# Economic Theory of Financial Markets

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## Chapter 6:

## Multiperiod Models and Yield Curves

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# Economic Theory of Financial Markets

- **Chapter 6: Multiperiod Models and Yield Curves**

- **Multiperiod Cash Flows**

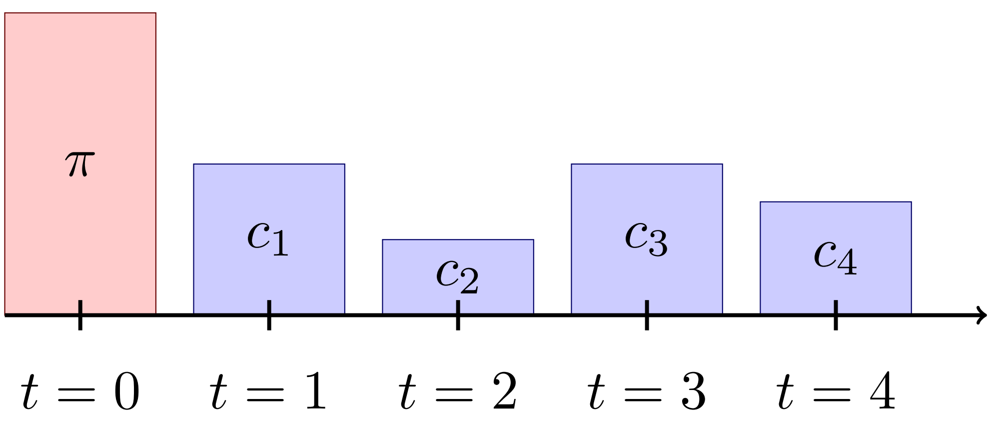
# $T + 1$ period models

- Choose discrete and finite time $t = 0, 1, \ldots, T$. Thus, we consider $T + 1$ periods.

- **Goal.** We would like to price the following cash flows at time 0

$$\boldsymbol{c} = (c_1, \ldots, c_T)^\top \in \mathbb{R}^T.$$

We will call this price $\pi = \pi(\boldsymbol{c}) \in \mathbb{R}$. We give an example for $T = 4$.



- **Deterministic Cash Flows**

# Definitions of positivity of vectors

Choose a vector $\boldsymbol{v} = (v_1, \ldots, v_n)^\top \in \mathbb{R}^n$ for some $n \geq 2$.

- $\boldsymbol{v} \geq 0$ (is non-negative) $\Longleftrightarrow$ $v_i \geq 0$ for all $1 \leq i \leq n$.
- $\boldsymbol{v} > 0$ (is positive) $\Longleftrightarrow$ $\boldsymbol{v} \geq 0$ and there exists $1 \leq k \leq n$ with $v_k > 0$.
- $\boldsymbol{v} \gg 0$ (is strictly positive) $\Longleftrightarrow$ $v_i > 0$ for all $1 \leq i \leq n$.

- We have

$$\boldsymbol{v} \gg 0 \implies \boldsymbol{v} > 0 \implies \boldsymbol{v} \geq 0.$$

- We use notation $\mathbb{R}^n_{++} = \{\boldsymbol{v} \in \mathbb{R}^n;\ \boldsymbol{v} \gg 0\}$.

# Security market

**Definition.** Choose $T, n \in \mathbb{N}$.
A security market is a pair $(\boldsymbol{\pi}, C)$ with $\boldsymbol{\pi} \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n \times T}$.

Each element $\pi_j$ of $\boldsymbol{\pi}$ is interpreted as the price at time 0 of the cash flow (security) $\boldsymbol{c}_j = (c_{j,1}, \ldots, c_{j,T})^\top \in \mathbb{R}^T$ which corresponds to the $j$-th row of $C \in \mathbb{R}^{n \times T}$:

$$\begin{pmatrix} \pi_1 \\ \vdots \\ \pi_j \\ \vdots \\ \pi_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_{1,1} & \cdots & c_{1,t} & \cdots & c_{1,T} \\ \vdots & \ddots & \vdots & & \vdots \\ c_{j,1} & \cdots & c_{j,t} & \cdots & c_{j,T} \\ \vdots & & \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,t} & \cdots & c_{n,T} \end{pmatrix} \in \mathbb{R}^{n \times T}.$$

Thus, we have $n$ different cash flows (securities) $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ with payments at times $t = 1, \ldots, T$ and prices $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)^\top \in \mathbb{R}^n$ at time 0.

**Question.** What are necessary assumptions to have a reasonable pricing system $\boldsymbol{\pi}$?

# Portfolio strategy

**Definition.**

- A portfolio strategy is a vector $\boldsymbol{x} \in \mathbb{R}^n$.

- A portfolio strategy $\boldsymbol{x} \in \mathbb{R}^n$ generates cash flow

$$C^\top \boldsymbol{x} = \left( \sum_{j=1}^n x_j c_{j,1}, \ldots, \sum_{j=1}^n x_j c_{j,T} \right)^\top \in \mathbb{R}^T.$$

- Portfolio strategy $\boldsymbol{x} \in \mathbb{R}^n$ has price at time 0: $\boldsymbol{\pi}^\top \boldsymbol{x}$.

# Arbitrage-free security market

**Definition.** A portfolio strategy $\boldsymbol{x} \in \mathbb{R}^n$ is an arbitrage opportunity if it satisfies one of the following two conditions:

(a) $\boldsymbol{\pi}^\top \boldsymbol{x} = 0$ and $C^\top \boldsymbol{x} > 0$, or

(b) $\boldsymbol{\pi}^\top \boldsymbol{x} < 0$ and $C^\top \boldsymbol{x} \geq 0$.

