3:

Mean-Variance Analysis

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Economic Theory of Financial Markets

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- **Chapter 3: Mean-Variance Analysis**

- **Financial Portfolios and Returns**

Financial assets in the two period problem

- Assume we have $n+1$ financial assets with with values $S_j^{(0)} > 0$ at time 0 and values $S_j^{(1)}$ at time 1, $0 \leq j \leq n$.

- We assume that the values at time 0 are known, and the values at time 1 are modeled by random variables on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- If a state price deflator $\varphi > 0$, $\mathbb{P}$-a.s., is given, these values are calculated as

$$S_j^{(0)} = \mathbb{E}[\varphi S_j^{(1)}] \qquad \text{for } 0 \leq j \leq n.$$

  For the moment, we just assume that these values $S_j^{(0)}$ are given "reasonably".

- Asset $j = 0$, will play the role of the riskless asset, i.e., its value $S_0^{(1)}$ is assumed to be perfectly known at time 0.

- Assets $j = 1, \ldots, n$ model the risky assets, and we make the assumption that $S_j^{(1)}$ are non-deterministic random variables seen from time 0.

Vector notation

- We distinguish vectors with n components and with $n+1$ components, respectively. This distinction will be highlighted by the following notation

$$\begin{aligned} \boldsymbol{x} &= (x_1, \ldots, x_n)^\top \in \mathbb{R}^n, \\ \widetilde{\boldsymbol{x}} &= (x_0, \ldots, x_n)^\top \in \mathbb{R}^{n+1}. \end{aligned}$$

The latter includes the riskless asset and the former does not.

Asset portfolios and returns (1/2)

- An asset portfolio is given by a vector $\widetilde{\boldsymbol{a}} \in \mathbb{R}^{n+1}$. It has initial value at time 0 (deterministic)

$$w_0 = w_0(\widetilde{\boldsymbol{a}}) = \sum_{j=0}^{n} a_j S_j^{(0)} = \widetilde{\boldsymbol{a}}^\top \widetilde{\boldsymbol{S}}^{(0)},$$

and it generates wealth at time 1 (random payoff)

$$W_1 = W_1(\widetilde{\boldsymbol{a}}) = \sum_{j=0}^{n} a_j S_j^{(1)} = \widetilde{\boldsymbol{a}}^\top \widetilde{\boldsymbol{S}}^{(1)}.$$

- The latter is usually reformulated for $w_0(\widetilde{\boldsymbol{a}}) \neq 0$ (by assumption $S_j^{(0)} > 0$)

$$\begin{aligned} W_1(\widetilde{\boldsymbol{a}}) &= \sum_{j=0}^{n} a_j S_j^{(1)} = w_0(\widetilde{\boldsymbol{a}}) + \sum_{j=0}^{n} a_j \left(S_j^{(1)} - S_j^{(0)} \right) \\ &= w_0(\widetilde{\boldsymbol{a}}) \left(1 + \sum_{j=0}^{n} \frac{a_j S_j^{(0)}}{w_0(\widetilde{\boldsymbol{a}})} \, \frac{S_j^{(1)} - S_j^{(0)}}{S_j^{(0)}} \right). \end{aligned}$$

Asset portfolios and returns (2/2)

- We define (relative) returns R_j and portfolio weights x_j of assets $0 \le j \le n$ as follows
$$R_j = \frac{S_j^{(1)} - S_j^{(0)}}{S_j^{(0)}} \qquad \text{and} \qquad x_j = \frac{a_j S_j^{(0)}}{w_0(\widetilde{\boldsymbol{a}})}.$$
Note that the x_j are indeed weights because $\sum_{j=0}^n x_j = 1$.

- Thus, for $w_0 = w_0(\widetilde{\boldsymbol{a}}) \neq 0$, we have
$$W_1 \;=\; w_0 \left(1 + \sum_{j=0}^n x_j R_j\right) = w_0 \left(1 + \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}}\right),$$
with
  - ⋆ initial value w_0, assumed to be different from 0;
  - ⋆ investment strategy $\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1}$ with $\widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1$ for $\widetilde{\boldsymbol{e}} = (1, \ldots, 1)^\top$; and
  - ⋆ random $(n+1)$-dimensional return vector $\widetilde{\boldsymbol{R}}$.

General assumptions throughout this chapter

- The initial value $w_0 \neq 0$.

- Set $\widetilde{\boldsymbol{e}} = (1, \ldots, 1)^\top \in \mathbb{R}^{n+1}$.

- The investment strategy $\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1}$ satisfies $\widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1$ (full investment).

- The random return vector $\widetilde{\boldsymbol{R}}$ has a first riskless component $R_0 = \mu_0 > 0$, and the n risky assets have finite second moments with

$$\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{R}] \in \mathbb{R}^n \qquad \text{and} \qquad \Sigma = \mathrm{Cov}(\boldsymbol{R}) \in \mathbb{R}^{n \times n} \quad \text{is positive definite.}$$

Simple properties of portfolio returns

- The expected portfolio wealth is given by

$$\mathbb{E}[W_1] = w_0 \left(1 + \mathbb{E}\left[\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}}\right]\right) = w_0 \left(1 + \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}}\right).$$

This motives to define the expected portfolio return of $\widetilde{\boldsymbol{x}}$ by

$$r = \frac{\mathbb{E}[W_1] - w_0}{w_0} = \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} \ \in \ \mathbb{R}.$$

- The portfolio variance is given by

$$\mathrm{Var}(W_1) = w_0^2 \ \mathrm{Var}\left(\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{R}}\right) = w_0^2 \ \mathrm{Var}\left(\boldsymbol{x}^\top \boldsymbol{R}\right) = w_0^2 \ \boldsymbol{x}^\top \Sigma \boldsymbol{x} \ > \ 0,$$

for $\boldsymbol{x} \neq 0$, because Σ is positive definite and $w_0 \neq 0$.

- **Motivation of Mean-Variance Portfolio Optimization**

Exponential utility and Gaussian returns

- This section aims at motivating mean-variance portfolio optimization, but it should not be used beyond that because the assumptions made in this section are not realistic in practice.

We make two rather restrictive additional assumptions:

- The random return vector follows a Gaussian distribution $\boldsymbol{R} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.
- The financial agent has exponential utility function with parameter $\alpha > 0$.

- These assumptions are *only* used in this section and they are not necessary for the general mean-variance optimization framework (which is a distribution-free approach relying on the first two moments).

On log-normal distributions

Lemma. Under multivariate Gaussian returns and exponential utility we have

$$\mathbb{E}[u(W_1)] = -\frac{1}{\alpha}\exp\left\{-\alpha w_0\left(1+\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}\right)+\frac{\alpha^2}{2}w_0^2\,\boldsymbol{x}^\top\Sigma\boldsymbol{x}\right\}.$$

Proof. The distribution of the wealth at time 1 is given by

$$W_1 = w_0\left(1+\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{R}}\right) \sim \mathcal{N}\left(w_0(1+\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}), w_0^2\,\boldsymbol{x}^\top\Sigma\boldsymbol{x}\right).$$

Therefore, $\exp\{-\alpha W_1\}$ has a log-normal distribution with mean and variance parameters

$$-\alpha w_0(1+\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}) \qquad \text{and} \qquad \alpha^2\, w_0^2\,\boldsymbol{x}^\top\Sigma\boldsymbol{x}.$$

This proves the lemma. $\square$

Utility maximization problem

Under the above multivariate Gaussian returns and exponential utility assumptions we can consider the following portfolio optimization problem

$$\widetilde{\boldsymbol{x}}_\rho = \underset{\widetilde{\boldsymbol{x}}\in\mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top\widetilde{\boldsymbol{x}}=1;\ \widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}=\rho}{\arg\max} \mathbb{E}[u(W_1)],$$

for a pre-defined expected return level $\rho \in \mathbb{R}$ (we come back to this on slide 67).

This optimization problem can be simplified using the above lemma

$$\begin{aligned}
\widetilde{\boldsymbol{x}}_\rho &= \underset{\widetilde{\boldsymbol{x}}\in\mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top\widetilde{\boldsymbol{x}}=1;\ \widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}=\rho}{\arg\max} -\frac{1}{\alpha}\exp\left\{-\alpha w_0\left(1+\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}\right)+\frac{\alpha^2}{2}w_0^2\ \boldsymbol{x}^\top\Sigma\boldsymbol{x}\right\} \\
&= \underset{\widetilde{\boldsymbol{x}}\in\mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top\widetilde{\boldsymbol{x}}=1;\ \widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}=\rho}{\arg\max} -\frac{1}{\alpha}\exp\left\{\frac{\alpha^2}{2}w_0^2\ \boldsymbol{x}^\top\Sigma\boldsymbol{x}\right\} \\
&= \underset{\widetilde{\boldsymbol{x}}\in\mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top\widetilde{\boldsymbol{x}}=1;\ \widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}=\rho}{\arg\min} \boldsymbol{x}^\top\Sigma\boldsymbol{x}.
\end{aligned}$$

Markowitz (1952) problem

Choose target returns $\rho, r \in \mathbb{R}$. The Markowitz problem is given by

$$\widetilde{\boldsymbol{x}}_\rho = \underset{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1;\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} = \rho}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

We can also study the following problem

$$\widetilde{\boldsymbol{x}}_r^+ = \underset{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1;\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} \geq r}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

Below we also discuss what happens

- if we drop the riskless asset, and/or
- if we drop the expected return constraint.

- **Convex Optimization Interlude**

Optimization: preliminaries (1/2)

- The subsequent derivations heavily rely on convex optimization. Therefore, we briefly recall the methods of Lagrange and of Karush, Kuhn and Tucker (KKT).

- A matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if $\boldsymbol{x}^\top A \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq 0$.

- A positive definite matrix A is invertible and its inverse A^{-1} is positive definite.

- Assume $\boldsymbol{R}$ is a random vector with covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. We have for matrices $A, C \in \mathbb{R}^{n \times n}$ and vectors $\boldsymbol{b}, \boldsymbol{d} \in \mathbb{R}^n$

$$\mathrm{Cov}(A\boldsymbol{R} + \boldsymbol{b}, C\boldsymbol{R} + \boldsymbol{d}) = A \Sigma C^\top.$$

Optimization: preliminaries (2/2)

- For a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, the gradient is

$$\nabla f(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{pmatrix} \frac{\partial f(\boldsymbol{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\boldsymbol{x})}{\partial x_n} \end{pmatrix} \in \mathbb{R}^n,$$

  and the Hessian is

$$\nabla^2 f(\boldsymbol{x}) = \frac{\partial^2 f(\boldsymbol{x})}{\partial \boldsymbol{x}^2} = \begin{pmatrix} \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f(\boldsymbol{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_n^2} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

- **Example.** Choose $f(\boldsymbol{x}) = \boldsymbol{x}^\top A \boldsymbol{x}$ for $A \in \mathbb{R}^{n \times n}$. Gradient and Hessian of f are

$$\nabla f(\boldsymbol{x}) = \left(A + A^\top\right) \boldsymbol{x} \qquad \text{and} \qquad \nabla^2 f(\boldsymbol{x}) = A + A^\top.$$

Unconstraint (local) maximum

- Choose a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. Unconstraint local maximums $\boldsymbol{x} \in \mathbb{R}^n$ of f are found by solving

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= 0, & \quad (1)\\ \boldsymbol{z}^\top \left(\nabla^2 f(\boldsymbol{x})\right) \boldsymbol{z} &< 0 \qquad \text{for all } \boldsymbol{z} \neq 0. & \quad (2) \end{aligned}$$

  Condition (1) guarantees that we have a critical point of f, and negative definiteness (2) guarantees that this critical point is a local maximum.

- If f is a concave function, then any critical point is *the* global maximum.

- **Example.** Choose $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{b}^\top \boldsymbol{x}$ with $A \in \mathbb{R}^{n\times n}$ symmetric and negative definite, and $\boldsymbol{b} \in \mathbb{R}^n$. Condition (1) gives

$$\nabla f(\boldsymbol{x}) = A\boldsymbol{x} - \boldsymbol{b} = 0, \qquad \text{thus} \qquad \boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

  Moreover, $\nabla^2 f(\boldsymbol{x}) = A$ is negative definite, and $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is *the* global maximum.

Maximum with equality constraint

- Choose twice differentiable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ and solve for given $a \in \mathbb{R}$

$$\underset{\boldsymbol{x} \in \mathbb{R}^n;\ g(\boldsymbol{x})=a}{\arg\max} \quad f(\boldsymbol{x}).$$

- Define Lagrange function for Lagrange multiplier $\lambda \in \mathbb{R}$

$$\mathcal{L}(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) - \lambda \left(g(\boldsymbol{x}) - a \right).$$

- Constraint local maximums $\boldsymbol{x} \in \mathbb{R}^n$ are found by solving

$$\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda)}{\partial \boldsymbol{x}} = \nabla f(\boldsymbol{x}) - \lambda \nabla g(\boldsymbol{x}) = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda)}{\partial \lambda} = -\left(g(\boldsymbol{x}) - a \right) = 0, \tag{4}$$

$+$ second order conditions.

