Extreme Value Theory and American Options

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When to Refinance a Mortgage?

Loan balance:	€500 000
Remaining term:	20 years.
Interest:	Fixed at 4% pa
Capital repayment: End of term	
Early redemption penalty: None	

- You might refinance if another bank offers you a rate substantially below 4%.
- But how far below? Would a rate of 3% be enough?
- Related problems:
  - When to cash in a guaranteed rate deposit
  - Surrender of term assurance with fixed premiums
  - Investment fund with fixed guarantee charges
  - Fixed price energy supply, phone contract, …

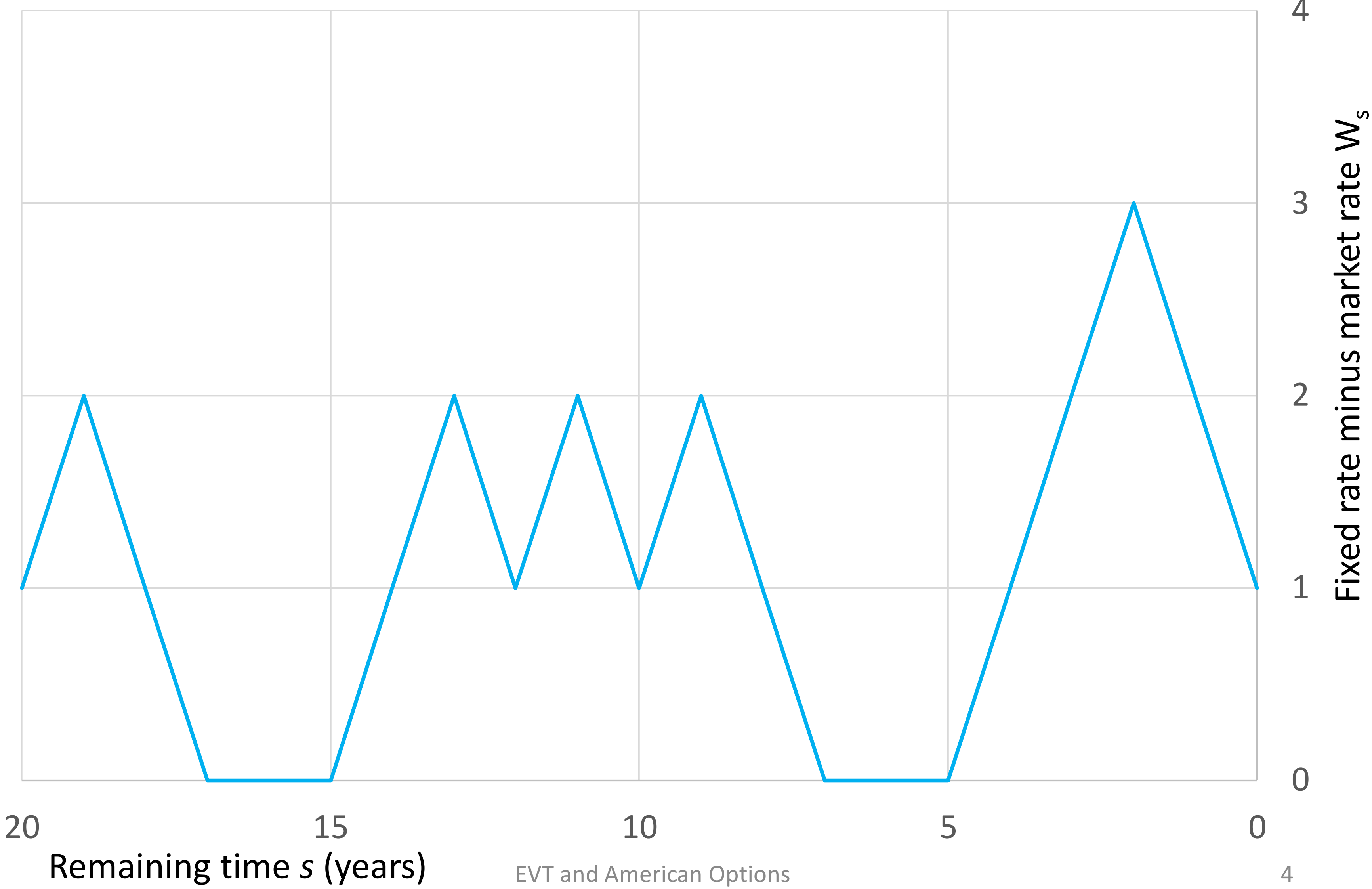
Optimal Refinancing

When I refinance at a new interest rate *m*:

