Variable Annuities with Guaranteed Minimum Withdrawal Benefits

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Risk Day 2016, ETH Zurich
Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed
Variable annuities with GMxB

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- Variable annuities with protection features have gained popularity
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- GMDB: Guaranteed minimum death benefits
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- GMAB: Guaranteed minimum accumulation benefits
- GMIB: Guaranteed minimum income benefits
- GMWB: Guaranteed minimum withdrawal benefits
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- GMWB: Guaranteed minimum withdrawal benefits

(See Bauer, Kling and Russ (2008) for an overview)
Global development of Variable Annuities

**USA**
- New guarantees for pension plans “401 k” (volume 5-7 bn USD)

**Canada**
- First mover in terms of VA regulation

**Japan**
- Growing business since 2002
- Word leader in terms of new business

**Growing “Variable Annuities”**
- Few “Variable Annuities” products in the market
- But many projects for new products going on
- Start looking at VA
VA sales in the US
VA sales in the US

**VA Sales vs. 10-year Treasury rate**

- **VA Sales ($bn)**
- **200**
- **150**
- **100**
- **50**
- **0**


- **Total Industry Sales ($BN) (left)**
- **10-year Treasury (mid-year) (right)**

**U.S. interest rate forecast: 2015 - 2021**

- **The end of the Fed's Quantitative Easing program in 2014 and a stronger economy have yet to push longer-term yields higher**

- **Yield (%)**
- **2.3**
- **2.9**
- **3.9**
- **4.2**
- **4.3**
- **4.3**
- **4.3**

Increasing popularity of GMWB with high water mark features (ratchets)

Description on the Vanguard Group website of one of their VA+GMWB

When your Secure Income rider becomes effective, your total withdrawal base (TWB) is set and locked in.

Should market performance cause the value of your investments to decline, your TWB remains locked in, and your withdrawal amount won’t decrease.

Each year, on the anniversary of your effective date, your TWB is assessed. If the value of your investments has increased due to market gains, your TWB increases and locks in at a higher rate—offering you the opportunity to take higher annual withdrawals.
What’s the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
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- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
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- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
- Solve the HJB equation numerically
- Gain insights into how the price depends on the risk free rate, volatility, ... the worst case policy holder behavior from the point of view of the issuer
- How insurance companies should set the fee structure
- How surrender penalties can be set to disincentivize early surrender
- Derive a hedging strategy
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The financial market

Dynamics of an underlying mutual fund

\[ dS_t = S_t(\mu dt + \sigma dW_t) = S_t(r dt + \sigma dB_t) \]
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where

- \( W \) is a Brownian motion under the real world probability measure \( \mathbb{P} \)
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Market filtration

\[ \mathbb{F} = (\mathcal{F}_t) \text{ generated by } (S_t) \]
The contract

- Fixed maturity $T$, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)
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Annuity account

$$dA_t = ((r - q)A_t - w_t)dt + \sigma A_t dB_t, \quad A_0 = c_1$$

with absorption at 0
Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $G = (G_t)$

Assume: $F$ and $G$ are independent under $Q$

Information of the policy holder: $H = F \lor G$
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- High water mark $M_t = \sup_{0 \leq s \leq t} A_t$
Withdrawal

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- **Constraints and withdrawal fees**
  - When withdrawing below $\alpha M_t$, fees are charged at rate $p_1$
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**Income rate**

$$f(w_t, M_t) = (1 - p_1) \min(w_t, \alpha M_t) + (1 - p_2) \max(w_t - \alpha M_t, 0), \quad w_t \leq \beta M_t$$
Surrender

- The holder can surrender the policy at an $\mathbb{H}$-stopping time $\theta \leq T$
Surrender

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- At time $\theta$ the issuer charges a penalty of $k(\theta)A_\theta$, returns $(1 - k(\theta))A_\theta$ to the holder and terminates the contract,

where $k: [0, T] \to [0, 1]$ is a surrender penalty function.
Surrender

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- Typically, \( k(T) = 0 \)
Policy holder behavior