Example: maximum with equality constraint

Choose $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{b}^\top\boldsymbol{x}$ with $A \in \mathbb{R}^{n\times n}$ symmetric and negative definite. Moreover, set side constraint $g(\boldsymbol{x}) = \boldsymbol{d}^\top\boldsymbol{x} = a$ for $\boldsymbol{d} \neq 0$. Conditions (3) and (4) are

$$\begin{aligned}
\frac{\partial \mathcal{L}(\boldsymbol{x},\lambda)}{\partial \boldsymbol{x}} &= A\boldsymbol{x} - \boldsymbol{b} - \lambda\boldsymbol{d} &= 0,\\
\frac{\partial \mathcal{L}(\boldsymbol{x},\lambda)}{\partial \lambda} &= -\left(\boldsymbol{d}^\top\boldsymbol{x} - a\right) &= 0.
\end{aligned}$$

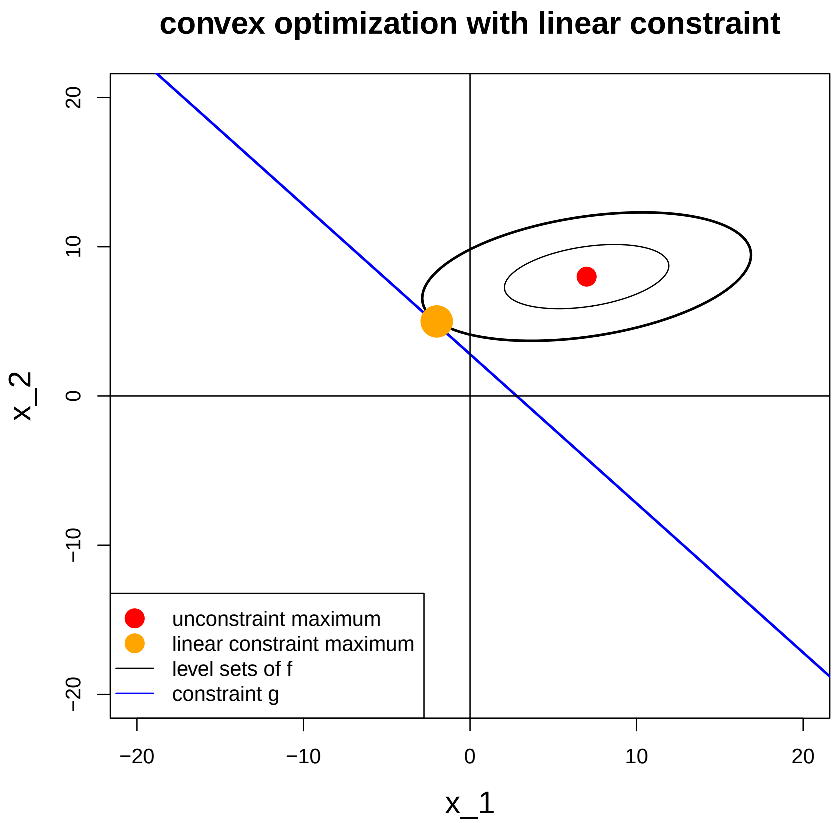
The first equation gives us $\boldsymbol{x} = A^{-1}\boldsymbol{b} + \lambda A^{-1}\boldsymbol{d}$. Plugging this into the second equation and using positive definiteness of A gives us

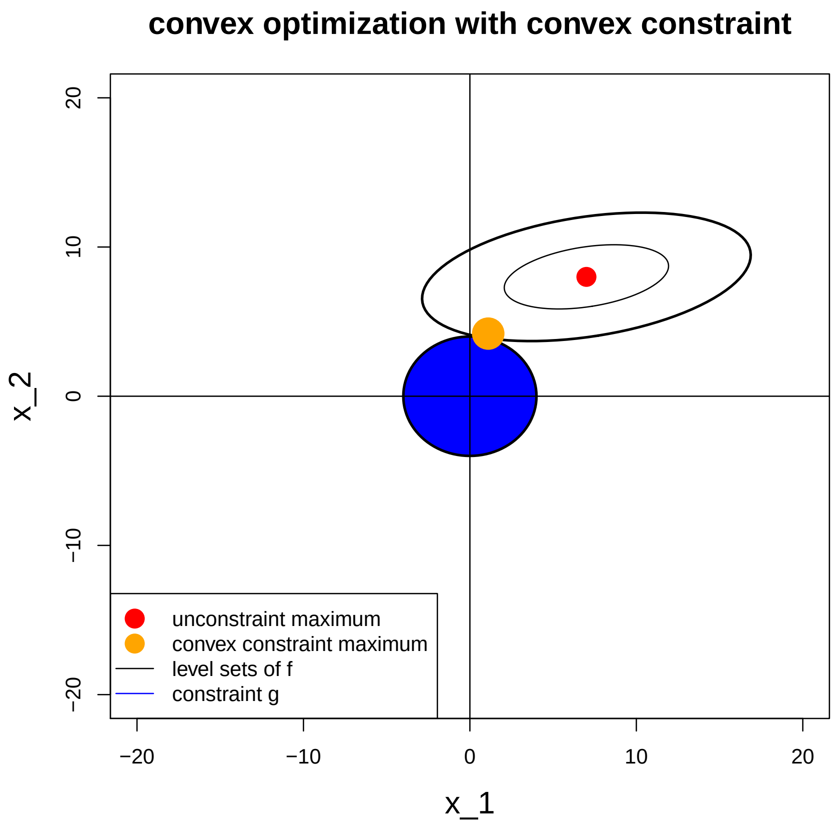
$$\lambda = \frac{a - \boldsymbol{d}^\top A^{-1}\boldsymbol{b}}{\boldsymbol{d}^\top A^{-1}\boldsymbol{d}}.$$

Thus, we have critical point

$$\boldsymbol{x} \;=\; A^{-1}\boldsymbol{b} + \frac{a - \boldsymbol{d}^\top A^{-1}\boldsymbol{b}}{\boldsymbol{d}^\top A^{-1}\boldsymbol{d}} A^{-1}\boldsymbol{d}.$$

Constraint convex optimization





Maximum with inequality constraint

Choose twice differentiable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ and solve for given $a \in \mathbb{R}$

$$\underset{\boldsymbol{x} \in \mathbb{R}^n;\; g(\boldsymbol{x}) \geq a}{\arg\max} \quad f(\boldsymbol{x}).$$

Define Lagrange function for Lagrange multiplier $\lambda \in \mathbb{R}$ and $b \geq a$

$$\mathcal{L}(\boldsymbol{x}, \lambda, b) = f(\boldsymbol{x}) - \lambda \left(g(\boldsymbol{x}) - b \right).$$

Constraint local maximums $\boldsymbol{x} \in \mathbb{R}^n$ are found by solving the KKT conditions

$$\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda, b)}{\partial \boldsymbol{x}} = \nabla f(\boldsymbol{x}) - \lambda \nabla g(\boldsymbol{x}) = 0, \tag{5}$$

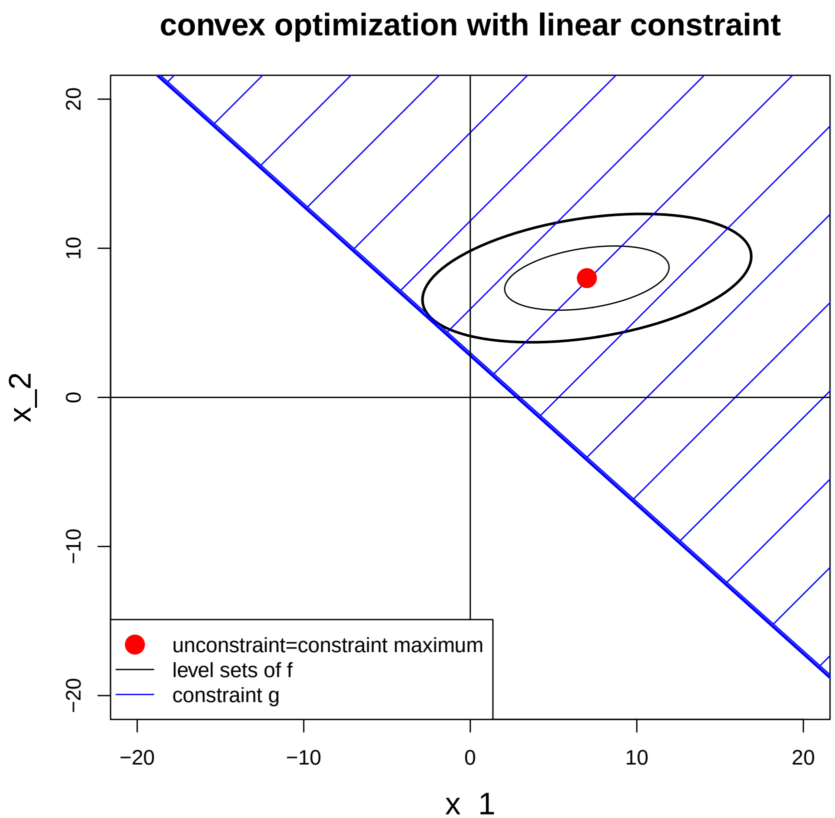
$$\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda, b)}{\partial \lambda} = -\left(g(\boldsymbol{x}) - b \right) = 0, \tag{6}$$

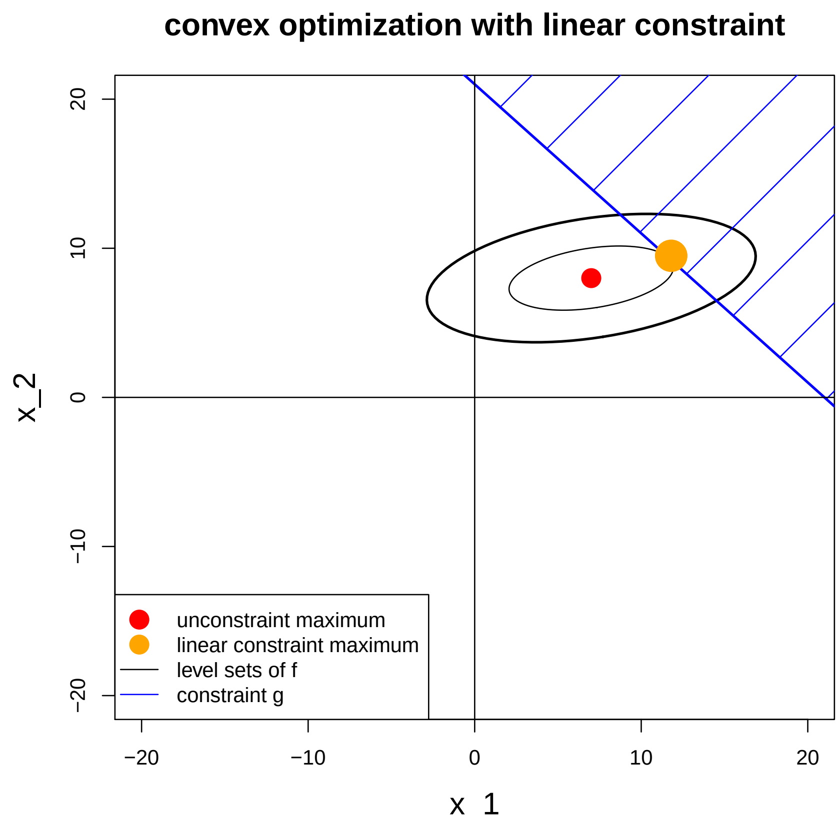
$$\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda, b)}{\partial b} = \lambda \leq 0, \tag{7}$$

$$(b - a)\lambda = 0, \tag{8}$$

$+$ second order conditions.

Inequality constraint convex optimization





Linear inequality constraint to the upper right of the blue line:
$\lambda = 0$: global (inner) maximum $b^* > a$
$\lambda < 0$: constraint (boundary) maximum $b^* = a$

- **Markowitz Without Riskless Asset**

Assumptions for model without riskless asset

Model Assumptions. There are n risky assets with returns $\boldsymbol{R} = (R_1, \ldots, R_n)^\top$ having finite second moments such that

(A1) $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{R}] \in \mathbb{R}^n$, and there exist $2 \leq j \leq n$ such that $\mu_j \neq \mu_1$;

(A2) $\Sigma = \mathrm{Cov}(\boldsymbol{R}) \in \mathbb{R}^{n \times n}$ is positive definite.

Terminology. An investment strategy $\boldsymbol{x} \in \mathbb{R}^n$ is normalized $\boldsymbol{e}^\top \boldsymbol{x} = 1$.

An investment strategy $\boldsymbol{z} \in \mathbb{R}^n$ is said to be efficient if there is no other investment strategy $\boldsymbol{x} \in \mathbb{R}^n$ with

$$\begin{aligned} \boldsymbol{x}^\top \boldsymbol{\mu} &\geq \boldsymbol{z}^\top \boldsymbol{\mu}, \\ \boldsymbol{x}^\top \Sigma \boldsymbol{x} &< \boldsymbol{z}^\top \Sigma \boldsymbol{z}. \end{aligned}$$

Markowitz (1952) problem without riskless asset

("=") Choose target return $\rho \in \mathbb{R}$. Determine the mean-variance portfolio

$$\boldsymbol{x}_\rho = \underset{\boldsymbol{x}\in\mathbb{R}^n;\ \boldsymbol{e}^\top\boldsymbol{x}=1;\ \boldsymbol{x}^\top\boldsymbol{\mu}=\rho}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

("$\geq$") We can also study the following problem for $r \in \mathbb{R}$

$$\boldsymbol{x}_r^+ = \underset{\boldsymbol{x}\in\mathbb{R}^n;\ \boldsymbol{e}^\top\boldsymbol{x}=1;\ \boldsymbol{x}^\top\boldsymbol{\mu}\geq r}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

$\boldsymbol{x}_r^+$ provides an efficient portfolio.

Calculation of the efficient portfolio

- Define the Lagrange function with Lagrange multipliers λ_1, λ_2 and with $\rho \geq r$

$$\mathcal{L}(\boldsymbol{x}, \lambda_1, \lambda_2, \rho) = -\frac{1}{2}\boldsymbol{x}^\top \Sigma \boldsymbol{x} - \lambda_1(\boldsymbol{e}^\top \boldsymbol{x} - 1) - \lambda_2(\boldsymbol{x}^\top \boldsymbol{\mu} - \rho).$$

- Solving the Markowitz problem "$\geq$" requires solving the following KKT conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda_1, \lambda_2, \rho)}{\partial \boldsymbol{x}} &= -\Sigma \boldsymbol{x} - \lambda_1 \boldsymbol{e} - \lambda_2 \boldsymbol{\mu} &= 0, && (9)\\
\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda_1, \lambda_2, \rho)}{\partial \lambda_1} &= -\left(\boldsymbol{e}^\top \boldsymbol{x} - 1\right) &= 0, && (10)\\
\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda_1, \lambda_2, \rho)}{\partial \lambda_2} &= -\left(\boldsymbol{x}^\top \boldsymbol{\mu} - \rho\right) &= 0, && (11)\\
\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda_1, \lambda_2, \rho)}{\partial \rho} &= \lambda_2 &\leq 0, && (12)\\
(\rho - r)\lambda_2 & &= 0. && (13)
\end{aligned}$$

Calculation of the mean-variance portfolio

- The mean-variance portfolio for $\boldsymbol{x}^\top \mu = \rho$ is obtained by solving (9)-(11). We start with this.
- KKT (9) gives us, note that Σ is positive definite,

$$\boldsymbol{x} = -\Sigma^{-1} \left(\lambda_1 \boldsymbol{e} + \lambda_2 \boldsymbol{\mu}\right) = -\Sigma^{-1}(\boldsymbol{e}, \boldsymbol{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{14}$$

- Inserting this into KKT (10)-(11) gives us

$$\begin{pmatrix} 1 \\ \rho \end{pmatrix} = (\boldsymbol{e}, \boldsymbol{\mu})^\top \boldsymbol{x} = -(\boldsymbol{e}, \boldsymbol{\mu})^\top \Sigma^{-1} (\boldsymbol{e}, \boldsymbol{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = A \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix},$$

  with matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e} & \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu} \\ \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu} & \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu} \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

- Claim. Matrix A is positive definite.
  Choose $\boldsymbol{z} = (z_1, z_2)^\top \neq 0$. Note that $\boldsymbol{y} \overset{\text{def.}}{=} (\boldsymbol{e}, \boldsymbol{\mu})\boldsymbol{z} = z_1 \boldsymbol{e} + z_2 \boldsymbol{\mu} \neq 0$ for all $\boldsymbol{z} \neq 0$ because of assumption (A1), saying that $\boldsymbol{e}$ and $\boldsymbol{\mu}$ cannot be collinear. This then implies the proof of positive definiteness of A

$$\boldsymbol{z}^\top A \boldsymbol{z} = \boldsymbol{y}^\top \Sigma^{-1} \boldsymbol{y}^\top > 0.$$