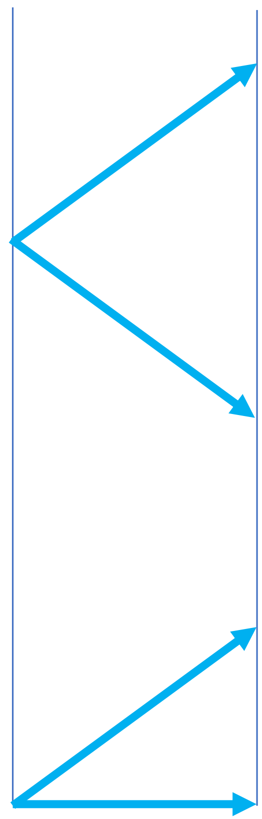
- I save interest at 4% pa
- I pay interest at *m* pa
- For *s* remaining years
- Overall saving = $s \times (4\% - m) \times 500000$

- The *optimal stopping problem* is to choose a refinancing (stopping) date to maximise the saving.

Simplified Interest Model



Non-Negative Random Walk



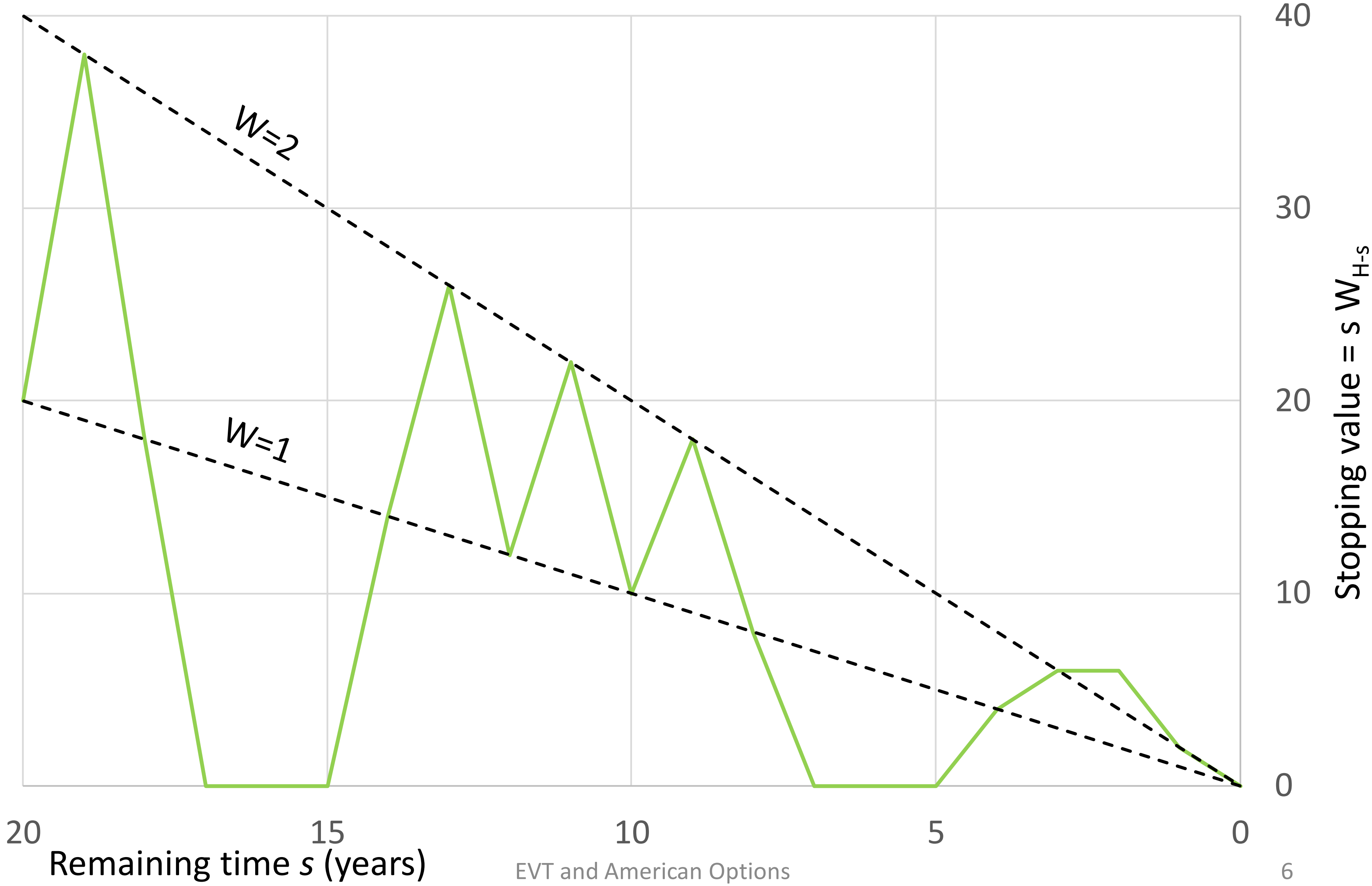
From a point (t-1, w)
The random walk can move to