**Interpretation.** Arbitrage opportunity (a) means to have a portfolio strategy $\boldsymbol{x}$ of price zero that has non-negative payouts and at least one strictly positive payout. Arbitrage opportunity (b) means to receive money at time 0 that we do not have to pay back.

**Definition.** A security market $(\boldsymbol{\pi}, C)$ is arbitrage-free if it does not contain any arbitrage opportunity.

# Auxiliary lemma from Linear Algebra

**Stiemke's Lemma (1915).** Let $A \in \mathbb{R}^{n \times m}$. Precisely one of the following two statements holds true:

i) there exists $\boldsymbol{y} \in \mathbb{R}^m_{++}$ such that $A\boldsymbol{y} = 0$, or

ii) there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x}^\top A > 0$.

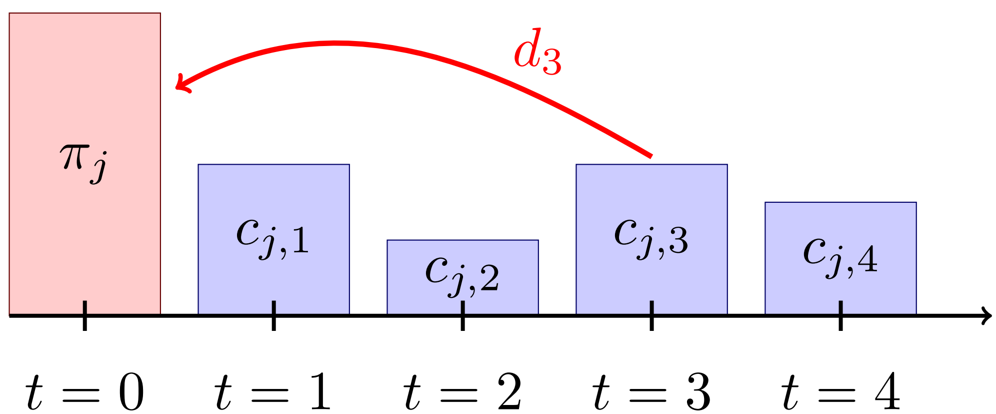
**Proof.** See lecture on Linear Algebra. $\square$

# No arbitrage theorem

**Theorem.** A security market $(\boldsymbol{\pi}, C)$ is arbitrage-free if and only if there exists $\boldsymbol{d} \in \mathbb{R}^T_{++}$ with $\boldsymbol{\pi} = C\boldsymbol{d}$.

**Interpretation.** $\boldsymbol{d} = (d_1, \ldots, d_T)^\top \in \mathbb{R}^T_{++}$ plays the role of discount factors

$$\pi_j = \sum_{t=1}^{T} d_t c_{j,t} \qquad \text{with } d_t > 0 \text{ for all } t.$$



# Proof: no arbitrage theorem

**Theorem.** A security market $(\boldsymbol{\pi}, C)$ is arbitrage-free if and only if there exists $\boldsymbol{d} \in \mathbb{R}^T_{++}$ with $\boldsymbol{\pi} = C\boldsymbol{d}$.

**Proof.** We define matrix

$$A = \begin{pmatrix} -\pi_1 & c_{1,1} & \cdots & c_{1,t} & \cdots & c_{1,T} \\ \vdots & \vdots & & \vdots & & \vdots \\ -\pi_j & c_{j,1} & \cdots & c_{j,t} & \cdots & c_{j,T} \\ \vdots & \vdots & & \vdots & & \vdots \\ -\pi_n & c_{n,1} & \cdots & c_{n,t} & \cdots & c_{n,T} \end{pmatrix} \in \mathbb{R}^{n \times (T+1)},$$

and apply Stiemke's Lemma to matrix $A$. Assume assertion i) of Stiemke's Lemma holds true, i.e., there exists $\boldsymbol{y} \in \mathbb{R}^{T+1}_{++}$ such that $A\boldsymbol{y} = 0$. This is equivalent to

$$\exists \boldsymbol{y} \in \mathbb{R}^{T+1}_{++} \qquad \text{such that} \qquad A\boldsymbol{y} = \left( -\pi_j y_0 + \sum_{t=1}^T c_{j,t} y_t \right)^\top_{1 \le j \le n} = 0.$$

Since $y_0 > 0$ this is equivalent to

$$\exists \boldsymbol{y} \in \mathbb{R}^{T+1}_{++} \qquad \text{such that} \qquad \sum_{t=1}^T c_{j,t} \frac{y_t}{y_0} = \pi_j \quad \text{for all } 1 \le j \le n.$$

From this we see that we have one redundancy providing that assertion i) of Stiemke's Lemma is equivalent to

$$\exists \boldsymbol{d} \in \mathbb{R}^T_{++} \qquad \text{such that} \qquad \sum_{t=1}^{T} c_{j,t} d_t = \pi_j \quad \text{for all } 1 \le j \le n.$$

Thus, if we can prove that assertion ii) of Stiemke's Lemma is equivalent to the existence of an arbitrage opportunity $\boldsymbol{x} \in \mathbb{R}^n$, the claim of the theorem follows. Assertion ii) of Stiemke's Lemma says that there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x}^\top A > 0$. This is equivalent to

$$\exists \boldsymbol{x} \in \mathbb{R}^n \qquad \text{such that} \qquad \left(-\boldsymbol{\pi}^\top \boldsymbol{x}, C^\top \boldsymbol{x}\right) > 0.$$

There are two cases: (1) $-\boldsymbol{\pi}^\top \boldsymbol{x} = 0$ and $C^\top \boldsymbol{x} > 0$, this is arbitrage opportunity (a), or (2) $-\boldsymbol{\pi}^\top \boldsymbol{x} > 0$ and $C^\top \boldsymbol{x} \ge 0$, this is arbitrage opportunity (b). Therefore, assertion ii) of Stiemke's Lemma is equivalent to the existence of an arbitrage opportunity. Because assertions i) and ii) of Stiemke's Lemma are mutually exclusive the proof is complete. $\square$

# Interpretation: no arbitrage theorem

**Theorem.** $(\boldsymbol{\pi}, C)$ is arbitrage-free if and only if there exists $\boldsymbol{d} \in \mathbb{R}^T_{++}$ with $\boldsymbol{\pi} = C\boldsymbol{d}$.