- As a consequence A is invertible and, henceforth,

$$\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ \rho \end{pmatrix}. \tag{15}$$

- This provides us with mean-variance portfolio for $\rho \in \mathbb{R}$, see (14),

$$\boldsymbol{x}_\rho = \Sigma^{-1}(\boldsymbol{e}, \boldsymbol{\mu}) A^{-1} \begin{pmatrix} 1 \\ \rho \end{pmatrix},$$

with

$$\begin{aligned} A^{-1} &= \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \\ &= \frac{1}{(\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e})(\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}) - (\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu})^2} \begin{pmatrix} \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu} & -\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu} \\ -\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu} & \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e} \end{pmatrix}. \end{aligned}$$

- We calculate the variance of the mean-variance portfolio $\boldsymbol{x}_\rho$

$$\begin{aligned} \mathrm{Var}(\boldsymbol{x}_\rho^\top \boldsymbol{R}) &= \boldsymbol{x}_\rho^\top \Sigma \boldsymbol{x}_\rho = (1, \rho) A^{-1} (\boldsymbol{e}, \boldsymbol{\mu})^\top \Sigma^{-1} \Sigma \Sigma^{-1} (\boldsymbol{e}, \boldsymbol{\mu}) A^{-1} \begin{pmatrix} 1 \\ \rho \end{pmatrix} \\ &= (1, \rho) A^{-1} \begin{pmatrix} 1 \\ \rho \end{pmatrix} = \frac{c - 2b\rho + a\rho^2}{ac - b^2}. \end{aligned}$$

Mean-variance portfolio without riskless asset

Theorem. The mean-variance portfolio $\boldsymbol{x}_\rho$ for target return $\rho \in \mathbb{R}$ is given by

$$\boldsymbol{x}_\rho = \Sigma^{-1}(\boldsymbol{e}, \boldsymbol{\mu}) A^{-1} \begin{pmatrix} 1 \\ \rho \end{pmatrix},$$

with $a = \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e}$, $c = \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}$, $b = \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu}$ and

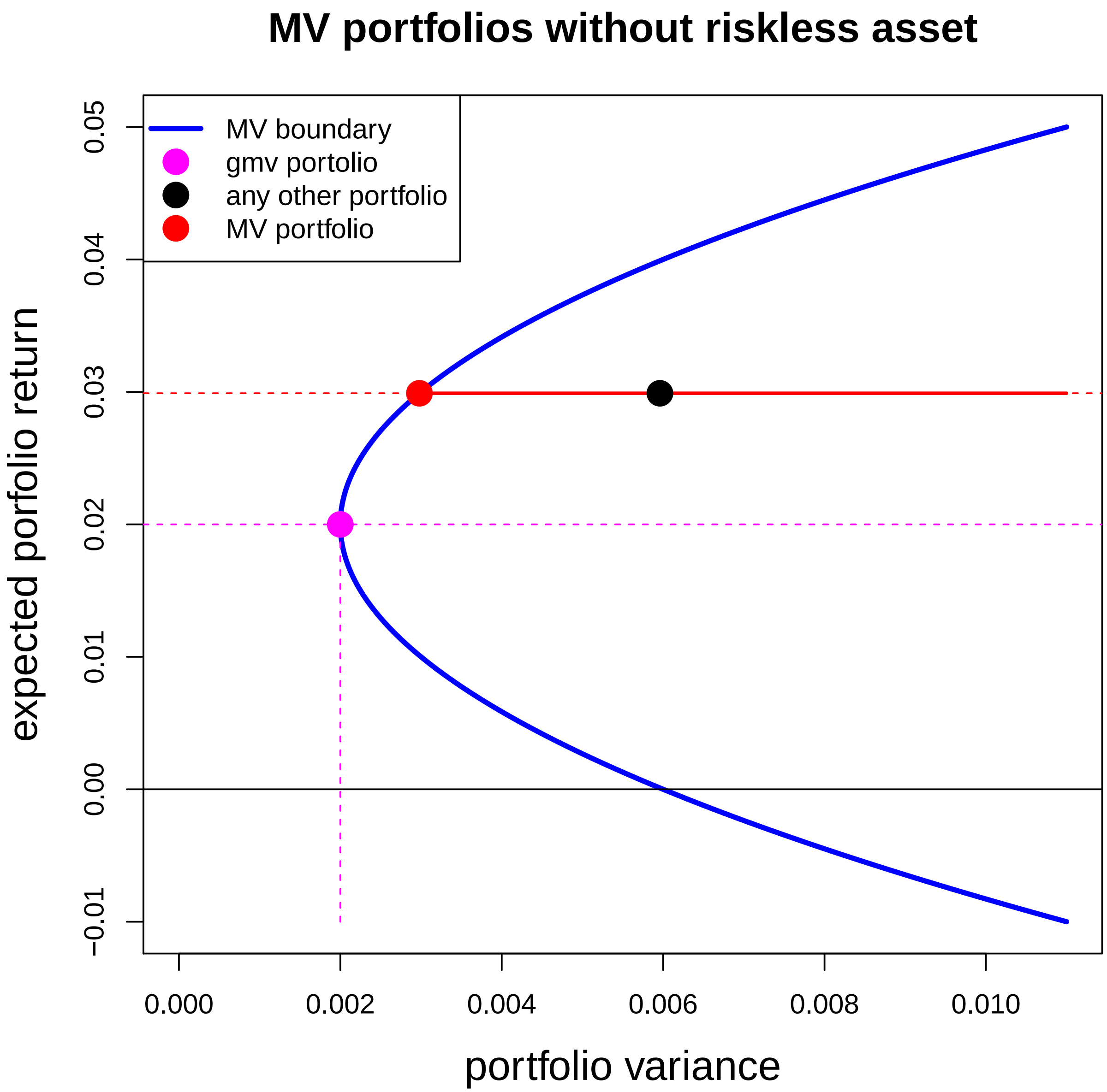
$$A^{-1} \quad = \quad \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}.$$

We have mean $\mathbb{E}[\boldsymbol{x}_\rho^\top \boldsymbol{R}] = \rho$ and variance

$$\operatorname{Var}(\boldsymbol{x}_\rho^\top \boldsymbol{R}) = \frac{c - 2b\rho + a\rho^2}{ac - b^2} \;\geq\; \frac{1}{a} = \frac{1}{\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e}} > 0.$$

The latter comes from the global minimum variance (gmv) portfolio with $\rho_{\text{gmv}} = b/a$.

Mean-variance (MV) boundary: parabola



Calculation of the efficient portfolio

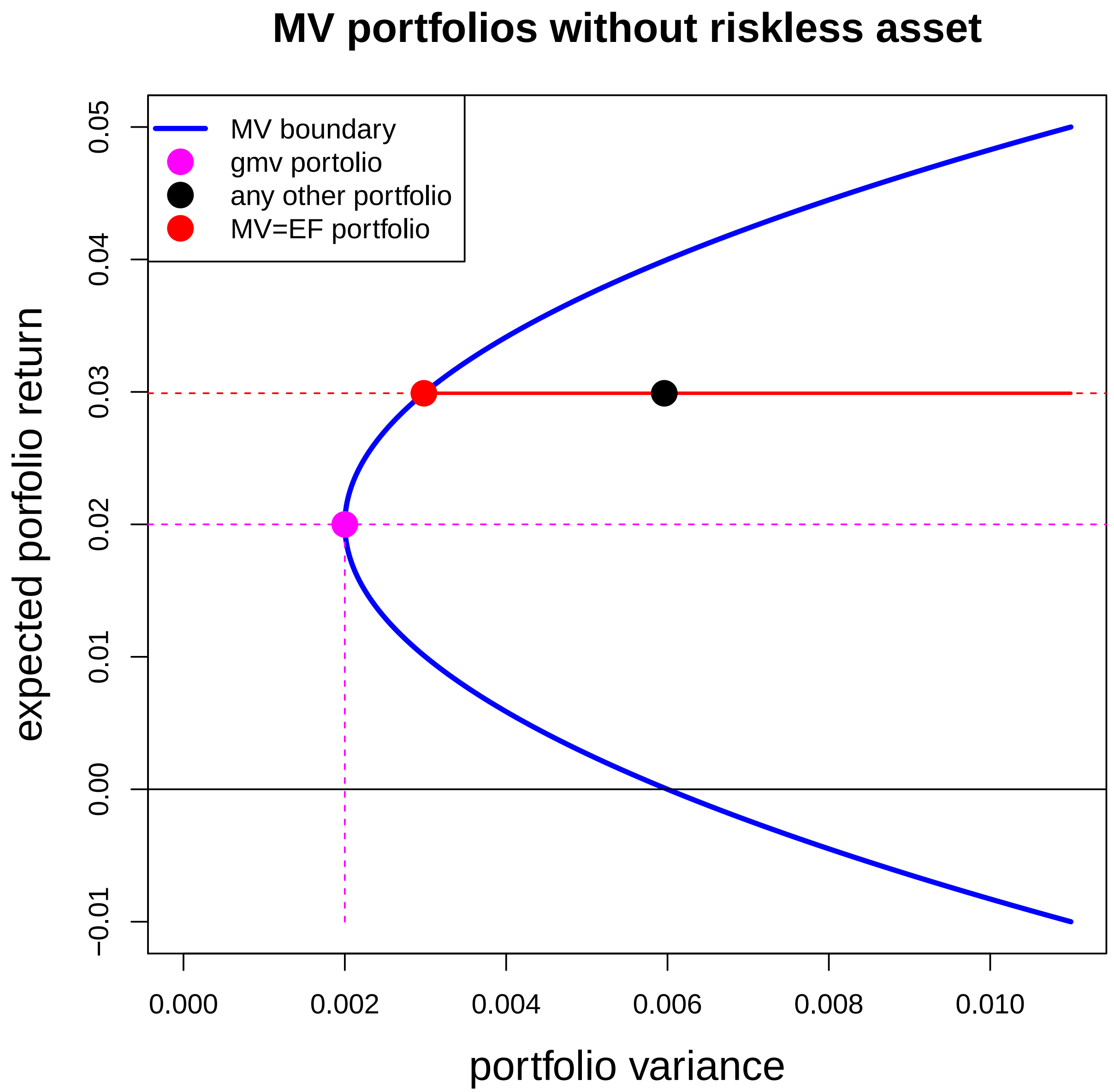
- To find the efficient portfolios $\boldsymbol{x}_r^+$ we still need to solve KKT (12)-(13), that is,

$$\begin{aligned} \frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda_1, \lambda_2, \rho)}{\partial \rho} = \lambda_2 = \frac{b - \rho a}{ac - b^2} &\leq 0, \\ (\rho - r)\lambda_2 &= 0, \end{aligned}$$

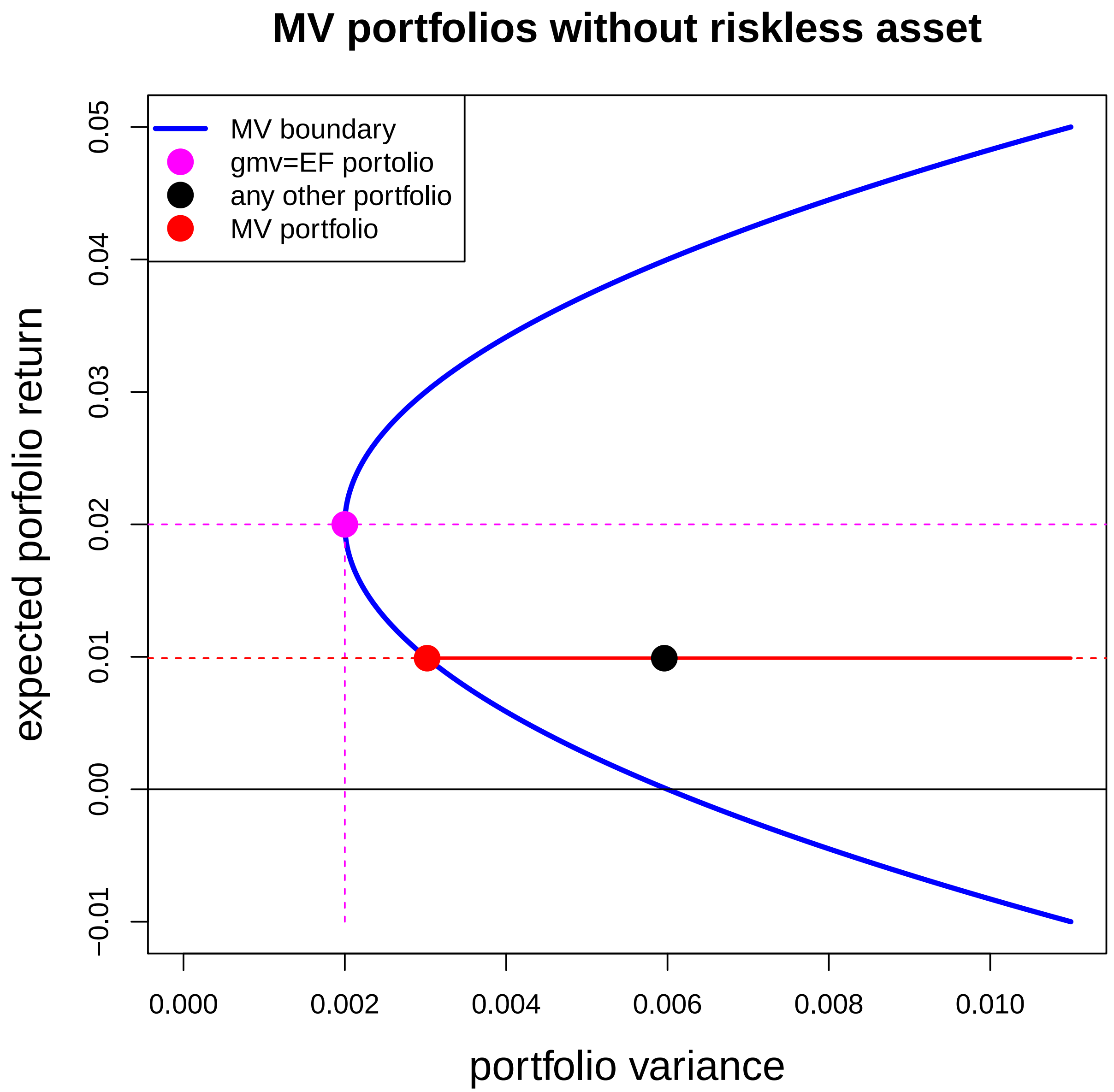
  for $\rho \geq r$.

- We can either do this formally, or just by looking at the MV boundary: every mean-variance portfolio $\boldsymbol{x}_\rho$ with an expected return ρ above the global minimum variance return $\rho_{\mathrm{gmv}} = b/a$ is an efficient portfolio $\boldsymbol{x}_\rho^+$, and below it is not.