- The policy holder chooses the withdrawal strategy $w$ and surrender time $\theta$


Policy holder behavior

- The policy holder chooses the withdrawal strategy $w$ and surrender time $\theta$
- Worst expected cost of the payments faced by the issuer

\[
D(A_0) = \sup_{w, \theta} \mathbb{E}^Q \int_0^\theta 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^Q e^{-r\theta} k(\theta) A_\theta
\]

\[
- \mathbb{E}^Q \int_0^\theta 1_{\{A_t>0\}} e^{-rs} [qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0)] ds
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Worst expected cost of the payments faced by the issuer

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$$- \mathbb{E}^Q \left[ \int_0^\theta 1_{\{A_t>0\}} e^{-rs} [qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0)] ds \right]$$

One has $A_0 + D(A_0) = E(A_0)$ for

$$E(A_0) = \sup_{w, \theta} \mathbb{E}^Q \left[ \int_0^\theta e^{-rs} f(w_s, M_s) ds + e^{-r\theta} (1 - k(\theta)) A_\theta \right]$$

worst case policy holder behavior ... not actual policy holder behavior!
Policy holder behavior

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\[
-\mathbb{E}^Q \int_0^\theta 1\{A_t>0\} e^{-rs} [qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0)] ds
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\]

**Worst case policy holder behavior** ... not actual policy holder behavior!

- The policy is correctly priced (from the issuer’s perspective) if

\[
D(c_1) = c_2 \iff E(c_1) = c_1 + c_2
\]
Policy holder behavior

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- **Worst expected cost** of the payments faced by the issuer

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D(A_0) = \sup_{w, \theta} \mathbb{E}^Q \int_0^\theta 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^Q e^{-r\theta} k(\theta) A_\theta \\
- \mathbb{E}^Q \int_0^\theta 1_{\{A_t>0\}} e^{-rs} \left[q A_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0)\right] ds
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- **Good news:** $E(A_0)$ is attained for $w, \theta$ only depending on market information $\mathbb{F}$
Policy holder behavior

- The policy holder chooses the withdrawal strategy $w$ and surrender time $\theta$

- **Worst expected cost** of the payments faced by the issuer

$$D(A_0) = \sup_{w, \theta} \mathbb{E}^Q \int_0^\theta 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^Q e^{-r\theta} k(\theta)A_\theta$$

$$-\mathbb{E}^Q \int_0^\theta 1_{\{A_t>0\}} e^{-rs} [qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0)] ds$$

- One has $A_0 + D(A_0) = E(A_0)$ for

$$E(A_0) = \sup_{w, \theta} \mathbb{E}^Q \left[ \int_0^\theta e^{-rs} f(w_s, M_s) ds + e^{-r\theta} (1 - k(\theta))A_\theta \right]$$

- **Worst case policy holder behavior** ... not actual policy holder behavior!

- The policy is correctly priced (from the issuer’s perspective) if

$$D(c_1) = c_2 \iff E(c_1) = c_1 + c_2$$

- **Good news**: $E(A_0)$ is attained for $w, \theta$ only depending on market information $\mathbb{F}$

- **Bad news**: The optimization problem $E(A_0)$ is not Markovian
Adding more state variables

For given $0 \leq t \leq T, 0 \leq a \leq m, 0 \leq z \leq \gamma m, w$, define

\[
\begin{align*}
    dA^{t,a,w}_s &= ((r - m)A^{t,a,w}_s - w_s)ds + \sigma A^{t,a,w}_s dB_s, \quad A^{t,a,w}_t = a \\
    M^{t,a,g,w}_s &= m \lor \sup_{t \leq u \leq s} A^{t,a,m,w}_u \\
    dZ^{t,z,w}_s &= w_s ds, \quad Z^{t,z,w}_t = z
\end{align*}
\]
Adding more state variables