**Interpretation.**

- $\boldsymbol{d} = (d_1, \ldots, d_T)^\top \in \mathbb{R}^T_{++}$ plays the role of discount factors

$$\pi_j = \sum_{t=1}^{T} d_t c_{j,t} \qquad \text{with } d_t > 0 \text{ for all } t,$$

  i.e. arbitrage-free prices are discounted net values in our security market.

- There are no probabilities involved, i.e., everything is deterministic here, and we only use Linear Algebra.

- Discount factors $d_t > 0$ are positive, and they can be bigger than 1.

- There may be infinitely many discount factors: note that if we have two discount factors $\boldsymbol{d}_1 \neq \boldsymbol{d}_2 \in \mathbb{R}^T_{++}$ for $\boldsymbol{\pi}$, then every convex combination $\alpha\boldsymbol{d}_1 + (1-\alpha)\boldsymbol{d}_2 \in \mathbb{R}^T_{++}$, $0 \leq \alpha \leq 1$, is a discount factor, too, for the same pricing system $\boldsymbol{\pi}$.

# Completeness of security market

**Theorem.** $(\boldsymbol{\pi}, C)$ is arbitrage-free if and only if there exists $\boldsymbol{d} \in \mathbb{R}^T_{++}$ with $\boldsymbol{\pi} = C\boldsymbol{d}$.

In general, under no arbitrage we can have infinitely many discount factors $\boldsymbol{d} \in \mathbb{R}^T_{++}$.

**Definition.** A security market $(\boldsymbol{\pi}, C)$ is complete if for every cash flow $\boldsymbol{c} \in \mathbb{R}^T$ there exists a portfolio strategy $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{c} = C^\top \boldsymbol{x}$.

- Completeness is equivalent to $C$ having full rank $T$ (rows of $C$ span $\mathbb{R}^T$).
- Completeness means: we can replicate every cash flow $\boldsymbol{c}$ with securities from $C$.
- Under completeness, we may, w.l.o.g., assume that the first $T$ rows of $C$ are linearly independent. Thus, in that case $\widetilde{C} = (\boldsymbol{c}_1^\top, \ldots, \boldsymbol{c}_T^\top)^T \in \mathbb{R}^{T \times T}$ has full rank $T$ and we can calculate its inverse. For any $\boldsymbol{c} \in \mathbb{R}^T$ we can then choose $\boldsymbol{x}_{\boldsymbol{c}} \in \mathbb{R}^T$ such that $\boldsymbol{c} = \widetilde{C}^\top \boldsymbol{x}_{\boldsymbol{c}}$, in particular, this portfolio strategy is found by $\boldsymbol{x}_{\boldsymbol{c}} = (\widetilde{C}^\top)^{-1} \boldsymbol{c}$.

# No arbitrage and completeness of security markets

**Corollary.** A security market $(\boldsymbol{\pi}, C)$ is arbitrage-free and complete if and only if there exists a unique $\boldsymbol{d} \in \mathbb{R}^T_{++}$ with $\boldsymbol{\pi} = C\boldsymbol{d}$.

**Sketch of proof.** No arbitrage is equivalent to the existence of a discount factor $\boldsymbol{d} \in \mathbb{R}^T_{++}$. The uniqueness of the discount factor is then equivalent to the full rank $T$ property of $C$. $\square$

**Outlook.**

- This finite dimensional security market $(\boldsymbol{\pi}, C)$ is the simplest multiperiod no-arbitrage set-up, and there are many extensions/generalizations, e.g.
  - ⋆ discrete time, finite horizon and infinite probability spaces (Dalang–Morton–Willinger, 1990)
  - ⋆ continuous-time and infinite probability spaces (Delbaen–Schachermayer, 1994)
- In general, these frameworks are summarized under the so-called Fundamental Theorem of Asset Pricing (FTAP) theory, relying on a suitable definition of no arbitrage (depending on the chosen market modeling framework).

- **Term Structures of Interest Rates**

# No arbitrage and completeness of security markets

**Corollary.** A security market $(\boldsymbol{\pi}, C)$ is arbitrage-free and complete if and only if there exists a unique $\boldsymbol{d} \in \mathbb{R}^T_{++}$ with $\boldsymbol{\pi} = C\boldsymbol{d}$.

- W.l.o.g. we may under completeness assume that $C \in \mathbb{R}^{T \times T}$ with full rank $T$.
- Under no-arbitrage and completeness we have $\boldsymbol{d} = C^{-1}\boldsymbol{\pi}$, and for any cash flow $\boldsymbol{c} \in \mathbb{R}^T$ the portfolio strategy $\boldsymbol{x} = (C^\top)^{-1}\boldsymbol{c}$ generates $\boldsymbol{c} \in \mathbb{R}^T$, thus, this cash flow has price at time 0

$$\pi(\boldsymbol{c}) = \boldsymbol{\pi}^\top \boldsymbol{x} = \boldsymbol{\pi}^\top (C^\top)^{-1}\boldsymbol{c} = (C\boldsymbol{d})^\top (C^\top)^{-1}\boldsymbol{c} = \boldsymbol{d}^\top \boldsymbol{c}.$$

**Corollary.** Assume the security market $(\boldsymbol{\pi}, C)$ is arbitrage-free and complete. The (unique) no arbitrage price of any cash flow $\boldsymbol{c} \in \mathbb{R}^T$ is given by $\pi(\boldsymbol{c}) = \boldsymbol{d}^\top \boldsymbol{c}$.