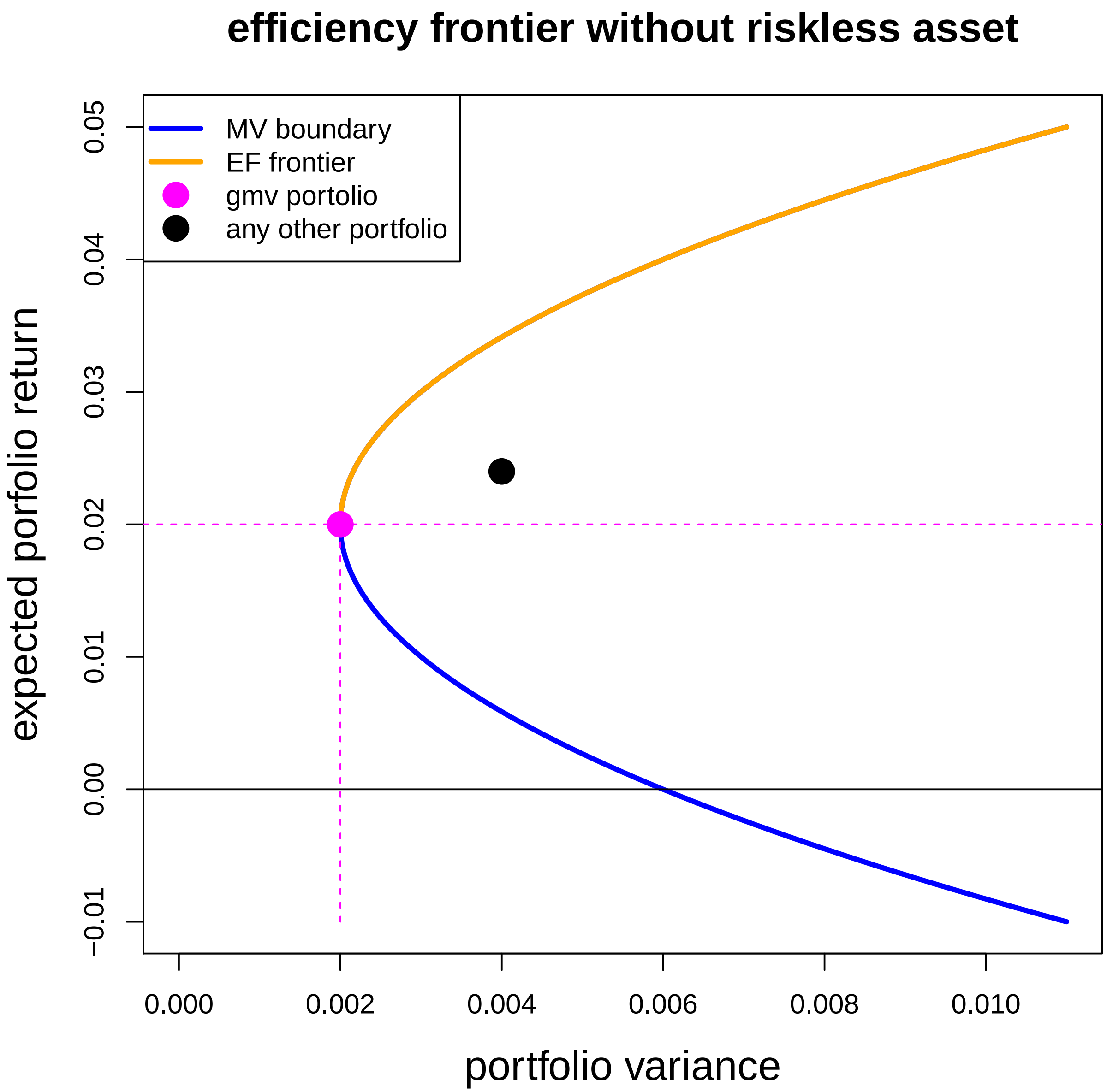
Efficiency frontier (EF): case 1



Efficiency frontier (EF): case 2



Efficiency frontier (EF)



Efficient portfolio without riskless asset

Theorem. The efficient portfolio $\boldsymbol{x}_r^+$ for target return r is given by

$$\boldsymbol{x}_r^+ = \Sigma^{-1}(\boldsymbol{e}, \boldsymbol{\mu})A^{-1}\begin{pmatrix} 1 \\ r \vee \rho_{\mathrm{gmv}} \end{pmatrix},$$

with $\rho_{\mathrm{gmv}} = b/a$, $a = \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e}$, $c = \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}$, $b = \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu}$ and

$$A^{-1} = \frac{1}{ac - b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}.$$

We have mean $\mathbb{E}[(\boldsymbol{x}_r^+)^\top \boldsymbol{R}] = r \vee \rho_{\mathrm{gmv}}$ and variance

$$\mathrm{Var}((\boldsymbol{x}_r^+)^\top \boldsymbol{R}) = \begin{cases} \frac{c - 2br + ar^2}{ac - b^2} & \text{if } r \geq \rho_{\mathrm{gmv}}, \\ 1/a & \text{if } r < \rho_{\mathrm{gmv}}. \end{cases}$$

- **Example Markowitz**

Example: Markowitz problem

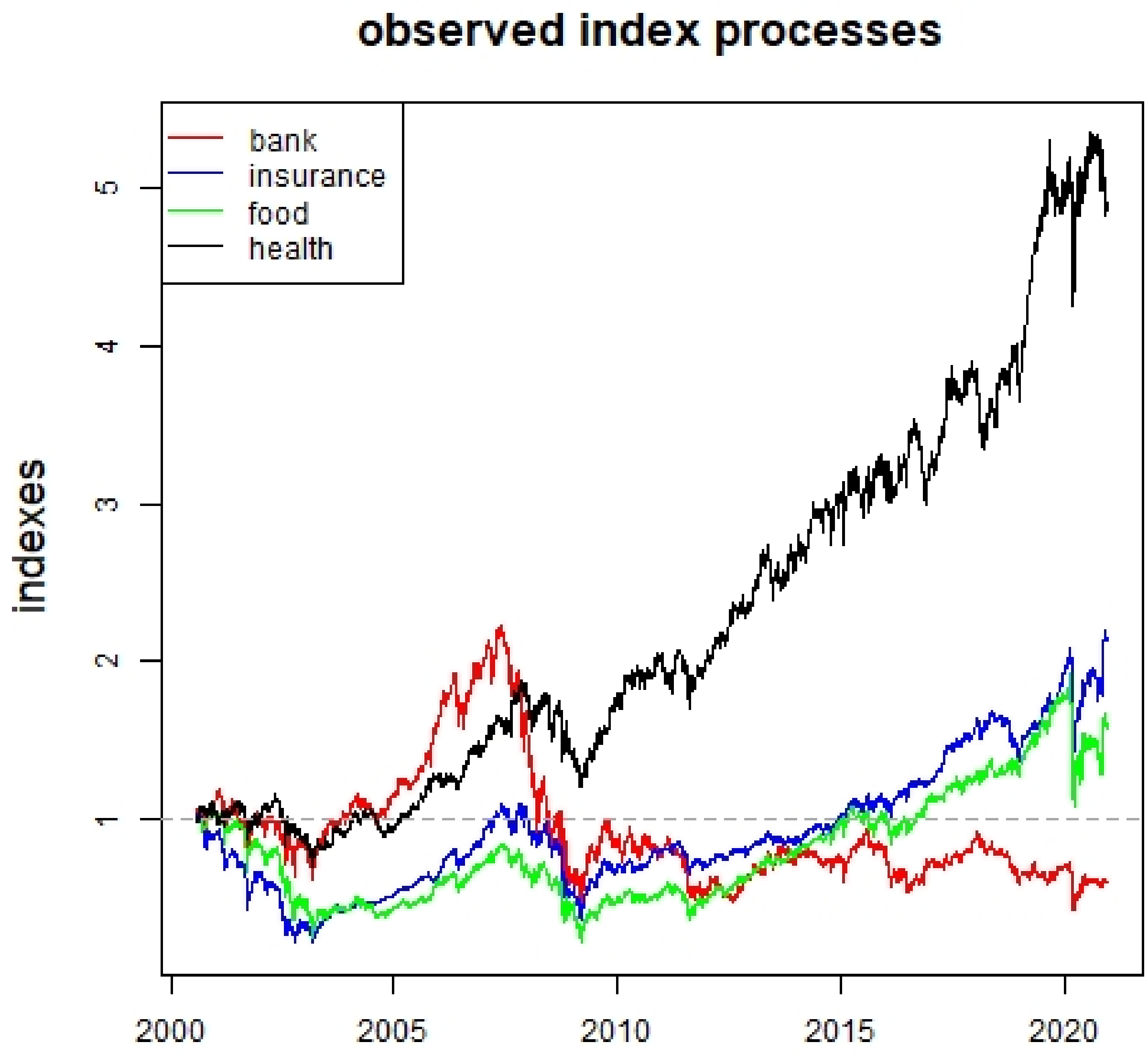
- We consider the daily stock market indexes of (1) CHF banks, (2) CHF insurance companies, (3) CHF food companies and (4) CHF health care companies.[1]

- Denote these daily indexes by $S_j^{(t)}$ for $j = 1, \ldots, 4$ and time points $t = 2000/08/02, \ldots, 2020/12/15$. This gives us $5'132$ daily weekday observations (without weekends and public holidays).

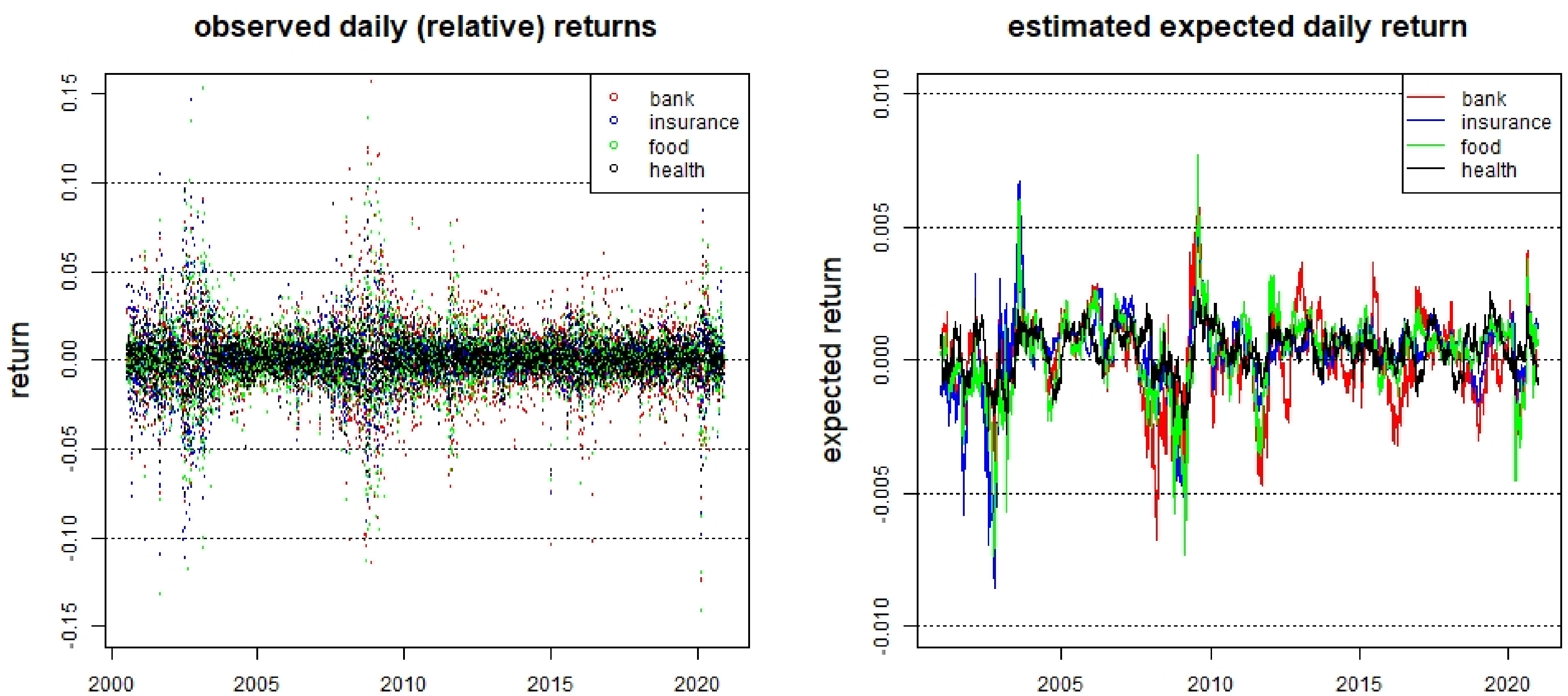
- The means $\boldsymbol{\mu}^{(t)}$ and the covariance matrices $\Sigma^{(t)}$ are estimated with the sample means and covariances using rolling windows of length $K = 100$ business days.

[1] Source: Swiss National Bank SNB, Kapitalmarkt, Schweizerische Aktienindizes: `https://data.snb.ch/de/topics/finma#!/cube/capchstocki`

