Variable Annuities with Guaranteed Minimum Withdrawal Benefits

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ETH Zurich and Princeton University

Risk Day 2016, ETH Zurich

Variable annuities with GMxB

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- GMWB: Guaranteed minimum withdrawal benefits

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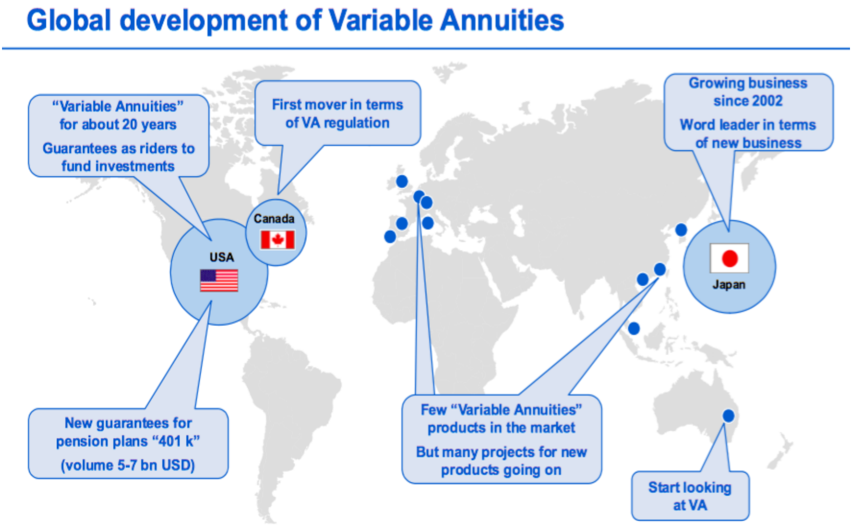
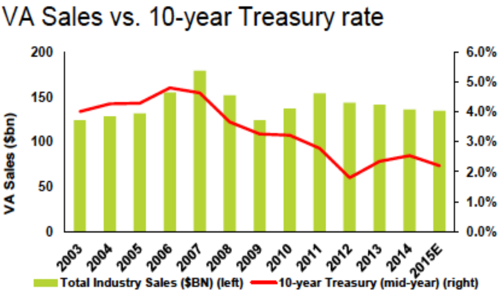
(See Bauer, Kling and Russ (2008) for an overview)

From Gerold Studer's talk at the SAA Annual Meeting 2010



VA sales in the US



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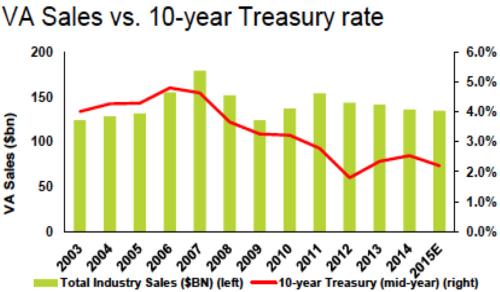

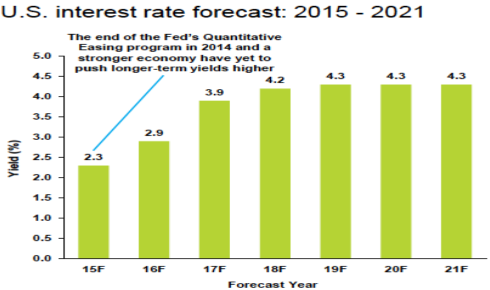

Increasing popularity of GMWB with high water mark features (ratchets)

Description on the Vanguard Group website of one of their VA+GMWB

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- Derive a hedging strategy

The financial market

Dynamics of an underlying mutual fund

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Market filtration

$$\mathbb{F} = (\mathcal{F}_t) \quad \text{generated by } (S_t)$$

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Annuity account

$$dA_t = ((r - q)A_t - w_t)dt + \sigma A_t dB_t, \quad A_0 = c_1$$

with absorption at 0

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

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Income rate

$$f(w_t, M_t) = (1 - p_1)\min(w_t, \alpha M_t) + (1 - p_2)\max(w_t - \alpha M_t, 0), \quad w_t \leq \beta M_t$$