# Zero-coupon bond (ZCB)

**Definition.** A (default-free) zero-coupon bond (ZCB) with maturity $m \in \{1, \ldots, T\}$ is given by the cash flow $\boldsymbol{e}_m = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^T$.

**Corollary.** Assume the security market $(\boldsymbol{\pi}, C)$ is arbitrage-free and complete. The (unique) no arbitrage price of ZCB $\boldsymbol{e}_m$ is given by $\pi(\boldsymbol{e}_m) = \boldsymbol{d}^\top \boldsymbol{e}_m = d_m > 0$.

**Remarks.**

- The above corollary gives unique ZCB prices at time 0 under no arbitrage *and* completeness. Having these (unique) prices of all ZCBs $0 \le m \le T$ we can uniquely price all cash flows (by forming appropriate linear combinations).

- If we do not have completeness we can, strictly speaking, only replicate and uniquely price cash flows $\boldsymbol{c}$ that are in the span of $C$.

- The discount factors may be $d_t > 1$ and $d_{t+1} > d_t$, no arbitrage only implies that they are strictly positive $\boldsymbol{d} \in \mathbb{R}^T_{++}$.

# Term structure of interest rates

Choose $\boldsymbol{d} \in \mathbb{R}^T_{++}$ such that $\boldsymbol{\pi} = C\boldsymbol{d}$ and, for simplicity, assume $1 \leq t \leq T$ are years.

- Spot rate/one-year interest rate at time 0

$$r_0 = d_1^{-1} - 1 \qquad \Longleftrightarrow \qquad d_1 = (1 + r_0)^{-1}.$$

- Forward rate at time 0 for maturity $t$ (annually compounded)

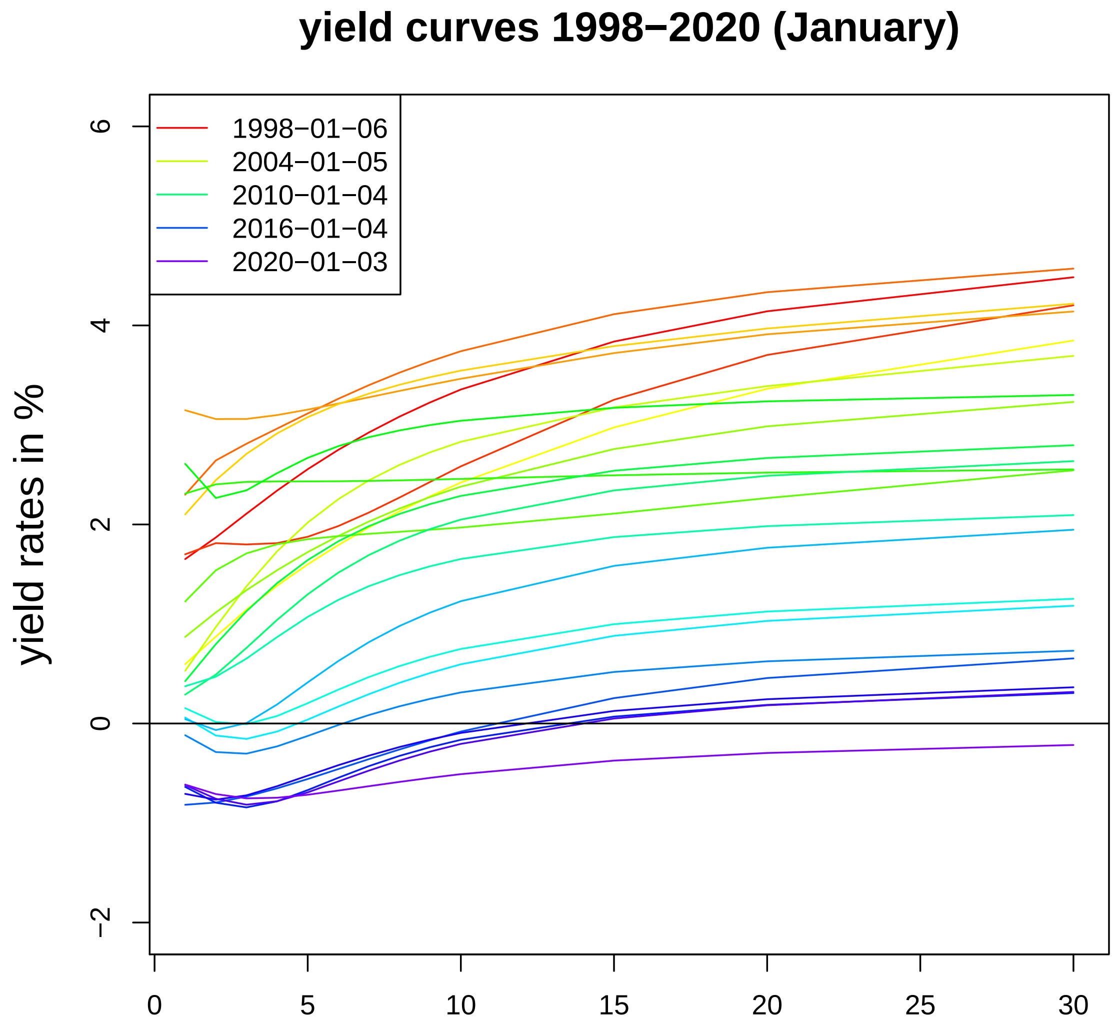
$$f_0(t) = \frac{d_{t-1}}{d_t} - 1 \qquad \Longleftrightarrow \qquad d_t = \frac{1}{1 + f_0(t)}\, d_{t-1}.$$

- Yield-to-maturity rate at time 0 for maturity $t$ (annually compounded)

$$y_0(t) = d_t^{-1/t} - 1 \qquad \Longleftrightarrow \qquad d_t = (1 + y_0(t))^{-t}.$$

- Yield curve at time 0 is given by $\boldsymbol{y}_0 = (y_0(1), \ldots, y_0(T))^\top \in \mathbb{R}^T$.