Example: Markowitz problem

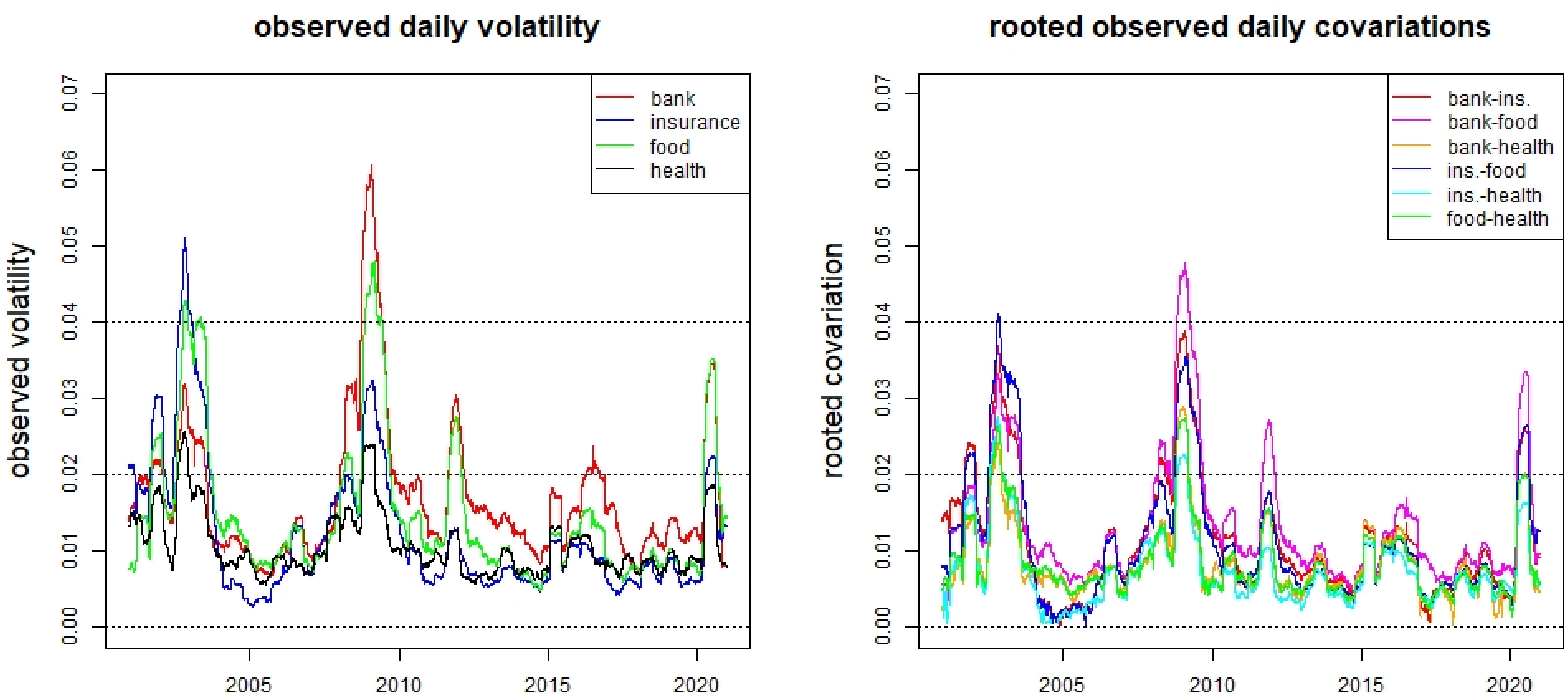


Example: Markowitz problem



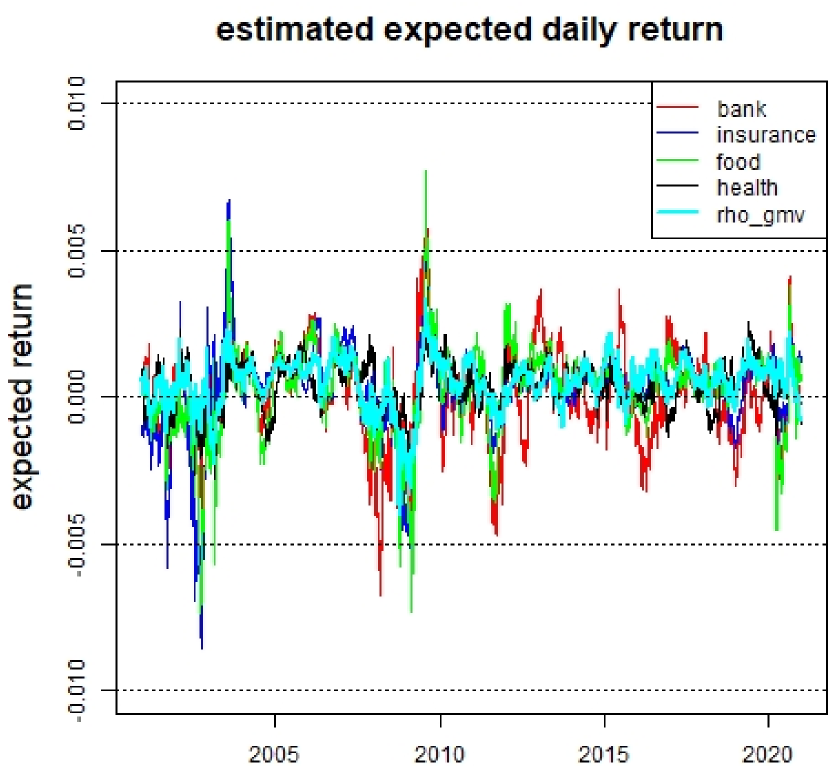
(lhs) Observed daily returns and (rhs) estimated expected daily returns $\boldsymbol{\mu}^{(t)}$ using the empirical mean of a rolling window of a length of $K = 100$ business days.

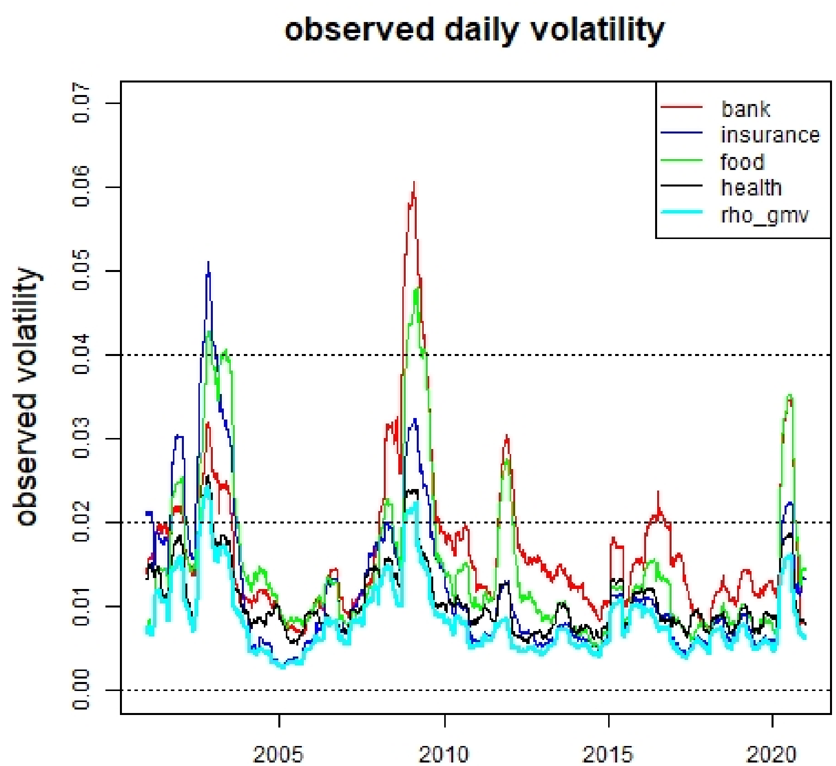
Example: Markowitz problem



(lhs) Observed daily volatilities and (rhs) rooted observed daily covariations using empirical covariances of a rolling window of a length of $K = 100$ business days. This provides calibration of covariance matrices $\Sigma^{(t)}$.

Example: Markowitz problem

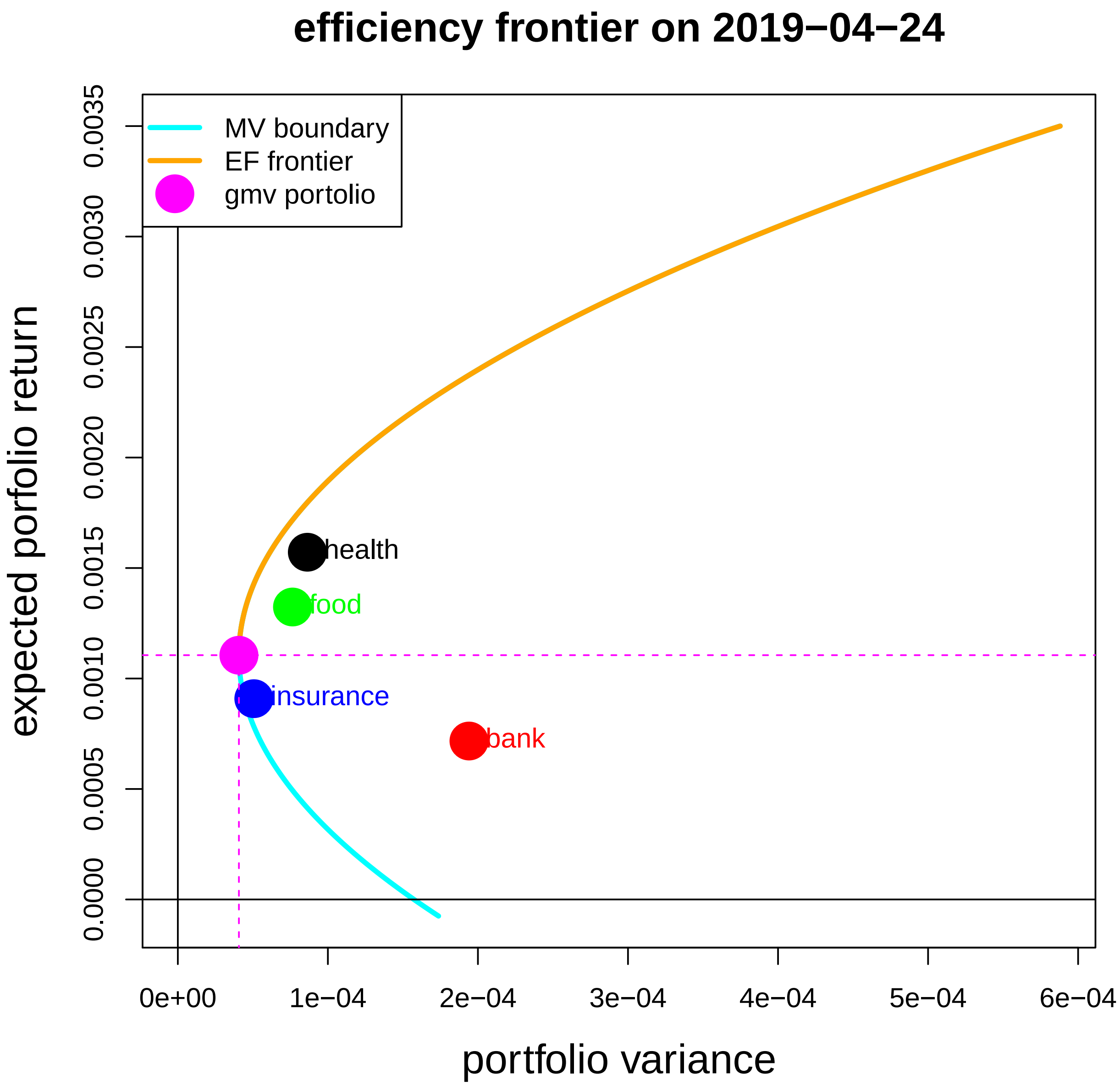




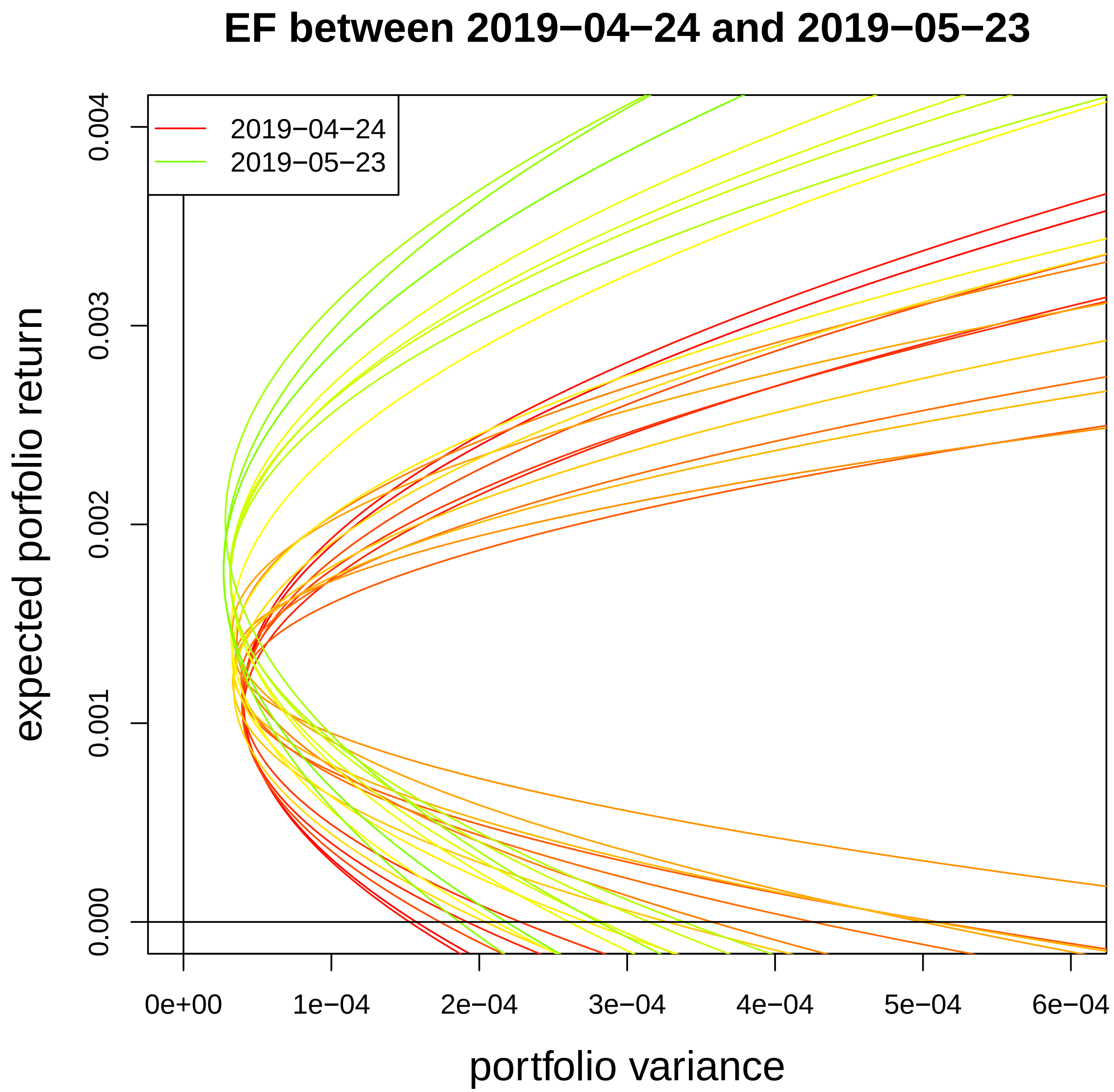
Global minimum variance portfolio:

(lhs) expected return $\rho_{\text{gmv}}^{(t)} = b^{(t)}/a^{(t)}$ and (rhs) volatility $\sigma_{\text{gmv}}^{(t)} = 1/\sqrt{a^{(t)}}$.