- (t,w-1), or to
- (t,w+1)

Both with probability ½

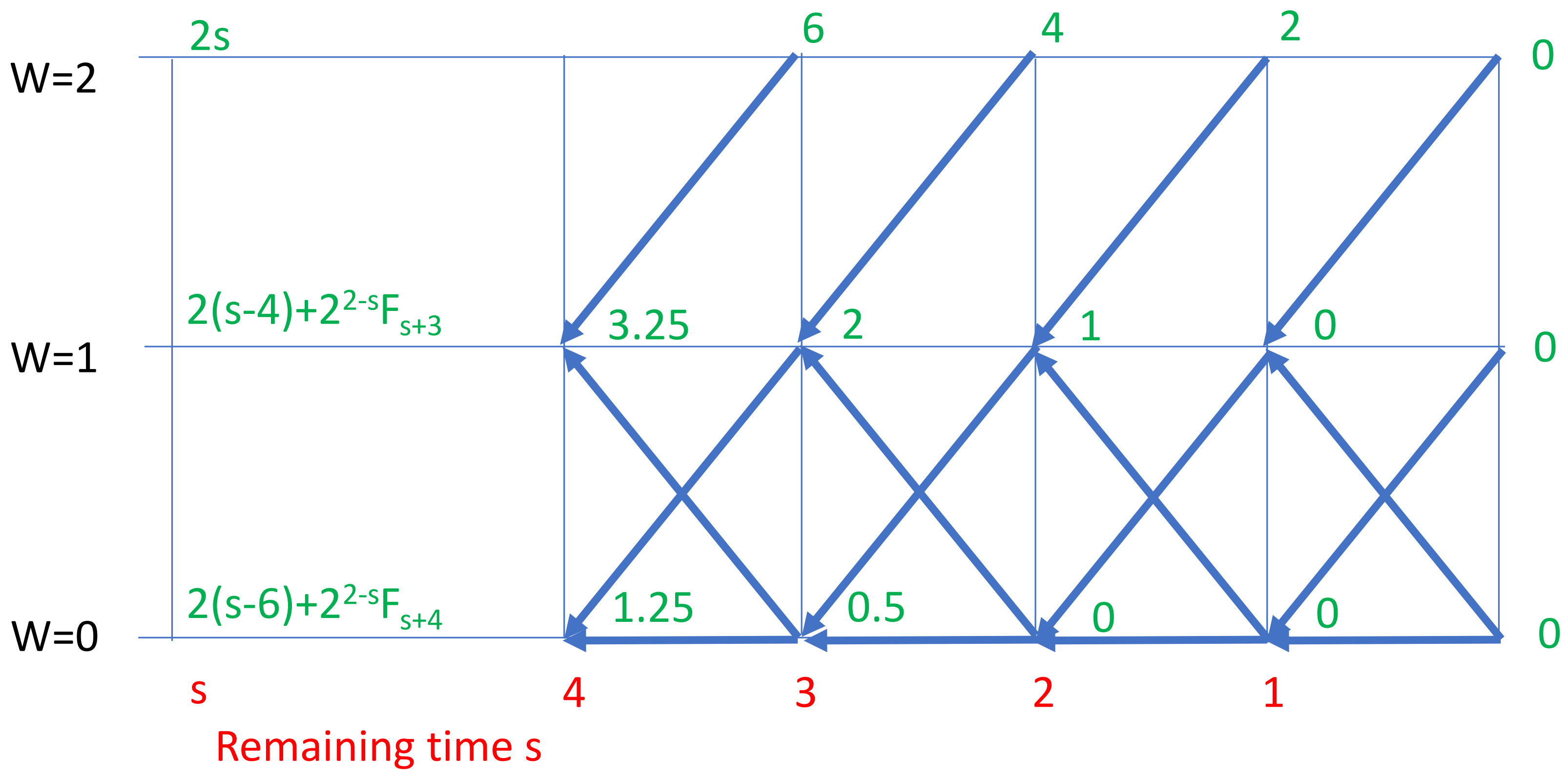
Unless $w_{t-1} = 0$
In which case the next moves are
(t, 0), or
(t, 1)
Both with probability ½