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- Typically, $k(T) = 0$

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&\quad - \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t>0\}} e^{-rs} \left[qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0) \right] ds
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- One has $A_0 + D(A_0) = E(A_0)$ for

$$E(A_0) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\theta e^{-rs} f(w_s, M_s) ds + e^{-r\theta}(1 - k(\theta)) A_\theta\right]$$

worst case policy holder behavior ... not actual policy holder behavior!

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- **Good news:** $E(A_0)$ is attained for w, θ only depending on market information $\mathbb{F}$
- **Bad news:** The optimization problem $E(A_0)$ is not Markovian

Adding more state variables

For given $0 \leq t \leq T$, $0 \leq a \leq m$, $0 \leq z \leq \gamma m$, w, define

$$\begin{aligned} dA_s^{t,a,w} &= ((r-m)A_s^{t,a,w} - w_s)ds + \sigma A_s^{t,a,w} dB_s, \quad A_t^{t,a,w} = a \\ M_s^{t,a,g,w} &= m \vee \sup_{t \leq u \leq s} A_u^{t,a,m,w} \\ dZ_s^{t,z,w} &= w_s ds, \quad Z_t^{t,z,w} = z \end{aligned}$$

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A "non-standard" standard stochastic control problem

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V(t,a,m,z) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T e^{-r(s-t)} f(w_s, M_s^{t,a,m,w}) ds + e^{-r(\theta-t)}(1-k(\theta)) A_\theta^{t,a,w} \right]
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where w and θ are adapted to $\mathcal{F}_s^t = \sigma(B_s - B_t)$, $s \in [t, T]$

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One has $V(0,a,a,0) = E(a)$

The HJB equation

Theorem

$V(t,a,m,z)$ is a viscosity solution of

$$\begin{aligned}
\min(-v_t - H(a,m,v,v_a,v_z,v_{aa}), v-(1-k)a) &= 0 \text{ for } 0<a<m,\ 0\leq z<\gamma m \\
\min(-v_t - H_0(a,v,v_a,v_{aa}), v-(1-k)a) &= 0 \text{ for } 0<a<m,\ z=\gamma m \\
v(t,0,m,z) &= \psi(t,m,z) \\
v_m(t,m,m,z) &= 0 \\
v(T,a,m,z) &= a,
\end{aligned}$$

where

$$H(a,m,z,v,v_a,v_z,v_{aa}) =$$

$$\sup_{0\leq w\leq \beta m}\{f(w,m)+w(v_z-v_a)\} - rv + (r-q)av_a + \frac{1}{2}\sigma^2 a^2 v_{aa}$$

$$H_0(a,v,v_a,v_{aa}) = -rv + (r-q)av_a + \frac{1}{2}\sigma^2 a^2 v_{aa}$$

$$\psi(t,m,z) = \sup_{0\leq w\leq \beta m}\int_t^T e^{-r(s-t)}f(w_s,m)ds \quad \text{such that} \int_t^T w_s ds \leq \gamma m - z.$$

Non-linear parabolic PDE on $[0,T]\times\mathbb{R}^3$ with a free boundary and unusual boundary conditions

Reducing the dimension

Theorem

$V(t,a,m,z) = mW(t,a/m,z/m) = mW(t,x,y)$, where W is a viscosity solution of

$$\begin{aligned}
\min(-v_t - G(x,y,v,v_x,v_y,v_{xx}), v-(1-k)x) &= 0 \quad \text{for } (x,y)\in(0,1)\times[0,\gamma) \\
\min(-v_t - G_0(x,v,v_x,v_{xx}), v-(1-k)x) &= 0 \quad \text{for } (x,y)\in(0,1)\times\{\gamma\} \\
v(t,0,y) &= \zeta(t,y) \\
v_x(t,1,y) + yv_y(t,1,y) &= v(t,1,y) \\
v(T,x,y) &= x,
\end{aligned}$$