# Yield curves[1]




[1]Source: Swiss National Bank `https://data.snb.ch/de/topics/ziredev#!/cube/rendoblid`

- **Duration and ALM**

# Macaulay duration (1938)

- In this section we assume a constant interest rate $r > 0$ which gives us yield rate $y_0(t) \equiv r$ and discount factors $d_t = (1+r)^{-t}$. The price of a cash flow $\boldsymbol{c} \in \mathbb{R}^T$ under these assumptions is

$$\pi(\boldsymbol{c}; r) = \boldsymbol{d}^\top \boldsymbol{c} = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t}.$$

- We consider the sensitivity of the log-price $\log(\pi(\boldsymbol{c}; r))$ in $r$ (assume $\boldsymbol{c} > 0$)

$$\frac{d \log(\pi(\boldsymbol{c}; r))}{dr} = \frac{1}{\pi(\boldsymbol{c}; r)} \sum_{t=1}^{T} (-t) \, \frac{c_t}{(1+r)^{t+1}} = -\frac{1}{1+r} \, \mathcal{D}(\boldsymbol{c}; r),$$

with Macaulay duration (present value of weighted maturities)

$$\mathcal{D}(\boldsymbol{c}; r) \;=\; \frac{\sum_{t=1}^{T} t \, \frac{c_t}{(1+r)^t}}{\sum_{t=1}^{T} \frac{c_t}{(1+r)^t}} \;=\; \sum_{t=1}^{T} t \, \frac{\frac{c_t}{(1+r)^t}}{\sum_{s=1}^{T} \frac{c_s}{(1+r)^s}}.$$

# Immunization and ALM

- The Macaulay duration is useful to describe first order changes in prices w.r.t. to interest rate changes $r \mapsto r + \Delta r$. Note (Taylor expansion)

$$\pi(\boldsymbol{c}; r + \Delta r) = \pi(\boldsymbol{c}; r) \left[ 1 - \frac{\mathcal{D}(\boldsymbol{c}; r)}{1 + r} \Delta r \right] + o(\Delta r), \qquad \text{as } \Delta r \to 0.$$

- Assume we have insurance liabilities $\boldsymbol{c}_L \in \mathbb{R}_+^T$ and financial assets $\boldsymbol{c}_A \in \mathbb{R}_+^T$. Typically, these two cash flows differ and we require that they have the same net present value at time 0: $\pi(\boldsymbol{c}_L; r) = \boldsymbol{d}^\top \boldsymbol{c}_L \stackrel{!}{=} \boldsymbol{d}^\top \boldsymbol{c}_A = \pi(\boldsymbol{c}_A; r)$.

- Immunization against an instantaneous small change $\Delta r$ in interest rate $r$, means that we should match the durations $\mathcal{D}(\boldsymbol{c}_L; r) \stackrel{!}{=} \mathcal{D}(\boldsymbol{c}_A; r)$ because in that case we have (under net present value equalization)

$$\pi(\boldsymbol{c}_L; r + \Delta r) - \pi(\boldsymbol{c}_A; r + \Delta r) = o(\Delta r), \qquad \text{as } \Delta r \to 0.$$

- **Term Structures in Continuous Time**

# Extension to continuous time

- We extend the above framework to continuous (and finite) time $t \in [0, T]$.
- The forward rate at time $s$ for maturities $t > s$ is a function $t > s \mapsto f(s, t)$.
- A (default-free) ZCB with maturity $t$ is a financial instrument that pays a fixed amount of size 1 at time $t$.
- The price at time $s \leq t$ of a ZCB with maturity $t$ is given by

$$P(s, t) = \exp\left\{ -\int_s^t f(s, u) du \right\},$$

assuming that the forward rate is integrable.

# Relating continuous time to discrete time

- The price at time 0 of a ZCB with maturity $t \in \{1, \ldots, \lfloor T \rfloor\}$ is given by

$$
\begin{aligned}
P(0,t) &= \exp\left\{-\int_0^t f(0,u)du\right\} = \exp\left\{-\sum_{k=1}^t \int_{k-1}^k f(0,u)du\right\} \\
&= \prod_{k=1}^t \exp\left\{-\int_{k-1}^k f(0,u)du\right\} \stackrel{\text{def.}}{=} \prod_{k=1}^t \frac{1}{1+f_0(k)} = d_t,
\end{aligned}
$$

  where $f_0(k)$ are exactly the discrete time forward rates at time 0 defined in the previous section, and $d_t$ is the resulting discount factor for maturity $t \in \mathbb{N}$.

- $(f(0,t))_{t>0}$ is called continuously compounded forward rate at time 0.

- $(f_0(t))_{t \in \mathbb{N}}$ is called annually compounded forward rate at time 0, supposed $t \in \mathbb{N}$ correspond to years.