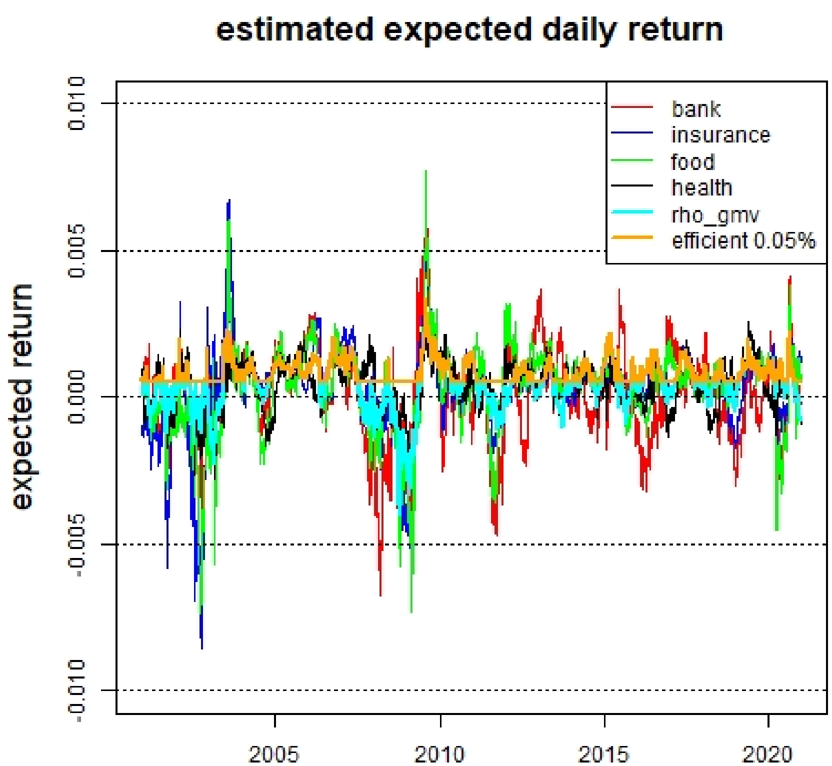
Example: Markowitz problem

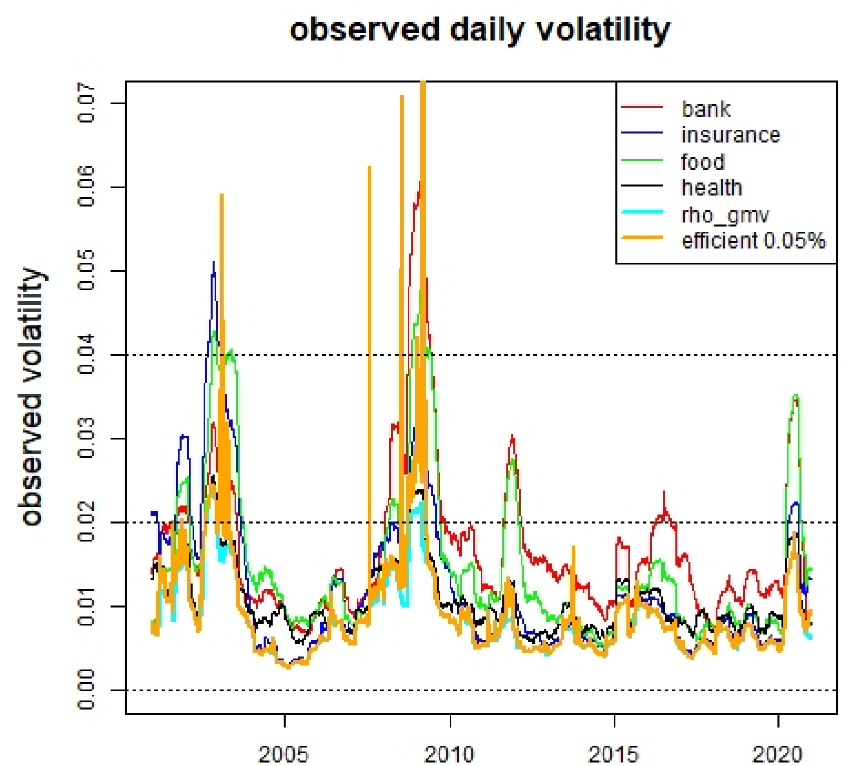


Example: Markowitz problem



Example: Markowitz problem





Efficient portfolios for return target $r = 0.05\%$:

(lhs) expected return $r_t^+ = \max\{r, \rho_{\text{gmv}}^{(t)}\}$ and (rhs) volatility of efficient portfolio

- **Further Analysis of Mean-Variance Portfolios**

Herding effect of mean-variance portfolios

Proposition. Assume $b = \boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu} \neq 0$.

- Every mean-variance portfolio $\boldsymbol{x}_\rho$, $\rho \in \mathbb{R}$, is a linear combination of the global minimum variance portfolio $\boldsymbol{x}_{\rho_{\mathrm{gmv}}}$ and the portfolio

$$\boldsymbol{x}^{(0)} = \frac{1}{\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu}} \Sigma^{-1} \boldsymbol{\mu}.$$

- Every weighted linear combination of $\boldsymbol{x}_{\rho_{\mathrm{gmv}}}$ and $\boldsymbol{x}^{(0)}$ is a mean-variance portfolio.

Proof. From the derivation of the mean-variance portfolio we know, see (14),

$$\boldsymbol{x}_\rho = -\Sigma^{-1} \left(\lambda_1 \boldsymbol{e} + \lambda_2 \boldsymbol{\mu}\right) = \Sigma^{-1} (\boldsymbol{e}, \boldsymbol{\mu}) \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix},$$

and the Lagrange multipliers are given by, see (15),

$$\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ \rho \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c - b\rho \\ -b + a\rho \end{pmatrix}.$$

Case 1: Choose $\rho = \rho_{\text{gmv}} = b/a$. This provides for the Lagrange multipliers

$$\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ \rho_{\text{gmv}} \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c - b^2/a \\ 0 \end{pmatrix} = \begin{pmatrix} 1/a \\ 0 \end{pmatrix}.$$

Therefore, the global minimum variance portfolio is given by

$$\boldsymbol{x}_{\rho_{\text{gmv}}} = \frac{1}{a}\Sigma^{-1}\boldsymbol{e} = \frac{1}{\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e}} \Sigma^{-1}\boldsymbol{e}.$$

Case 2: Choose $\rho = \rho_0 = c/b$, which, by assumption, is well-defined. This provides for the Lagrange multipliers

$$\begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ \rho_0 \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} 0 \\ -b + ac/b \end{pmatrix} = \begin{pmatrix} 0 \\ 1/b \end{pmatrix},$$

and

$$\boldsymbol{x}^{(0)} = \boldsymbol{x}_{\rho_0} = \frac{1}{b}\Sigma^{-1}\boldsymbol{\mu} = \frac{1}{\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{\mu}} \Sigma^{-1}\boldsymbol{\mu}.$$

Remark that $\boldsymbol{x}_{\rho_{\text{gmv}}}$ and $\boldsymbol{x}^{(0)}$ are mean-variance portfolios and $\rho_{\text{gmv}} \neq \rho_0$, because A is positive definite ($ac - b^2 \neq 0$).

We are now ready to prove the proposition. Choose $\rho \in \mathbb{R}$. There exist $\alpha \in \mathbb{R}$ such that

$\rho = \alpha\rho_0 + (1-\alpha)\rho_{\rm gmv}$. We then have by linearity

$$\begin{aligned}
\boldsymbol{x}_\rho &= \Sigma^{-1}(\boldsymbol{e},\boldsymbol{\mu})A^{-1}\begin{pmatrix}1\\ \rho\end{pmatrix} = \Sigma^{-1}(\boldsymbol{e},\boldsymbol{\mu})A^{-1}\begin{pmatrix}1\\ \alpha\rho_0+(1-\alpha)\rho_{\rm gmv}\end{pmatrix}\\
&= \alpha\Sigma^{-1}(\boldsymbol{e},\boldsymbol{\mu})A^{-1}\begin{pmatrix}1\\ \rho_0\end{pmatrix} + (1-\alpha)\Sigma^{-1}(\boldsymbol{e},\boldsymbol{\mu})A^{-1}\begin{pmatrix}1\\ \rho_{\rm gmv}\end{pmatrix}\\
&= \alpha\boldsymbol{x}^{(0)} + (1-\alpha)\boldsymbol{x}_{\rho_{\rm gmv}}.
\end{aligned}$$

$\square$

Remarks.

- We have proved that all mean-variance optimizers hold the same two portfolios $\boldsymbol{x}^{(0)}$ and $\boldsymbol{x}_{\rho_{\rm gmv}}$, only their shares α and $1-\alpha$ in these two portfolios differ according to their required target returns ρ. We can interpret this as a herding effect because all these financial agents have the same asset strategy.
- This behavior will also be the basis of the capital asset pricing model (CAPM) formula in the next chapter.
- Note that implicitly we assume that all these financial agents work with the same parameters $\boldsymbol{\mu}$ and Σ.

- **Markowitz With Riskless Asset**

Assumptions for model with riskless asset

Model Assumptions. There is one riskless asset $R_0 = \mu_0 \in \mathbb{R}$ and there are n risky assets with returns $\boldsymbol{R} = (R_1, \ldots, R_n)^\top$ having finite second moments such that

(A1) $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{R}] \in \mathbb{R}^n$, and there exist $1 \leq j \leq n$ such that $\mu_j \neq \mu_0$;

(A2) $\Sigma = \mathrm{Cov}(\boldsymbol{R}) \in \mathbb{R}^{n \times n}$ is positive definite.

(“$=$”) Choose target return $\rho \in \mathbb{R}$, the mean-variance portfolio is

$$\widetilde{\boldsymbol{x}}_\rho = \underset{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}}=1;\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}}=\rho}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

(“$\geq$”) We can also study the efficient portfolio for $r \in \mathbb{R}$

$$\widetilde{\boldsymbol{x}}_r^{+} = \underset{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}}=1;\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}}\geq r}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

Reformulation of the problem with riskless asset

Note that the riskless asset does not appear in the objective function $\boldsymbol{x}^\top \Sigma \boldsymbol{x}$:

$$\widetilde{\boldsymbol{x}}_r^+ = \underset{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1;\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} \geq r}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

Therefore, we can solve this problem in two steps:

▷ **Step 1.** Solve the following problem for expected excess return $r^e = r - \mu_0$

$$\boldsymbol{x}_r^+ = \underset{\boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{x}^\top \boldsymbol{\mu}^e \geq r^e}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x},$$

that is, we drop component x_0 and normalization $\widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1$. This is taken care off in Step 2. Moreover, we only consider excess returns above the riskless rate μ_0

$$\widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} = \widetilde{\boldsymbol{x}}^\top \left(\widetilde{\boldsymbol{\mu}} - \mu_0 \widetilde{\boldsymbol{e}} + \mu_0 \widetilde{\boldsymbol{e}} \right) = \boldsymbol{x}^\top \boldsymbol{\mu}^e + \mu_0,$$

under normalization $\widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1$ and where we set excess returns $\boldsymbol{\mu}^e = \boldsymbol{\mu} - \mu_0 \boldsymbol{e}$.

▷ **Step 2.** Set $(\widetilde{\boldsymbol{x}}_r^+)_0 = 1 - \boldsymbol{e}^\top \boldsymbol{x}_r^+$.

Reformulation of the Markowitz problems

(“$=$”) Choose target return $\rho = \rho^e + \mu_0 \in \mathbb{R}$, the mean-variance portfolio is

$$\boldsymbol{x}_\rho = \underset{\boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{x}^\top \boldsymbol{\mu}^e = \rho^e}{\arg\min}\ \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

Moreover, portfolio normalization is achieved by $(\widetilde{\boldsymbol{x}}_\rho)_0 = 1 - \boldsymbol{e}^\top \boldsymbol{x}_\rho$.

(“$\geq$”) We can also study the efficient portfolio for $r = r^e + \mu_0 \in \mathbb{R}$

$$\boldsymbol{x}_r^+ = \underset{\boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{x}^\top \boldsymbol{\mu}^e \geq r^e}{\arg\min}\ \boldsymbol{x}^\top \Sigma \boldsymbol{x}.$$

Moreover, portfolio normalization is achieved by $(\widetilde{\boldsymbol{x}}_r^+)_0 = 1 - \boldsymbol{e}^\top \boldsymbol{x}_r^+$.

Calculation of the efficient portfolio

- Define the Lagrange function with Lagrange multiplier λ and with $\rho^e \geq r^e$

$$\mathcal{L}(\boldsymbol{x}, \lambda, \rho^e) = -\frac{1}{2}\boldsymbol{x}^\top \Sigma \boldsymbol{x} - \lambda(\boldsymbol{x}^\top \boldsymbol{\mu}^e - \rho^e).$$

- Solving the Markowitz problem "$\geq$" requires solving the following KKT conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda, \rho^e)}{\partial \boldsymbol{x}} &= -\Sigma \boldsymbol{x} - \lambda \boldsymbol{\mu}^e &= 0, \\
\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda, \rho^e)}{\partial \lambda} &= -\left(\boldsymbol{x}^\top \boldsymbol{\mu}^e - \rho^e\right) &= 0, \\
\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda, \rho^e)}{\partial \rho^e} &= \lambda &\leq 0, \\
(\rho^e - r^e)\lambda &&= 0.
\end{aligned}$$

Again, we only solve the Lagrange problem formally which corresponds to the first two KKT conditions, and the full KKT solution is obtained graphically.

Calculation of the mean-variance portfolio

- The first KKT condition gives us, note that Σ is positive definite,

$$\boldsymbol{x} = -\lambda \Sigma^{-1} \boldsymbol{\mu}^e.$$

- Inserting this into the second KKT condition gives us

$$\rho^e = (\boldsymbol{\mu}^e)^\top \boldsymbol{x} = -\lambda (\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e.$$

- Because of assumption (A1), $\boldsymbol{\mu}^e$ cannot be the zero vector, which allows us to calculate Lagrange multiplier λ. As a consequence

$$\boldsymbol{x}_\rho = \frac{\rho^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e.$$

- We calculate the variance of the mean-variance portfolio $\boldsymbol{x}_\rho$

$$\begin{aligned} \mathrm{Var}(\boldsymbol{x}_\rho^\top \boldsymbol{R}) &= \boldsymbol{x}_\rho^\top \Sigma \boldsymbol{x}_\rho = \left(\frac{\rho^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \right)^2 (\boldsymbol{\mu}^e)^\top \Sigma^{-1} \Sigma \Sigma^{-1} \boldsymbol{\mu}^e \\ &= \frac{(\rho^e)^2}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e}. \end{aligned}$$

Mean-variance portfolio with riskless asset

Theorem. The mean-variance portfolio $\widetilde{\boldsymbol{x}}_\rho$ for target return $\rho = \mu_0 + \rho^e$ is given by

$$\boldsymbol{x}_\rho = \frac{\rho^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e \qquad \text{and} \qquad (\widetilde{\boldsymbol{x}}_\rho)_0 = 1 - \boldsymbol{e}^\top \boldsymbol{x}_\rho.$$