The Game Value on Stopping

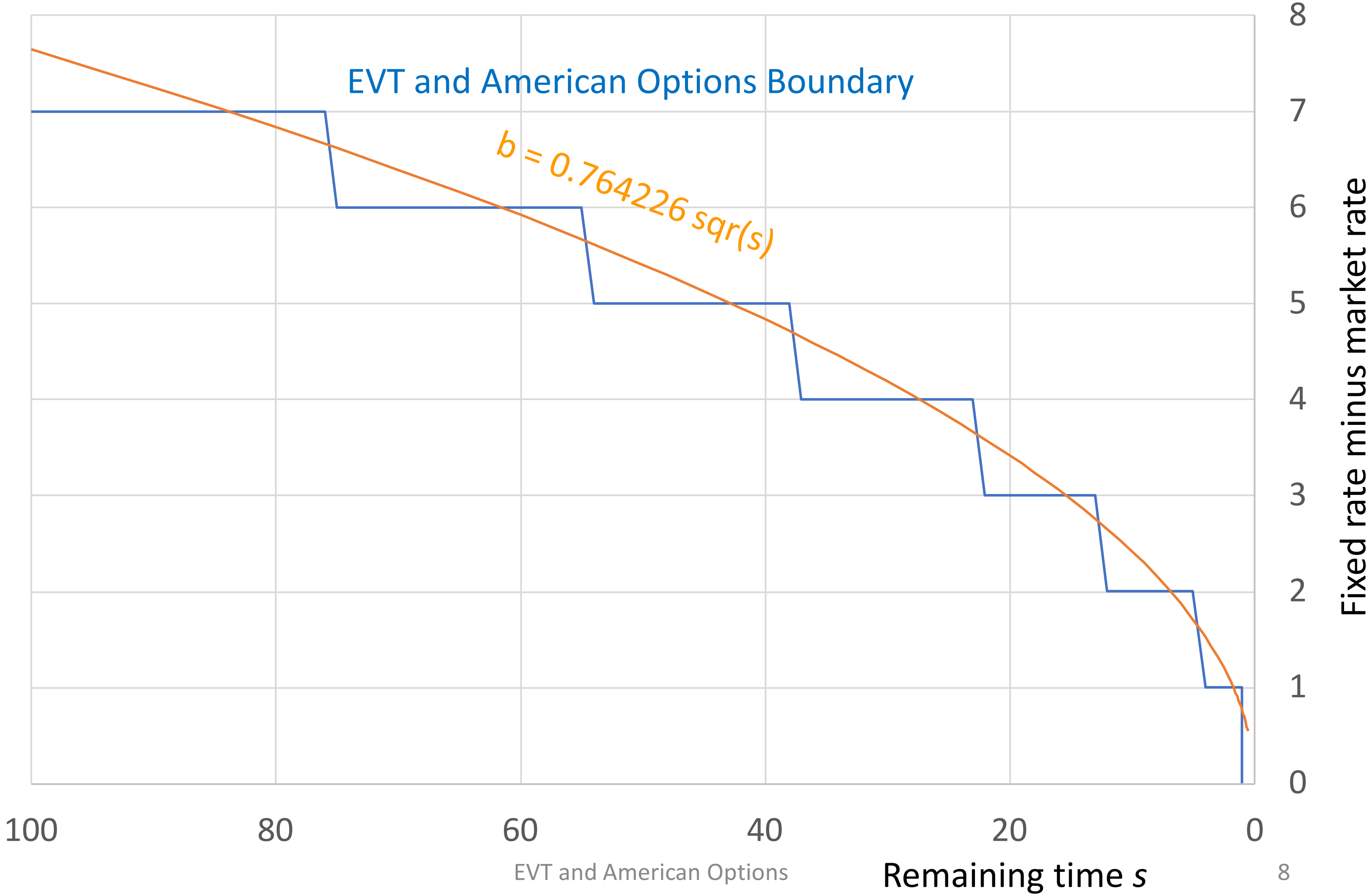


Stopping the First Time W = 2

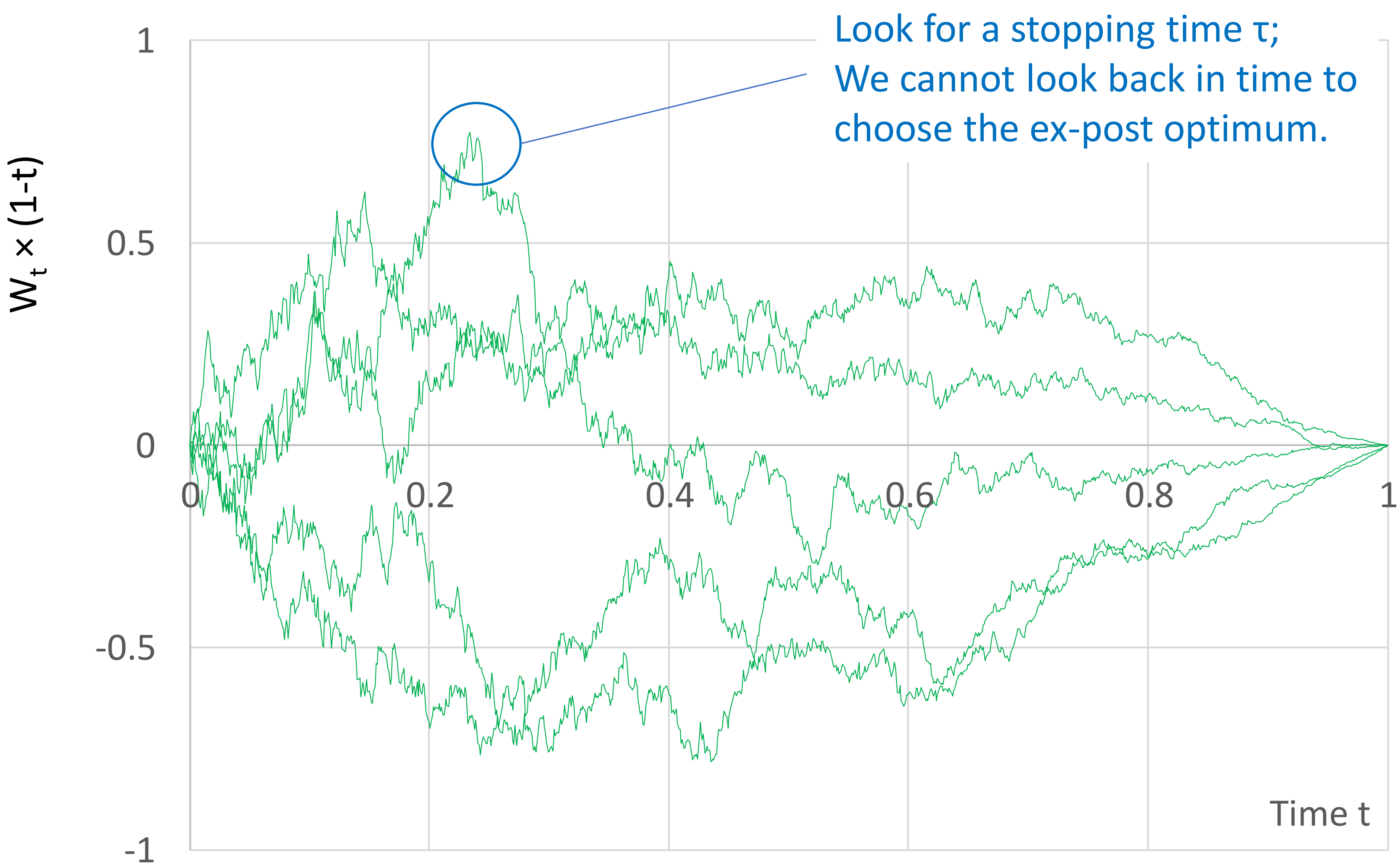
Game values in green



Theoretically Optimal Behaviour

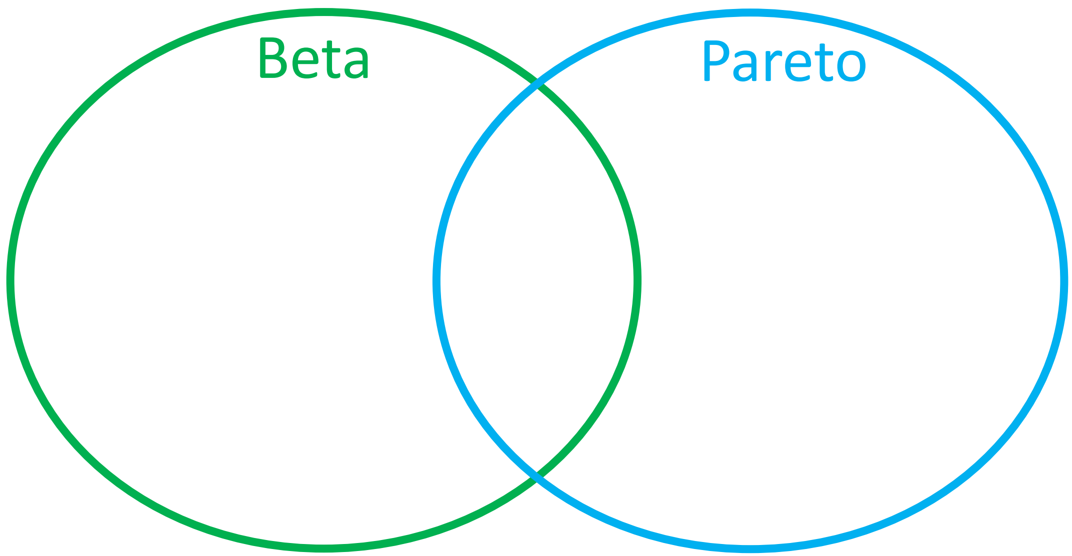


Generally: Maximise **E**{ $W_\tau s(\tau)$ }



Nested Solutions for general $s(t)$

Maximise $\mathbf{E}[W_\tau s(\tau)]$ for a survival function $s(\tau)$