# Term structure of interest rates

- Forward rates at time $s$ are obtained from ZCBs by (subject to existence)

$$f(s,t) = -\frac{\partial}{\partial t} \log P(s,t).$$

- Spot/short rate at time $s$ (subject to existence)

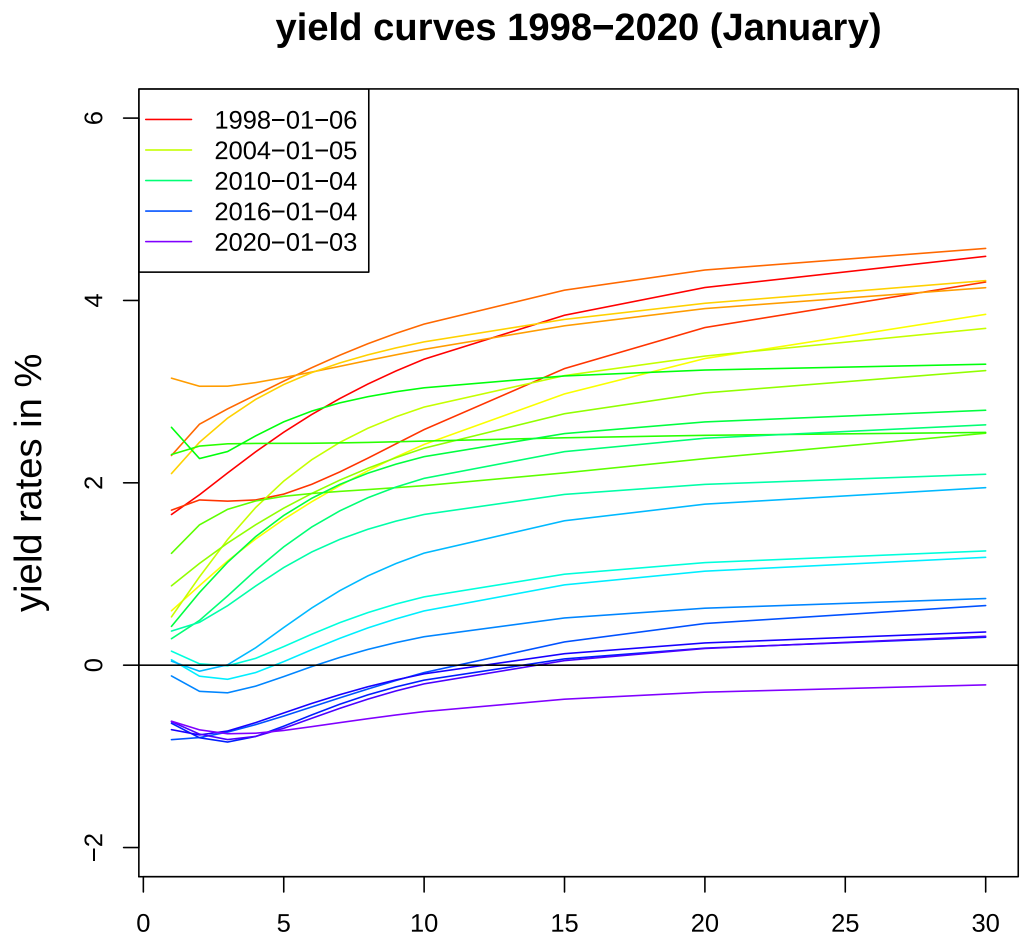
$$r_s = \lim_{t \downarrow s} f(s,t).$$

- Continuously compounded yield rate at time $s$ for maturity $t > s$

$$y(s,t) = -\frac{1}{t-s} \log P(s,t) = -\frac{1}{t-s} \int_s^t f(s,u)du.$$

- Continuously compounded yield curve at time $s$ is given by $u > 0 \mapsto y(s, s+u)$.

# Yield curves




In fact, these are continuously compounded yield curves $u > 0 \mapsto y(s, s+u)$!

- **Interest Rate Shocks in Continuous Time**

# Interest rate shocks

- Set forward rate $f = (f(t))_{t>0} = (f(0,t))_{t>0}$ at time 0. Then,

$$P(0,t;f) \quad = \quad \exp\left\{-\int_0^t f(u)du\right\}.$$

- **1st Goal.** Study an instantaneous constant interest rate shock $\Delta$

$$P(0,t;f+\Delta) \quad = \quad \exp\left\{-\int_0^t f(u)+\Delta du\right\} \;=\; P(0,t;f)\exp\{-t\Delta\}.$$

- Similarly to the duration section we would like to understand immunization for these instantaneous constant interest rate shocks (parallel shifts).

# Reservation about this setup

**Claim.** In general, instantaneous constant interest rate shocks $\Delta$ (parallel shifts) induce "arbitrage" (we did not define the right version of arbitrage here).

**Example.** Consider 3 ZCBs with maturities $m = 1, 2, 3$ and prices $P(0, 1)$, $P(0, 2)$ and $P(0, 3)$. We can then choose $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{x} \neq 0$ such that

$$x_1 P(0, 1) + x_2 P(0, 2) + x_3 P(0, 3) = 0. \tag{1}$$

Thus, $\boldsymbol{x}$ is a zero net investment portfolio.