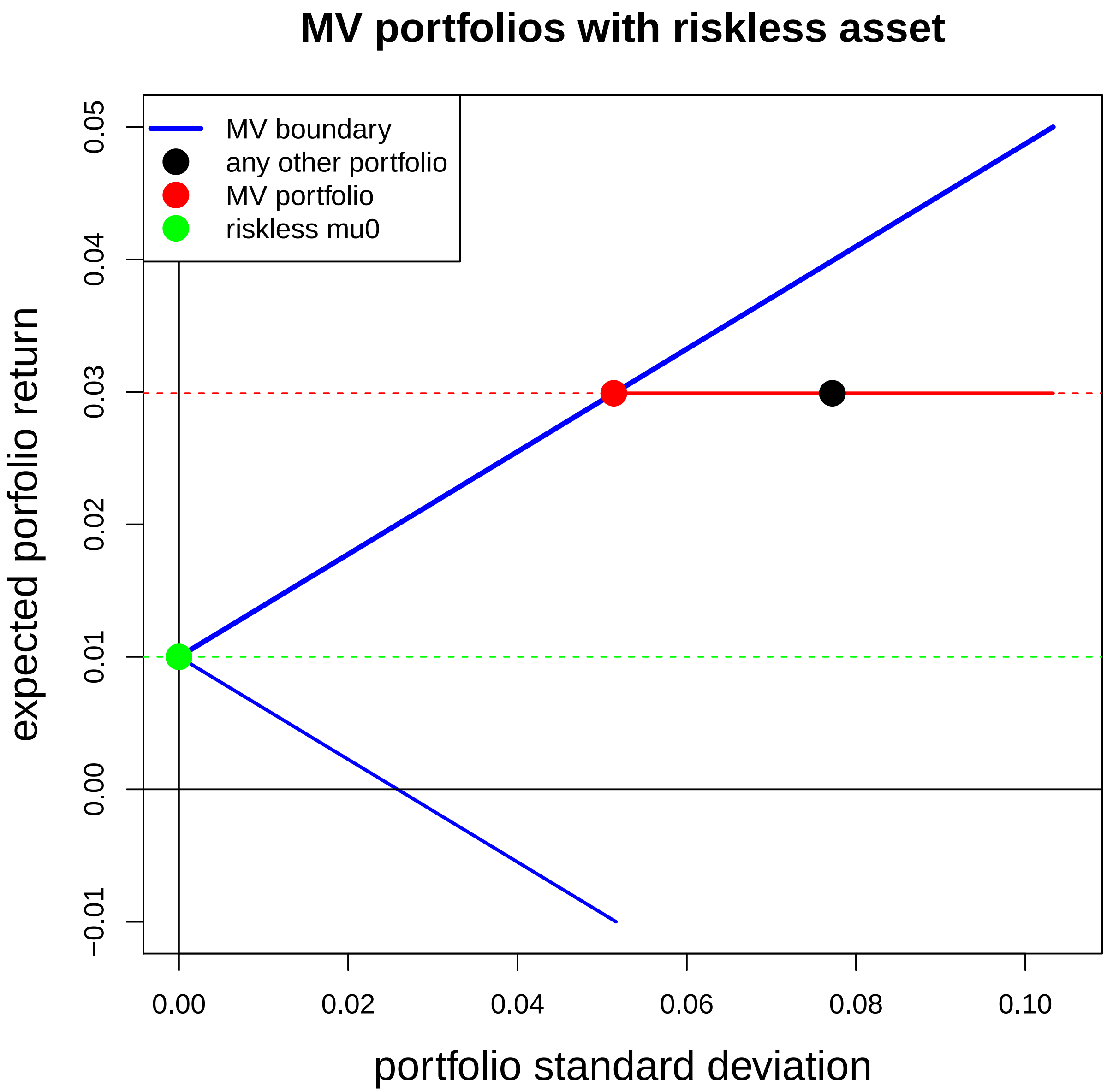
We have mean $\mathbb{E}[\widetilde{\boldsymbol{x}}_\rho^\top \widetilde{\boldsymbol{R}}] = \widetilde{\boldsymbol{x}}_\rho^\top \widetilde{\boldsymbol{\mu}} = \rho$ and $\mathbb{E}[\widetilde{\boldsymbol{x}}_\rho^\top \widetilde{\boldsymbol{R}}^e] = \boldsymbol{x}_\rho^\top \boldsymbol{\mu}^e = \rho^e$, and variance

$$\mathrm{Var}(\widetilde{\boldsymbol{x}}_\rho^\top \widetilde{\boldsymbol{R}}) = \mathrm{Var}(\boldsymbol{x}_\rho^\top \boldsymbol{R}) = \frac{(\rho^e)^2}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \geq 0.$$

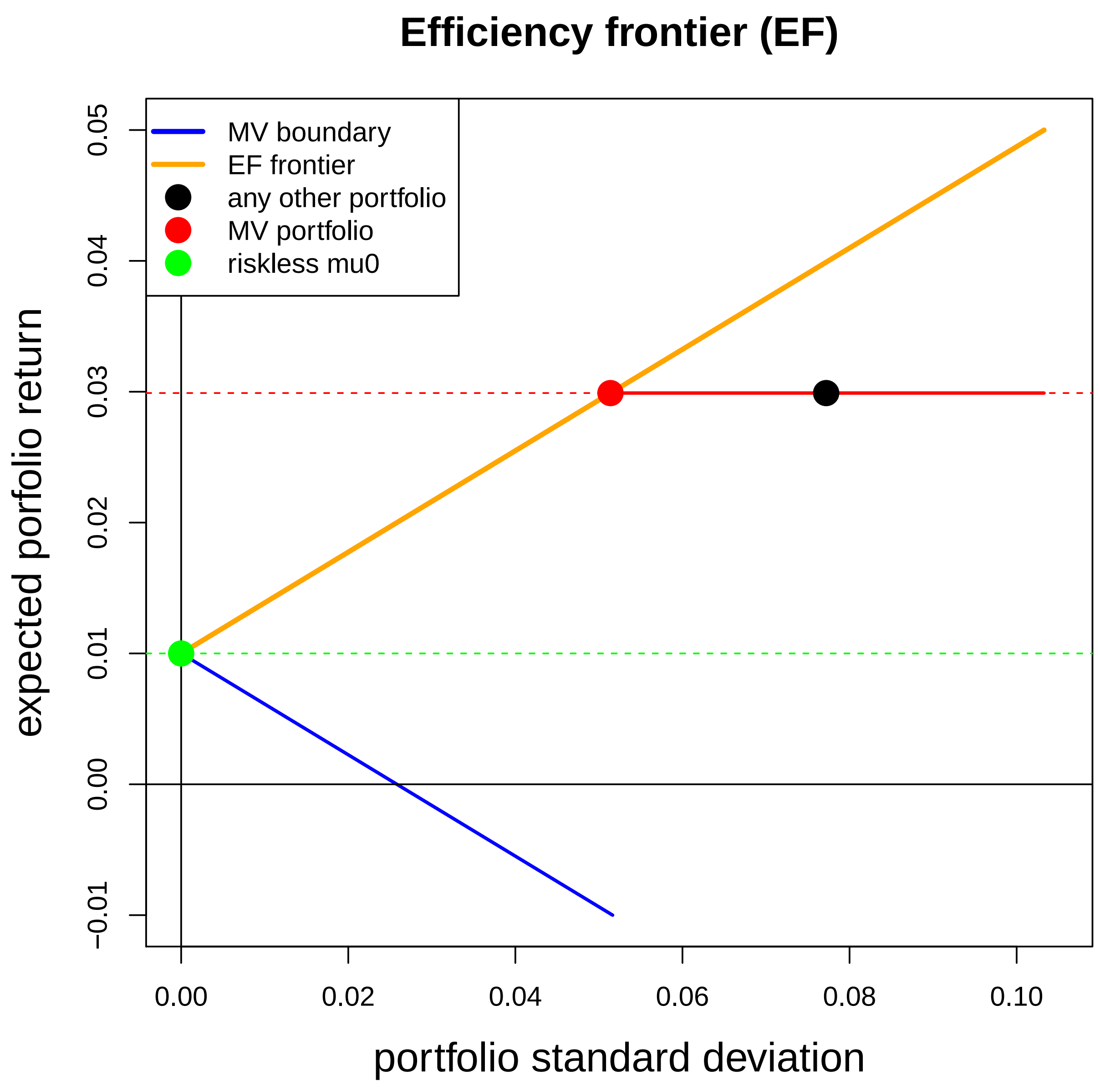
- **Remark:** The variance of the mean-variance portfolio $\widetilde{\boldsymbol{x}}_\rho^\top \widetilde{\boldsymbol{R}}$ is a quadratic function in the required expected excess return ρ^e.
- To compare the two cases (with riskless versus without riskless) we will need

$$(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e = (\boldsymbol{\mu} - \mu_0 \boldsymbol{e})^\top \Sigma^{-1} (\boldsymbol{\mu} - \mu_0 \boldsymbol{e}) = \mu_0^2 a - 2\mu_0 b + c.$$

Mean-variance (MV) boundary



Efficiency frontier (EF)



Efficient portfolio with riskless asset

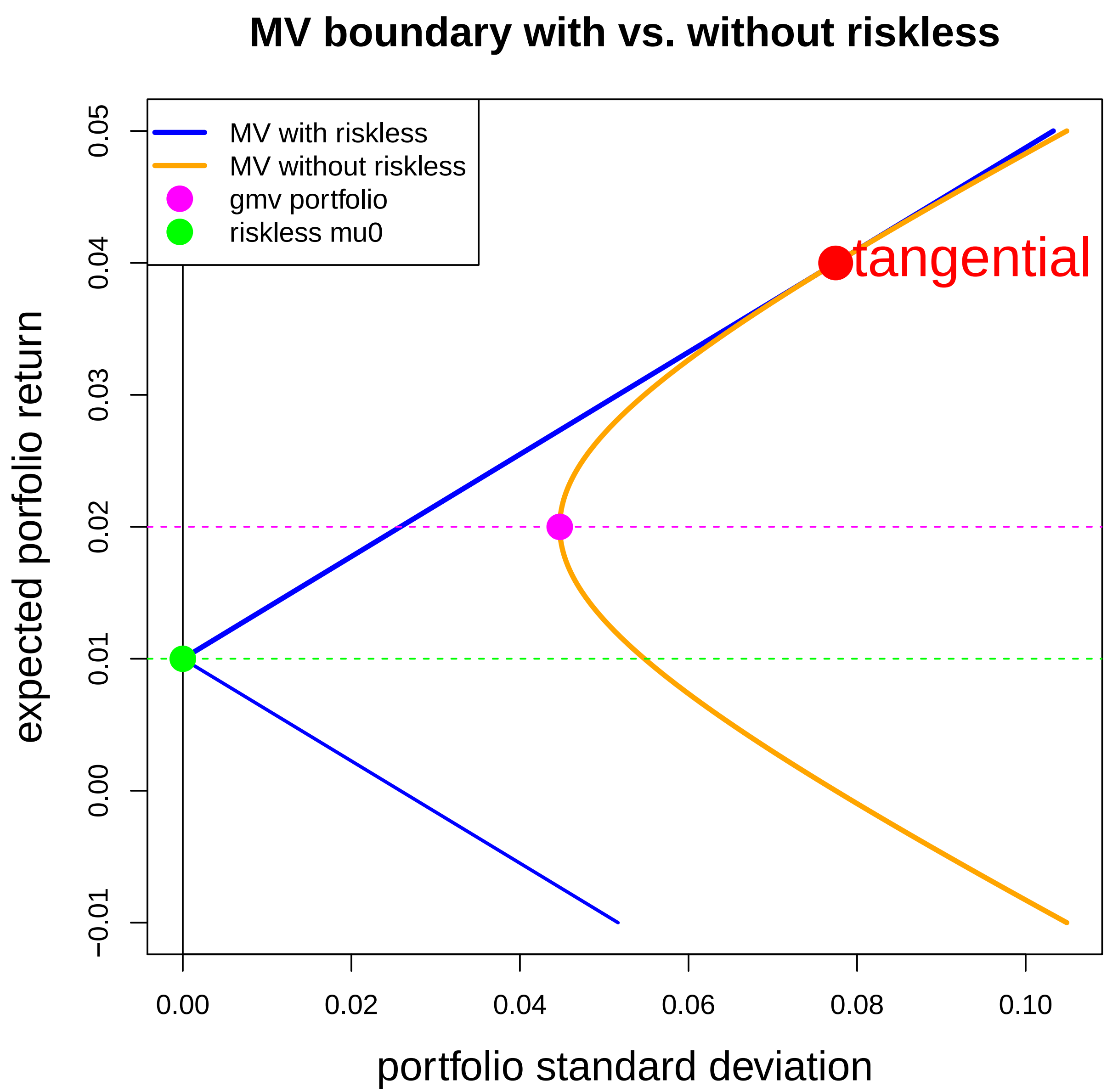
Theorem. The efficient portfolio $\widetilde{\boldsymbol{x}}_r^+$ for target return $r = \mu_0 + r^e$ is given by

$$\boldsymbol{x}_r^+ = \frac{r^e \vee 0}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e \qquad \text{and} \qquad (\widetilde{\boldsymbol{x}}_r^+)_0 = 1 - \boldsymbol{e}^\top \boldsymbol{x}_r^+.$$

We have mean $\mathbb{E}[(\widetilde{\boldsymbol{x}}_r^+)^\top \widetilde{\boldsymbol{R}}] = (\widetilde{\boldsymbol{x}}_r^+)^\top \widetilde{\boldsymbol{\mu}} = r \vee \mu_0$ and variance

$$\mathrm{Var}((\widetilde{\boldsymbol{x}}_r^+)^\top \widetilde{\boldsymbol{R}}) = \mathrm{Var}((\boldsymbol{x}_r^+)^\top \boldsymbol{R}) = \frac{(r^e \vee 0)^2}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \geq 0.$$

MV boundary: with vs. without riskless asset



Tangential portfolio

Definition. The mean-variance portfolio $\widetilde{\boldsymbol{x}}_\rho \in \mathbb{R}^{n+1}$ with $\boldsymbol{x}_\rho^\top \boldsymbol{e} = 1$ is called tangential portfolio and its return is denoted by ρ_{tan}.

- The above implies $(\widetilde{\boldsymbol{x}}_\rho)_0 = 0$, i.e., zero investments into the riskless asset.
- We need to prove existence and uniqueness.

Proposition. Assume $\rho_{\mathrm{gmv}} = b/a \neq \mu_0$. There exists a unique tangential portfolio $\widetilde{\boldsymbol{x}}_{\mathrm{tan}} = \widetilde{\boldsymbol{x}}_{\rho_{\mathrm{tan}}}$ with

$$\rho_{\mathrm{tan}} = \mu_0 + \frac{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{e}} = \frac{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{e}} = \frac{c - \mu_0 b}{b - \mu_0 a}.$$

Proof. If tangential portfolio exists, it has to be of the form

$$\boldsymbol{x}_\rho = \frac{\rho^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \Sigma^{-1} \boldsymbol{\mu}^e \qquad \text{and} \qquad \boldsymbol{x}_\rho^\top \boldsymbol{e} = 1,$$

for some $\rho = \mu_0 + \rho^e$. Merging these two identities gives us requirement

$$1 \quad = \quad \frac{\rho^e}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, (\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{e}.$$

The claim follows, once we prove that the last term is different from zero. We have

$$(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{e} = (\boldsymbol{\mu} - \mu_0 \boldsymbol{e})^\top \Sigma^{-1} \boldsymbol{e} = b - \mu_0 a \neq 0,$$

where the last step follows from the assumptions.