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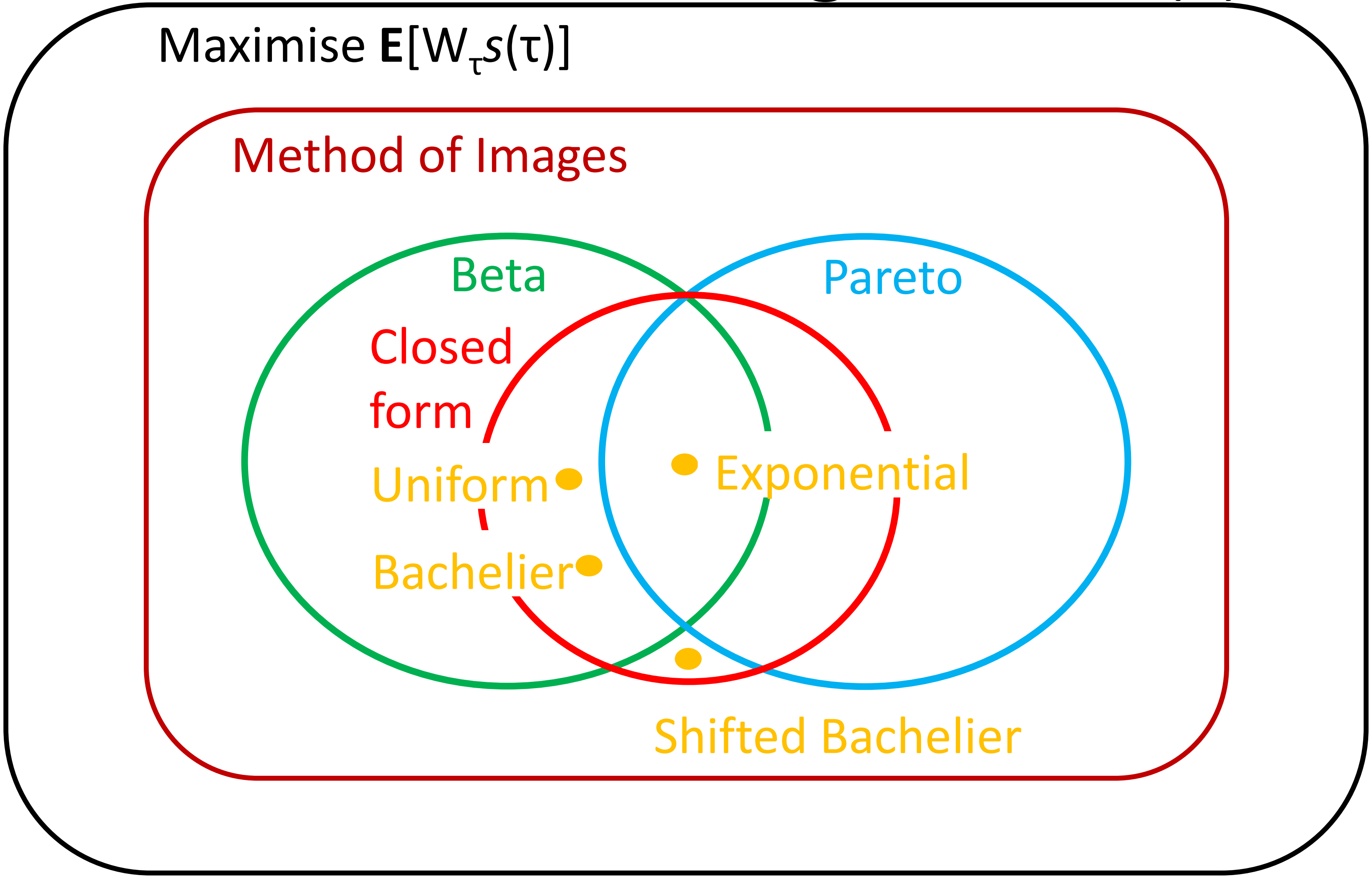

Generalised Pareto Distributions

Original Distribution *T*			**Conditional distribution of T-1, given that *T*>1**		
Mean	Stdev	Type	Mean	Stdev	Type
2	0	Point			
2	2/√3	Uniform[0,4]			

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2	2/√3	Uniform[0,4]	3/2	√3/2	Uniform[0,3]
2	√(12/5)	8 × Beta(1,3)	7/4	7/8×√(12/5)	7 × Beta(1,3)
2	2	Exponential	2	2	Exponential
2	√8	Pareto(4,6)	7/3	7/6×√8	Pareto(4,7)

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The scaling property of the GPD, combined with the scaling property of Brownian motion, permit us to reduce the PDE for the optimal stopping problem to an ODE and hence find closed-form solutions to movable boundary problems.