where

$$G(x,y,v,v_x,v_y,v_{xx}) =$$

$$\sup_{0\le u\le\beta}\{u(v_y - v_x) + f(u,1)\} - rv + (r-q)xv_x + \frac{1}{2}\sigma^2x^2v_{xx}$$

$$G_0(x,v,v_x,v_{xx}) = -rv + (r-q)xv_x + \frac{1}{2}\sigma^2x^2v_{xx}$$

$$\zeta(t,y) = \sup_{0\le u\le\beta}\int_t^T e^{-r(s-t)}f(u_s,1)ds \quad \text{such that} \int_t^T u_s ds \le \gamma - y.$$

Non-linear parabolic PDE on $[0,T]\times[0,1]\times[0,\gamma]$ with a free boundary and (even more) unusual boundary conditions

Worst case strategy

- The worst case withdrawal strategy $\hat{w}(t, A_t, M_t, Z_t)$ is given by the maximizer of

$$w \mapsto f(w, M_t) + w\left[V_z(t, A_t, M_t, Z_t) - V_a(t, A_t, M_t, Z_t)\right]$$

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- The issuer can set $k(t)$ so that the worst case policy holder never surrenders early

Numerical scheme

- Semi-Lagrangian scheme with an obstacle
- Backwards in time
- Solves an optimization problem in every time-step
- Converges to the true solution if the mesh size of the discretization goes to zero
- Gives approximations to $V(t, a, m, z)$ and the worst case behavior $\hat{w}$ and $\hat{\theta}$

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The issuer can **super-hedge** the contract by trading in S_t and the money market account

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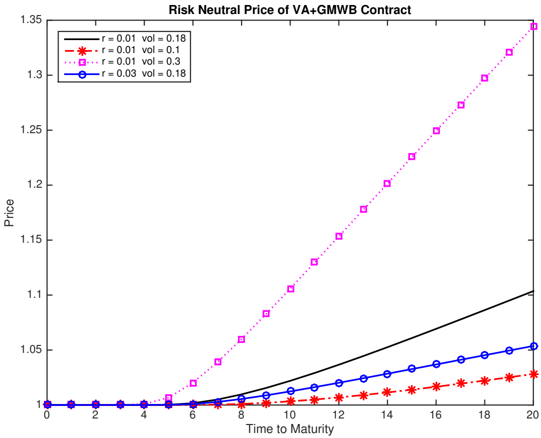
Proof: Itô's formula

Numerical results



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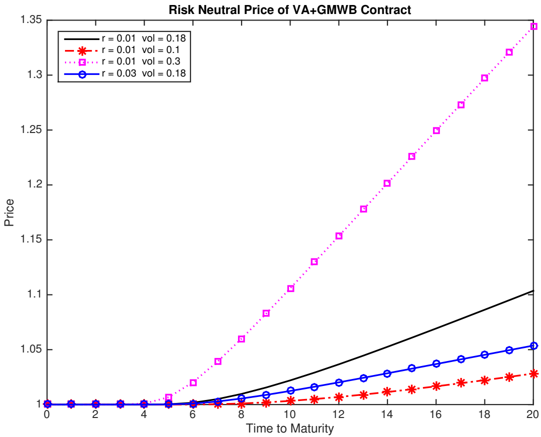
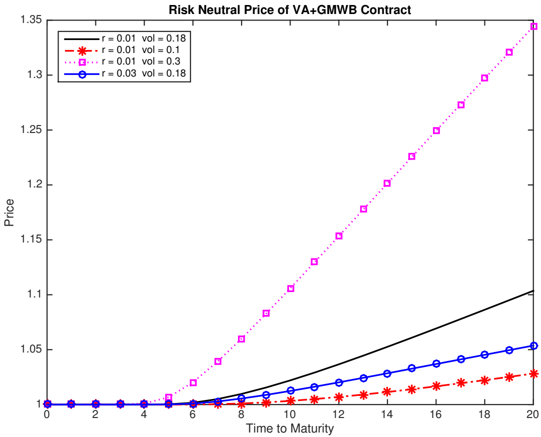