Consider an instantaneous interest rate shock $\Delta \neq 0$. This provides us with portfolio value

$$V(\Delta) = x_1 P(0, 1)e^{-\Delta} + x_2 P(0, 2)e^{-2\Delta} + x_3 P(0, 3)e^{-3\Delta}.$$

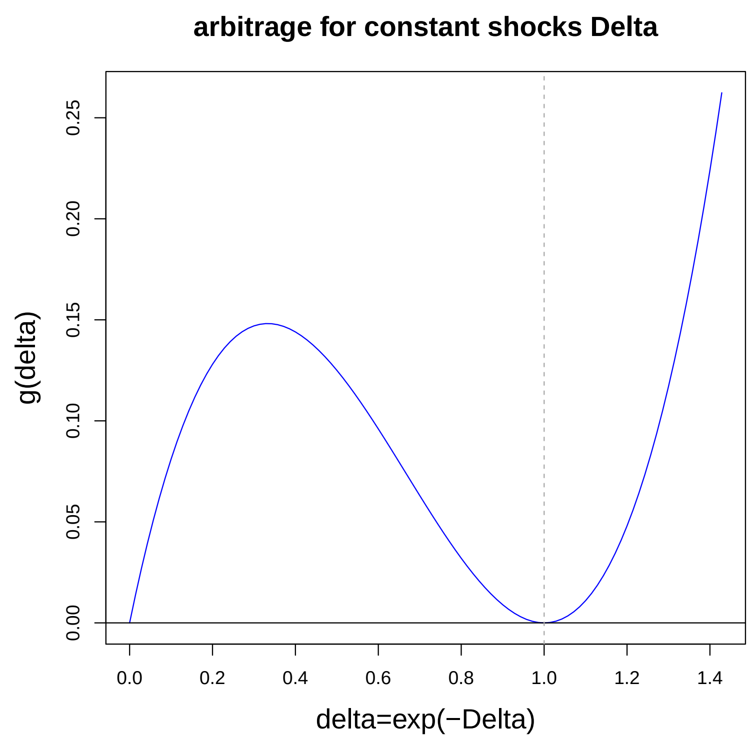
This is a polynomial of degree 3 for variable $\delta = e^{-\Delta} > 0$.

Thus, we can consider function on $\delta \in (0, \infty)$

$$g(\delta) = x_1 P(0,1)\delta + x_2 P(0,2)\delta^2 + x_3 P(0,3)\delta^3,$$

with the following requirements

$$\begin{aligned} g(1) &= 0 && \text{net investment zero (1),} \\ g'(1) &= 0 && \text{net investment zero (1) is a local minimum,} \\ \lim_{\delta \downarrow 0} g(\delta) &= 0, \\ \lim_{\delta \to \infty} g(\delta) &= \infty. \end{aligned}$$




A solution is
$x_1 P(0,1) = 1$,
$x_3 P(0,3) = 1$ and
$x_2 P(0,2) = -2x_3 P(0,3)$.

This provides $g(\delta) > 0$ for all $\delta \neq 1$, i.e. *any* interest shock $\Delta \neq 0$ gives a positive value $V(\Delta) > 0$. □

# Fisher–Weil immunization (1971)

- Assume we have an insurance liability with a fixed payment $L > 0$ a time $T$ (here, time can run beyond $T$). Its present value at time 0 is given by

$$V(L, f) = L\ P(0, T; f) = L \exp\left\{ -\int_0^T f(u) du \right\}.$$

- Assume we can buy ZCBs with $n$ different maturities $m_1, \ldots, m_n > 0$. If we hold a cash flow $A = (A_1, \ldots, A_n)$ with $A_k \geq 0$ units of ZCB with maturity $m_k$ we have asset value at time 0

$$V(A, f) = \sum_{k=1}^{n} A_k\ P(0, m_k; f) = \sum_{k=1}^{n} A_k \exp\left\{ -\int_0^{m_k} f(u) du \right\}.$$

- The equivalence principle requires $V(L, f) = V(A, f)$ at time 0.

- **2nd Goal.** Study general instantaneous interest rate shocks $f(\cdot) \mapsto f(\cdot) + \Delta(\cdot)$.

# Shiu (1987) result on Fisher–Weil immunization

Fisher–Weil immunization is given by

$$\mathcal{D}(A;f) = \frac{\sum_{k=1}^{n} m_k A_k \; P(0,m_k;f)}{\sum_{k=1}^{n} A_k \; P(0,m_k;f)} \stackrel{!}{=} T = \frac{T \; L \; P(0,T;f)}{L \; P(0,T;f)}. \tag{2}$$

**Theorem.** Assume that we have the equivalence principle and Fisher–Weil immunization (2) for positive liability $L > 0$ and non-negative cash flows $A \geq 0$. Assume the instantaneous interest rate shock $u \mapsto \Delta(u)$ is continuously differentiable.

- If $\Delta(u)^2 - \Delta'(u) \geq 0$ for all $u$, then $V(A, f + \Delta) \geq V(L, f + \Delta)$.
- If $\Delta(u)^2 - \Delta'(u) \leq 0$ for all $u$, then $V(A, f + \Delta) \leq V(L, f + \Delta)$.

The 2nd case $\Delta' \geq \Delta^2 \geq 0$ (counter-clockwise turns) is dangerous.

# Small lemma from analysis

For the proof of this theorem we first provide a small lemma from analysis.

**Lemma.** Assume $\Delta$ is continuously differentiable and define

$$g(t) = \exp\left\{-\int_T^t \Delta(u)du\right\}.$$

$g$ is twice continuously differentiable with

1. $g''(t) = g(t)\left[\Delta^2(t) - \Delta'(t)\right]$;

2. $g(T) = 1$ and $g'(T) = -\Delta(T)$;

3. $g(t) = g(T) + g'(T)(t-T) + \int_T^t (t-w)g''(w)dw$.