Solutions: Beta Example

β	value(w) proportional to:	Boundary b_1	
-1/2	$\Phi(w)$	N/A	
0 Bachelier	$w\Phi(w) + \phi(w)$	infinite	
1/2	$\frac{1}{2}(w^2 + 1)\Phi(w) + \frac{w}{2}\phi(w)$	0.839 924	
1 Uniform	$\frac{1}{6}(w^3 + 3w)\Phi(w) + \frac{1}{6}(w^2 + 2)\phi(w)$	0.638 833	

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2		0.472 862	

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These are solutions with $v(-\infty) = 0$. For the reflecting problem, use $v(w)+v(-w)$.

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β	value(w) proportional to:	b_1	b_2
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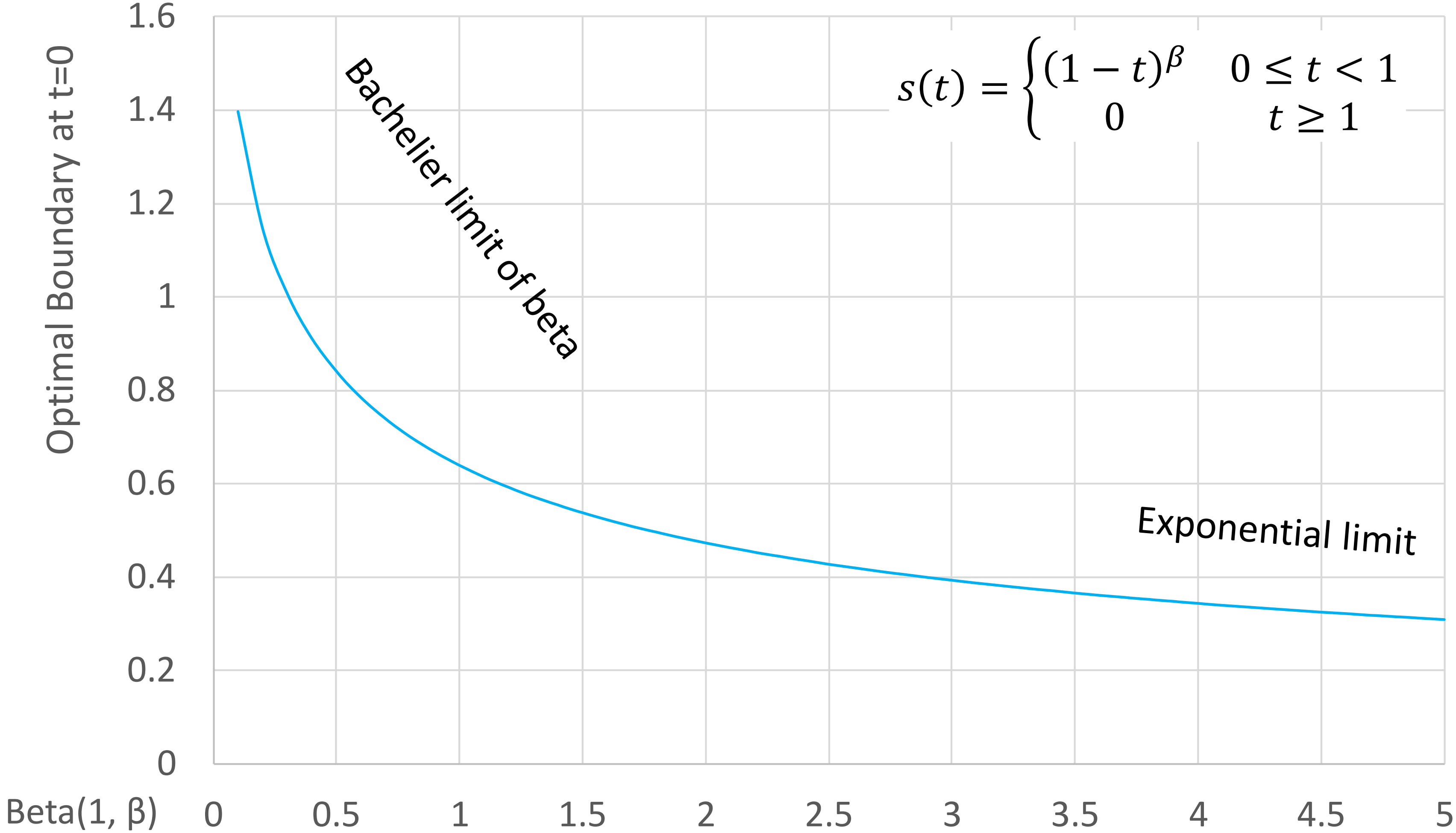
1 Uniform	$\frac{1}{6}(w^3 + 3w)\Phi(w) + \frac{1}{6}(w^2 + 2)\phi(w)$	0.638 833	0.764 226
3/2	… each solution is the integral of the row above it …	0.537 259	0.643 594
2		0.472 862	0.566 776