- Price is increasing in σ

Numerical results



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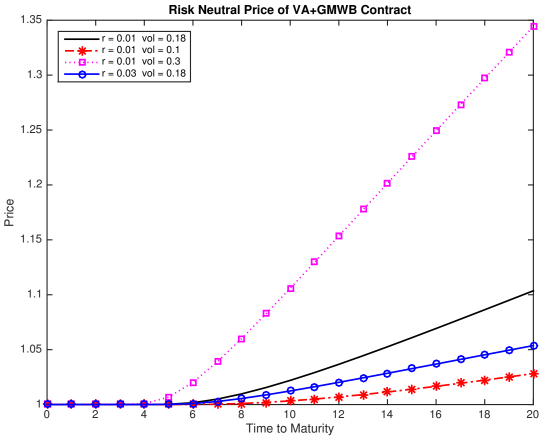

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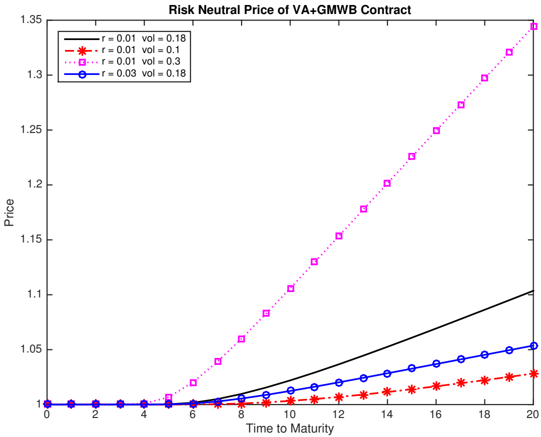
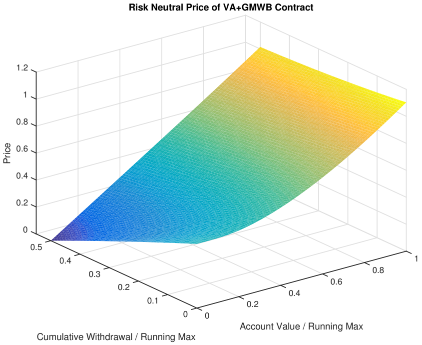
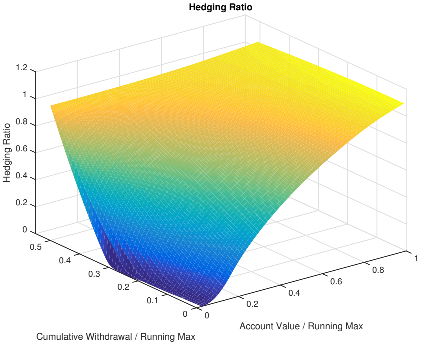


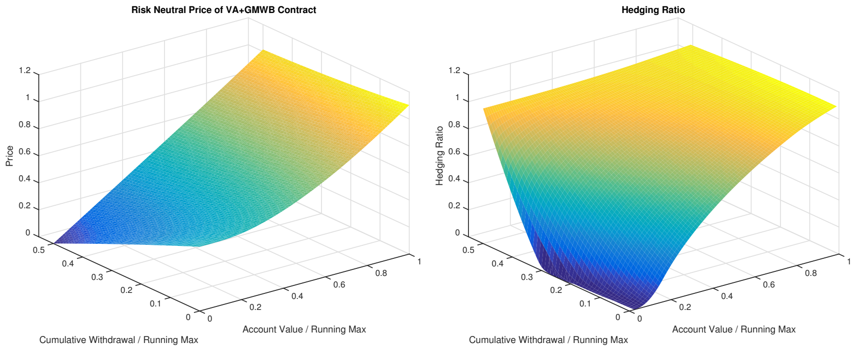
- Price is increasing in σ ... not taken into account by insurance companies
- Price is decreasing in r ... makes these products difficult to sell in a low interest rate environment