For given $0 \leq t \leq T$, $0 \leq a \leq m$, $0 \leq z \leq \gamma m$, $w$, define

$$dA_s^{t,a,w} = ((r - m)A_s^{t,a,w} - w_s)ds + \sigma A_s^{t,a,w} dB_s, \quad A_t^{t,a,w} = a$$

$$M_s^{t,a,g,w} = m \vee \sup_{t \leq u \leq s} A_u^{t,a,m,w}$$

$$dZ_s^{t,z,w} = w_s ds, \quad Z_t^{t,z,w} = z$$

A "non-standard" standard stochastic control problem

$$V(t, a, m, z) = \sup_{w, \theta} \mathbb{E}^Q \left[ \int_t^T e^{-r(s-t)} f(w_s, M_s^{t,a,m,w}) ds + e^{-r(\theta-t)}(1 - k(\theta))A_{\theta}^{t,a,w} \right]$$

where $w$ and $\theta$ are adapted to $\mathcal{F}_s^t = \sigma(B_s - B_t)$, $s \in [t, T]$
Adding more state variables

For given $0 \leq t \leq T$, $0 \leq a \leq m$, $0 \leq z \leq \gamma m$, $w$, define

$$
dA_{s}^{t,a,w} = ((r - m)A_{s}^{t,a,w} - w_{s})ds + \sigma A_{s}^{t,a,w} dB_{s}, \quad A_{t}^{t,a,w} = a
$$

$$
M_{s}^{t,a,g,w} = m \vee \sup_{t \leq u \leq s} A_{u}^{t,a,m,w}
$$

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V(t, a, m, z) = \sup_{w, \theta} \mathbb{E}^{Q} \left[ \int_{t}^{T} e^{-(s-t)} f(w_{s}, M_{s}^{t,a,m,w}) ds + e^{-(\theta-t)}(1 - k(\theta)) A_{\theta}^{t,a,w} \right]
$$

where $w$ and $\theta$ are adapted to $\mathcal{F}_{s}^{t} = \sigma(Bs - B_{t})$, $s \in [t, T]$

One has $V(0, a, a, 0) = E(a)$
The HJB equation

**Theorem**

$V(t, a, m, z)$ is a viscosity solution of

$$
\begin{align*}
\min(-v_t - H(a, m, v, v_a, v_z, v_{aa}), v - (1 - k)a) &= 0 \text{ for } 0 < a < m, 0 \leq z < \gamma m \\
\min(-v_t - H_0(a, v, v_a, v_{aa}), v - (1 - k)a) &= 0 \text{ for } 0 < a < m, z = \gamma m \\
v(t, 0, m, z) &= \psi(t, m, z) \\
v_m(t, m, m, z) &= 0 \\
v(T, a, m, z) &= a,
\end{align*}
$$

where

$$
H(a, m, z, v, v_a, v_z, v_{aa}) = \sup_{0 \leq w \leq \beta m} \{f(w, m) + w(v_z - v_a)\} - rv + (r - q)a v_a + \frac{1}{2} \sigma^2 a^2 v_{aa}
$$

$$
H_0(a, v, v_a, v_{aa}) = -rv + (r - q)a v_a + \frac{1}{2} \sigma^2 a^2 v_{aa}
$$

$$
\psi(t, m, z) = \sup_{0 \leq w \leq \beta m} \int_t^T e^{-r(s-t)} f(w_s, m) ds \text{ such that } \int_t^T w_s ds \leq \gamma m - z.
$$

Non-linear parabolic PDE on $[0, T] \times \mathbb{R}^3$ with a free boundary and unusual boundary conditions
Reducing the dimension

**Theorem**

\[ V(t, a, m, z) = mW(t, a/m, z/m) = mW(t, x, y), \] where \( W \) is a viscosity solution of

\[
\begin{align*}
\min(-v_t - G(x, y, v, v_x, v_y, v_{xx}), v - (1 - k)x) &= 0 \quad \text{for } (x, y) \in (0, 1) \times [0, \gamma) \\
\min(-v_t - G_0(x, v, v_x, v_{xx}), v - (1 - k)x) &= 0 \quad \text{for } (x, y) \in (0, 1) \times \{\gamma\} \\
v(t, 0, y) &= \zeta(t, y) \\
v_x(t, 1, y) + yv_y(t, 1, y) &= v(t, 1, y) \\
v(T, x, y) &= x,
\end{align*}
\]