**Proof.** The proof is immediate; the last statement uses integration by parts. $\square$

# Proof of the Shiu (1987) result

**Proof.** The following equivalences also hold with the opposite sign $\le$ instead of $\ge$.

$$
\begin{aligned}
V(A, f+\Delta) &\ge V(L, f+\Delta) \\
&\Longleftrightarrow \\
L &\le \sum_{k=1}^{n} A_k \exp\left\{-\int_T^{m_k} f(u)+\Delta(u)du\right\} \\
&= \sum_{k=1}^{n} A_k \exp\left\{-\int_T^{m_k} f(u)du\right\} \exp\left\{-\int_T^{m_k} \Delta(u)du\right\} \\
&\stackrel{\text{def.}}{=} \sum_{k=1}^{n} a_k \, g(m_k),
\end{aligned}
$$

with $\sum_{k=1}^{n} a_k = L$ (equivalence principle). Thus,

$$
V(A, f+\Delta) \ge V(L, f+\Delta) \quad \Longleftrightarrow \quad \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} a_k g(m_k). \tag{3}
$$

Using the equivalence principle and Fisher–Weil immunization (2) we receive

$$\sum_{k=1}^{n} m_k a_k = \mathcal{D}(A;f) \sum_{k=1}^{n} a_k = T \sum_{k=1}^{n} a_k,$$

and as a result

$$\sum_{k=1}^{n} (m_k - T) a_k = 0. \tag{4}$$

The above lemma provides us with

$$g(t) = 1 - \Delta(T)(t-T) + \int_T^t (t-w) g''(w) dw.$$

We use this for the right-hand side of (3), and in the 3rd step we use (4),

$$\begin{aligned}
\sum_{k=1}^{n} a_k g(m_k) - \sum_{k=1}^{n} a_k &= \sum_{k=1}^{n} a_k \left(g(m_k) - 1\right) \qquad (5) \\
&= \sum_{k=1}^{n} a_k \left( -\Delta(T)(m_k - T) + \int_T^{m_k} (m_k - w) g''(w) dw \right) \\
&= \sum_{k=1}^{n} a_k \int_T^{m_k} (m_k - w) g''(w) dw.
\end{aligned}$$

Note that in the last integral the difference $m_k - w$ always has the same sign, so we can apply the mean value theorem that gives us some $w_k$ either in $[T, m_k]$ or in $[m_k, T]$ (depending on the sign) such that the last term is equal to

$$\begin{aligned} \sum_{k=1}^n a_k g(m_k) - \sum_{k=1}^n a_k &= \sum_{k=1}^n a_k g''(w_k) \int_T^{m_k} (m_k - w) dw \\ &= \sum_{k=1}^n a_k g''(w_k) \frac{(m_k - T)^2}{2}. \end{aligned}$$

Since that cash flows $A \geq 0$, we have for all $k$: $a_k (m_k - T)^2/2 \geq 0$. Therefore, the sign in (3) is determined by

$$g''(w_k) = g(w_k) \left[ \Delta^2(w_k) - \Delta'(w_k) \right],$$

which, by assumption on $\Delta$, is either positive or negative for all $k$. This finishes the proof. $\square$

# Alternative by Fong–Vasiček (1984) (1/2)

Fong–Vasiček (1984) use $e^x \geq 1 + x$ to get lower bound $g(t) = \exp\{-\int_T^t \Delta(u)du\} \geq 1 - \int_T^t \Delta(u)du$. Inserting this into (5) gives us

$$\sum_{k=1}^n a_k g(m_k) - \sum_{k=1}^n a_k \quad \geq \quad \sum_{k=1}^n a_k \int_{m_k}^{T} \Delta(u)du \stackrel{\text{def.}}{=} \sum_{k=1}^n a_k h(m_k).$$

Using the mean value theorem we have

$$\begin{aligned} h(m_k) &= h(T) + h'(T)(m_k - T) + h''(w_k)\frac{(m_k - T)^2}{2} \\ &= -\Delta(T)(m_k - T) - \Delta'(w_k)\frac{(m_k - T)^2}{2}. \end{aligned}$$

Using (4) we have

$$\begin{aligned} \sum_{k=1}^n a_k g(m_k) - \sum_{k=1}^n a_k &\geq -\frac{1}{2}\sum_{k=1}^n a_k \Delta'(w_k)(m_k - T)^2 \\ &\geq -\frac{1}{2}\left(\max_{w_k \in [m_k, T]} \Delta'(w_k)\right)\sum_{k=1}^n a_k (m_k - T)^2. \end{aligned}$$

# Alternative by Fong–Vasiček (1984) (2/2)

**Theorem.** Assume that we have the equivalence principle and Fisher–Weil immunization (2) for positive liability $L > 0$ and cash flows $A \geq 0$. Assume the instantaneous interest rate shock $u \mapsto \Delta(u)$ is continuously differentiable. If

$$-\left(\max_{w_k \in [m_k, T]} \Delta'(w_k)\right) \sum_{k=1}^{n} A_k \exp\left\{-\int_T^{m_k} f(u)du\right\} \frac{(m_k - T)^2}{2} \geq 0,$$

we receive

$$V(A, f + \Delta) \geq V(L, f + \Delta).$$

This result also involves the convexity (beyond duration) defined by, see also (2),

$$\mathcal{C}(A; f) \;=\; \frac{\sum_{k=1}^{n} m_k^2 A_k \; P(0, m_k; f)}{\sum_{k=1}^{n} A_k \; P(0, m_k; f)},$$

and the last term in the above theorem can be controlled by convexity matching.

# Outlook

- The above framework can now be generalized to stochastic interest rate modeling in discrete time. This is an essential next step in valuation and solvency modeling. We refer to the book of Föllmer–Schied *Stochastic Finance - An Introduction in Discrete Time*, to the ETH lecture *Mathematical Foundations for Finance* and to our lecture notes *Market-Consistent Actuarial Valuation*.

- One can introduce insurance technical risk beyond financial risks, in general, this will lead to incompleteness of markets.

- One can go over to continuous time modeling, using the Black–Scholes model, the Vasiček short rate model, or a more complex stochastic process. We refer to lectures in mathematical finance.

- One can study different asset classes that have different stochastic behavior.

- **Thank you for attending the lecture!**