Price and hedging ratio as functions of $x = a/m$ and $y = z/m$





Price and hedging ratio as functions of $x = a/m$ and $y = z/m$



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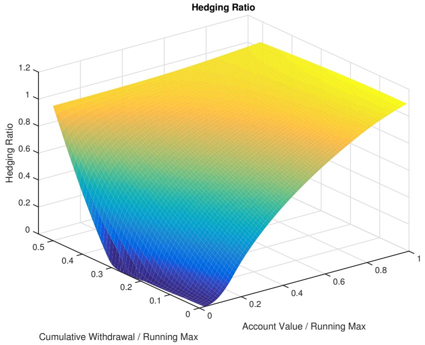
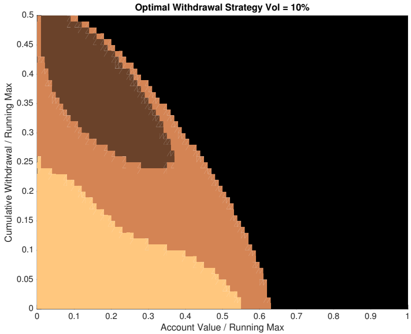


- Price is increasing in $x = a/m$ and decreasing in $y = z/m$
- The hedge is always long in S_t in contrast to the hedge of a put option)

Worst case withdrawal and surrender



- $T - t = 5$ years, $r = 1\%$, $\sigma = 18\%$, $q = 0.8\%$, $p_1 = 0$, $p_2 = 5\%$, $k \equiv 0$

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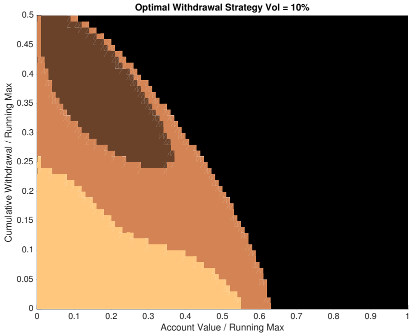

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- yellow = full withdrawal (at rate βM_t)

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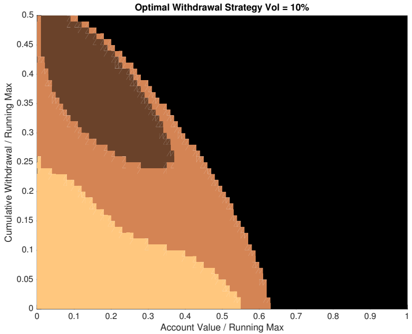

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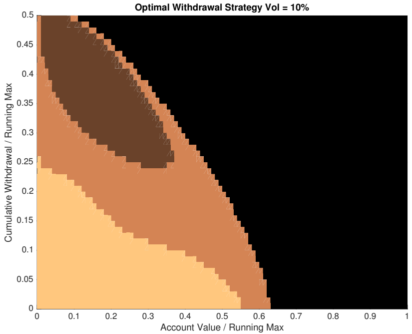

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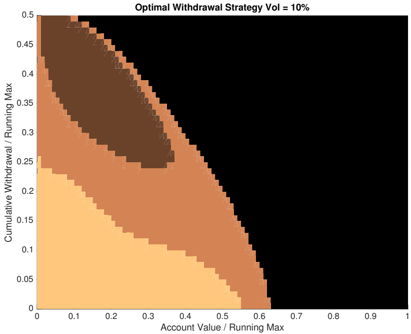

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- yellow = full withdrawal (at rate βM_t)
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- black = surrender

Discouraging early surrender

Set the surrender penalty function such that

$$k(t) \geq (T - t)q$$

Thank You