where

\[
G(x, y, v, v_x, v_y, v_{xx}) = \sup_{0 \leq u \leq \beta} \{u(v_y - v_x) + f(u, 1)\} - rv + (r - q)xv_x + \frac{1}{2} \sigma^2 x^2 v_{xx}
\]

\[
G_0(x, v, v_x, v_{xx}) = -rv + (r - q)xv_x + \frac{1}{2} \sigma^2 x^2 v_{xx}
\]

\[
\zeta(t, y) = \sup_{0 \leq u \leq \beta} \int_t^T e^{-r(s-t)} f(u_s, 1) ds \quad \text{such that } \int_t^T u_s ds \leq \gamma - y.
\]

Non-linear parabolic PDE on \([0, T] \times [0, 1] \times [0, \gamma]\) with a free boundary and (even more) unusual boundary conditions.
Worst case strategy

The worst case withdrawal strategy $\hat{w}(t, A_t, M_t, Z_t)$ is given by the maximizer of

$$w \mapsto f(w, M_t) + w [V_z(t, A_t, M_t, Z_t) - V_a(t, A_t, M_t, Z_t)]$$
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- The worst case surrender time is

$$\theta = \inf \{t \geq 0 : V(t, A_t, M_t, Z_t) = (1 - k(t))A_t\}$$
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- The issuer can set \( k(t) \) so that the worst case policy holder never surrenders early
Numerical scheme

- Semi-Lagrangian scheme with an obstacle
- Backwards in time
- Solves an optimization problem in every time-step
- Converges to the true solution if the mesh size of the discretization goes to zero
- Gives approximations to $V(t, a, m, z)$ and the worst case behavior $\hat{w}$ and $\hat{\theta}$
Hedging

The issuer can **super-hedge** the contract by trading in $S_t$ and the money market account.

Proof: Itô's formula
Hedging

The issuer can **super-hedge** the contract by trading in $S_t$ and the money market account.

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
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- This will super-hedge the contract
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**Proof:** Itô’s formula
Numerical results

The Risk Neutral Price of a VA+GMWB Contract

- Price is increasing in $\sigma$
- Price is decreasing in $r$

- Not taken into account by insurance companies
- Makes these products difficult to sell in a low interest rate environment
Price is increasing in $\sigma$.
Numerical results

- Price is increasing in $\sigma$ … not taken into account by insurance companies
Numerical results

- Price is increasing in $\sigma$ ... not taken into account by insurance companies
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Numerical results

- Price is increasing in $\sigma$ ... not taken into account by insurance companies
- Price is decreasing in $r$ ... makes these products difficult to sell in a low interest rate environment
Price and hedging ratio as functions of $x = a/m$ and $y = z/m$
Price and hedging ratio as functions of $x = a/m$ and $y = z/m$

- Price is increasing in $x = a/m$ and decreasing in $y = z/m$
Price and hedging ratio as functions of $x = a/m$ and $y = z/m$

- Price is increasing in $x = a/m$ and decreasing in $y = z/m$
- The hedge is always long in $S_t$ in contrast to the hedge of a put option.
Worst case withdrawal and surrender

- $T - t = 5$ years, $r = 1\%$, $\sigma = 18\%$, $q = 0.8\%$, $p_1 = 0$, $p_2 = 5\%$, $k \equiv 0$
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- $T - t = 5$ years, $r = 1\%$, $\sigma = 18\%$, $q = 0.8\%$, $p_1 = 0$, $p_2 = 5\%$, $k \equiv 0$
- yellow = full withdrawal (at rate $\beta M_t$)
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Worst case withdrawal and surrender

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- yellow = full withdrawal (at rate $\beta M_t$)
- light brown = intermediate withdrawal (at rate $\alpha M_t$)
- brown = no withdrawal
- black = surrender
Discouraging early surrender

Set the surrender penalty function such that

\[ k(t) \geq (T - t)q \]
Thank You