$\square$

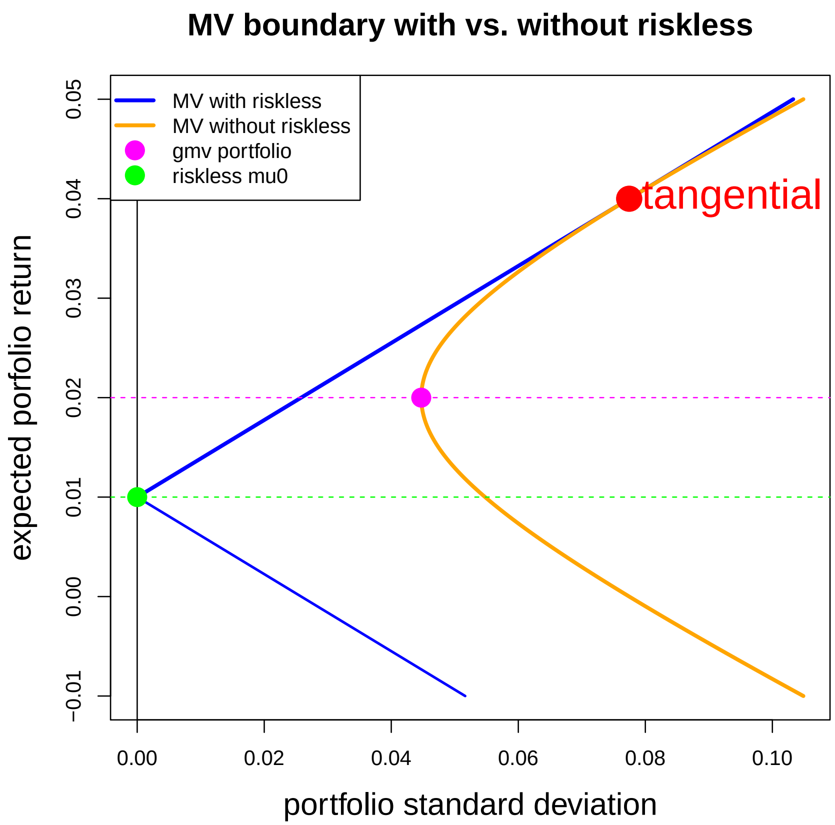
Remarks.

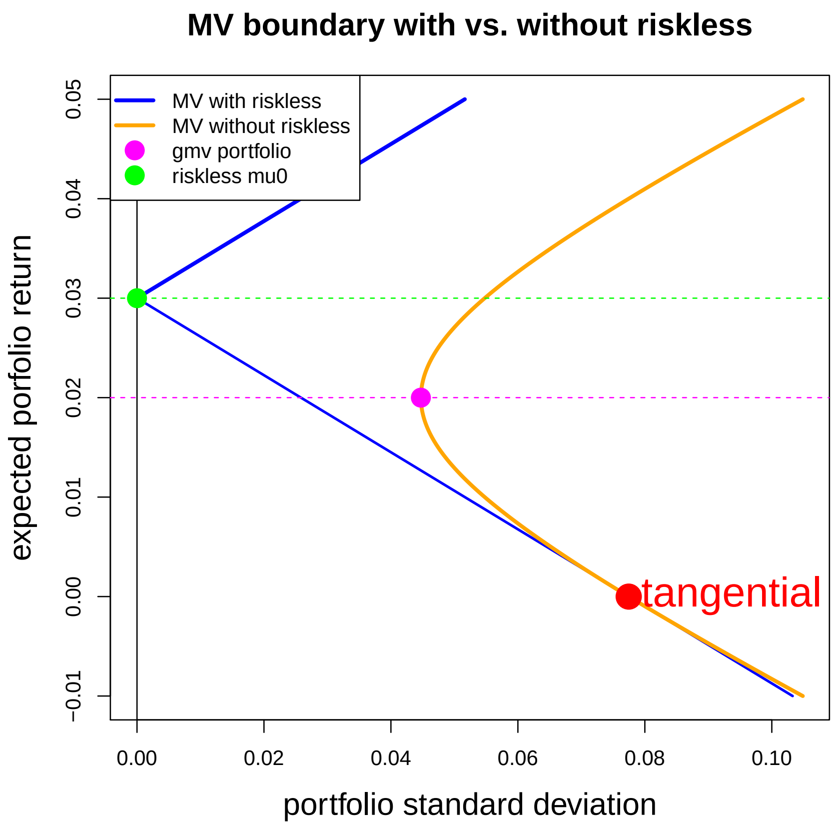
- There is a singularity in parametrization $\rho_{\text{gmv}} = b/a = \mu_0$.
- We have under $\rho_{\text{gmv}} \neq \mu_0$:

$$\rho_{\text{gmv}} = \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{e}}{\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e}} = \mu_0 + \frac{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{e}}{\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e}} = \mu_0 + \frac{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e}{\boldsymbol{e}^\top \Sigma^{-1} \boldsymbol{e}} \frac{1}{\rho^e_{\text{tan}}},$$

thus, $\rho_{\text{gmv}} > \mu_0$ if and only if $\rho^e_{\text{tan}} > 0$.

Riskless rate vs. global minimum variance rate





Both models are mathematically valid, however, economically only $\rho_{\text{gmv}} > \mu_0$ is sensible. Therefore, model calibration should provide $b/a > \mu_0$.

- **Further Considerations**

Further budget constraints in portfolio optimization

- Choose target return $r \in \mathbb{R}$, the efficient portfolio is

$$\widetilde{\boldsymbol{x}}_r^{\mathcal{C}} = \underset{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}}=1;\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} \geq r,\ \widetilde{\boldsymbol{x}} \in \mathcal{C}}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x},$$

  where $\mathcal{C} \subset \mathbb{R}^{n+1}$ is a set that poses further restrictions.

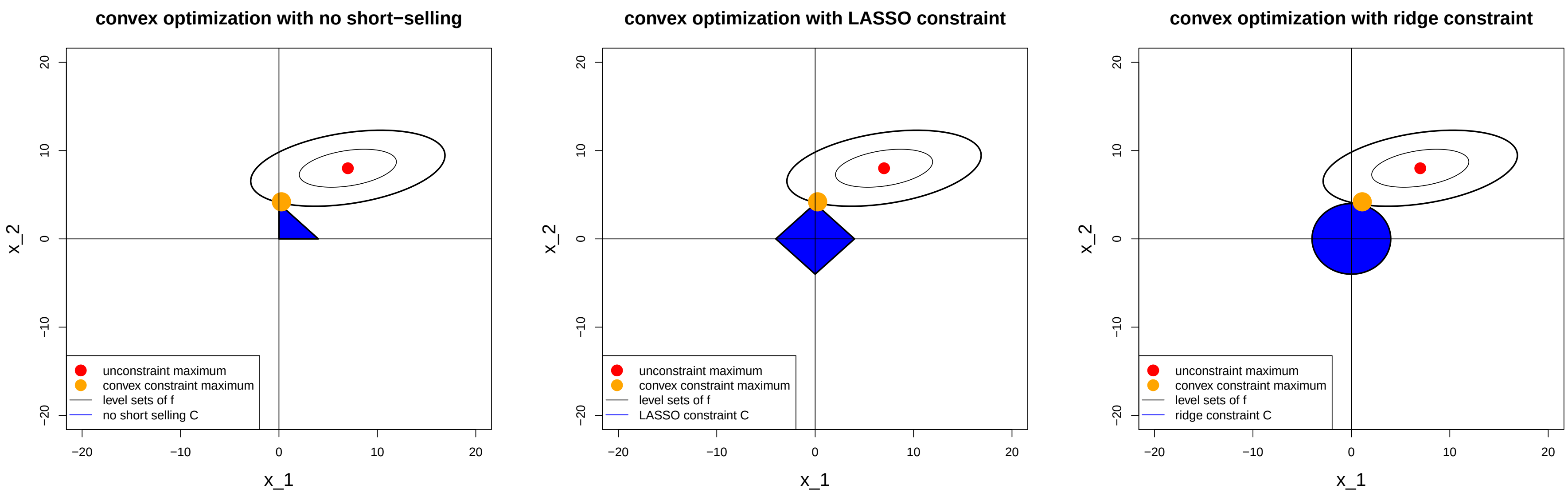
- Typically, $\mathcal{C}$ is a convex set which makes optimization still feasible. We give some examples:

  ⋆ no short-selling $\mathcal{C} = \{\boldsymbol{x} \in [0,1]^n; \sum_{j=1}^n x_j \leq 1\}$ (unit simplex);
  ⋆ bounded L^1-norm $\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^n; \sum_{j=1}^n |x_j| \leq 1\}$ (LASSO regularization);
  ⋆ bounded L^2-norm $\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^n; \sum_{j=1}^n x_j^2 \leq 1\}$ (ridge regularization),

  the latter two are borrowed from regression modeling in statistics.

  Note that $\mathcal{C}$ needs to be seen in conjunction with normalization $\widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1$.

Further budget constraints in portfolio optimization



An L^2 constraint is fundamentally different from an L^1 constraint because the former is differentiable. The latter leads to sparsity in portfolio selection, meaning, that certain assets will not be chosen at all, whereas under the former the weights can be very small. We refer to lectures on regularization in regression modeling.

Revisit of exponential utility and Gaussian returns

- As a motivation (and a motivation, only) we have started from the exponential utility function and multivariate Gaussian returns $\boldsymbol{R}$. This provides us with (note: we drop the expected return condition)

$$
\begin{aligned}
\widetilde{\boldsymbol{x}} &= \underset{\widetilde{\boldsymbol{x}}\in\mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top\widetilde{\boldsymbol{x}}=1}{\arg\max} \quad \mathbb{E}[u(W_1)] \\
&= \underset{\widetilde{\boldsymbol{x}}\in\mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top\widetilde{\boldsymbol{x}}=1}{\arg\max} \quad -\frac{1}{\alpha}\exp\left\{-\alpha w_0\left(1+\widetilde{\boldsymbol{x}}^\top\widetilde{\boldsymbol{\mu}}\right)+\frac{\alpha^2}{2}w_0^2\ \boldsymbol{x}^\top\Sigma\boldsymbol{x}\right\} \\
&= \underset{\widetilde{\boldsymbol{x}}_\rho\in\mathbb{R}^{n+1};\ \rho\in\mathbb{R}}{\arg\max} \quad -\frac{1}{\alpha}\exp\left\{-\alpha w_0\left(1+\widetilde{\boldsymbol{x}}_\rho^\top\widetilde{\boldsymbol{\mu}}\right)+\frac{\alpha^2}{2}w_0^2\ \boldsymbol{x}_\rho^\top\Sigma\boldsymbol{x}_\rho\right\} \\
&= \underset{\widetilde{\boldsymbol{x}}_\rho\in\mathbb{R}^{n+1};\ \rho\in\mathbb{R}}{\arg\max} \quad -\frac{1}{\alpha}\exp\left\{-\alpha w_0\left(1+\rho\right)+\frac{\alpha^2}{2}w_0^2\ \frac{(\rho^e)^2}{(\boldsymbol{\mu}^e)^\top\Sigma^{-1}\boldsymbol{\mu}^e}\right\}.
\end{aligned}
$$

This shows that the optimal return $\rho = \mu_0 + ((\boldsymbol{\mu}^e)^\top\Sigma^{-1}\boldsymbol{\mu}^e)/(\alpha w_0) > \mu_0$ is a decreasing function of risk aversion α and initial wealth w_0, i.e. select ρ vs. α.

- **Asset and Liability Management in Insurance**

Asset and Liability Management (ALM) in Insurance

- We are considering the idealized situation of fully diversified insurance technical risk, so that we are only left with a financial portfolio: assume that the liabilities can be described by a portfolio $\widetilde{\boldsymbol{z}} \in \mathbb{R}^{n+1}$ with $\widetilde{\boldsymbol{z}}^\top \widetilde{\boldsymbol{e}} = 1$.

- Assume we want to choose an investment strategy $\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1}$ that provides an extra return $r \in \mathbb{R}_+$ above $\widetilde{\boldsymbol{z}}^\top \widetilde{\boldsymbol{\mu}}$ at "minimal ALM risk".

- This motivates ALM strategy

$$\begin{aligned}
\widetilde{\boldsymbol{x}}_r &= \underset{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{n+1};\ \widetilde{\boldsymbol{e}}^\top \widetilde{\boldsymbol{x}} = 1;\ \widetilde{\boldsymbol{x}}^\top \widetilde{\boldsymbol{\mu}} \geq \widetilde{\boldsymbol{z}}^\top \widetilde{\boldsymbol{\mu}} + r}{\arg\min} (\boldsymbol{x} - \boldsymbol{z})^\top \Sigma (\boldsymbol{x} - \boldsymbol{z}) \\
&= \underset{\boldsymbol{x} \in \mathbb{R}^{n};\ \boldsymbol{x}^\top \boldsymbol{\mu}^e \geq \boldsymbol{z}^\top \boldsymbol{\mu}^e + r}{\arg\min} \boldsymbol{x}^\top \Sigma \boldsymbol{x} - 2\boldsymbol{z}^\top \Sigma \boldsymbol{x},
\end{aligned}$$

the latter optimization drops the first component by setting $(\widetilde{\boldsymbol{x}}_r)_0 = 1 - \boldsymbol{x}_r^\top \boldsymbol{e}$.

Lagrange problem

- We start with the Lagrange problem. The Lagrange function is given by

$$\mathcal{L}(\boldsymbol{x}, \lambda, r) = -\frac{1}{2}\boldsymbol{x}^\top \Sigma \boldsymbol{x} + \boldsymbol{z}^\top \Sigma \boldsymbol{x} - \lambda(\boldsymbol{x}^\top \boldsymbol{\mu}^e - (\boldsymbol{z}^\top \boldsymbol{\mu}^e + r)).$$

- The first KKT condition gives us, note that Σ is positive definite,

$$\boldsymbol{x} = \boldsymbol{z} - \lambda \Sigma^{-1} \boldsymbol{\mu}^e.$$

- Inserting this into the second KKT condition gives us

$$\boldsymbol{z}^\top \boldsymbol{\mu}^e + r = (\boldsymbol{\mu}^e)^\top \boldsymbol{x} = \boldsymbol{z}^\top \boldsymbol{\mu}^e - \lambda (\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e.$$

- Because of assumption (A1), $\boldsymbol{\mu}^e$ cannot be the zero vector, which allows us to calculate Lagrange multiplier λ. As a consequence

$$\boldsymbol{x}_r = \boldsymbol{z} + \frac{r}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e.$$

ALM problem solution

- The portfolio with extra return $r \geq 0$ at minimal ALM risk is given by

$$\boldsymbol{x}_r = \boldsymbol{z} + \frac{r}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e} \, \Sigma^{-1} \boldsymbol{\mu}^e.$$

- We calculate the ALM risk for $r \geq 0$

$$\mathrm{Var}\left((\boldsymbol{x}_r - \boldsymbol{z})^\top \boldsymbol{R} \right) \quad = \quad \frac{r^2}{(\boldsymbol{\mu}^e)^\top \Sigma^{-1} \boldsymbol{\mu}^e}.$$

- In solvency considerations extra return through ALM mismatch means extra risk, and the latter formula quantifies the size of this extra risk.