These are solutions with $v(-\infty) = 0$. For the reflecting problem, use $v(w)+v(-w)$.
The constant b_1 = 0.638 833 in the uniform case also arises in the asymptotic expansion of American options on GBM assets; see Dewynne et al (1993).
When 2β not an integer, solution involves confluent hypergeometric functions.

GPD shape and Optimal Boundary



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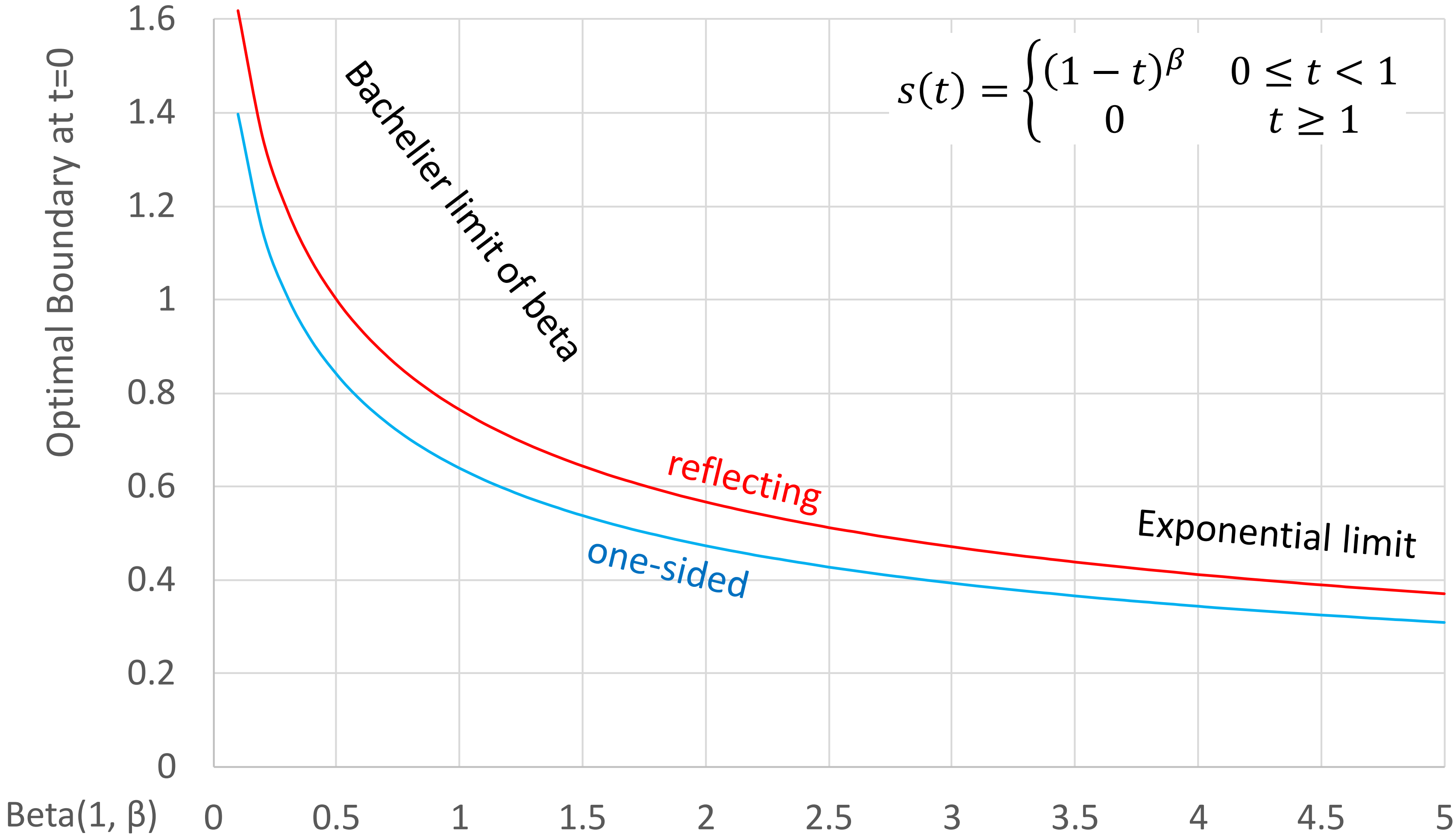

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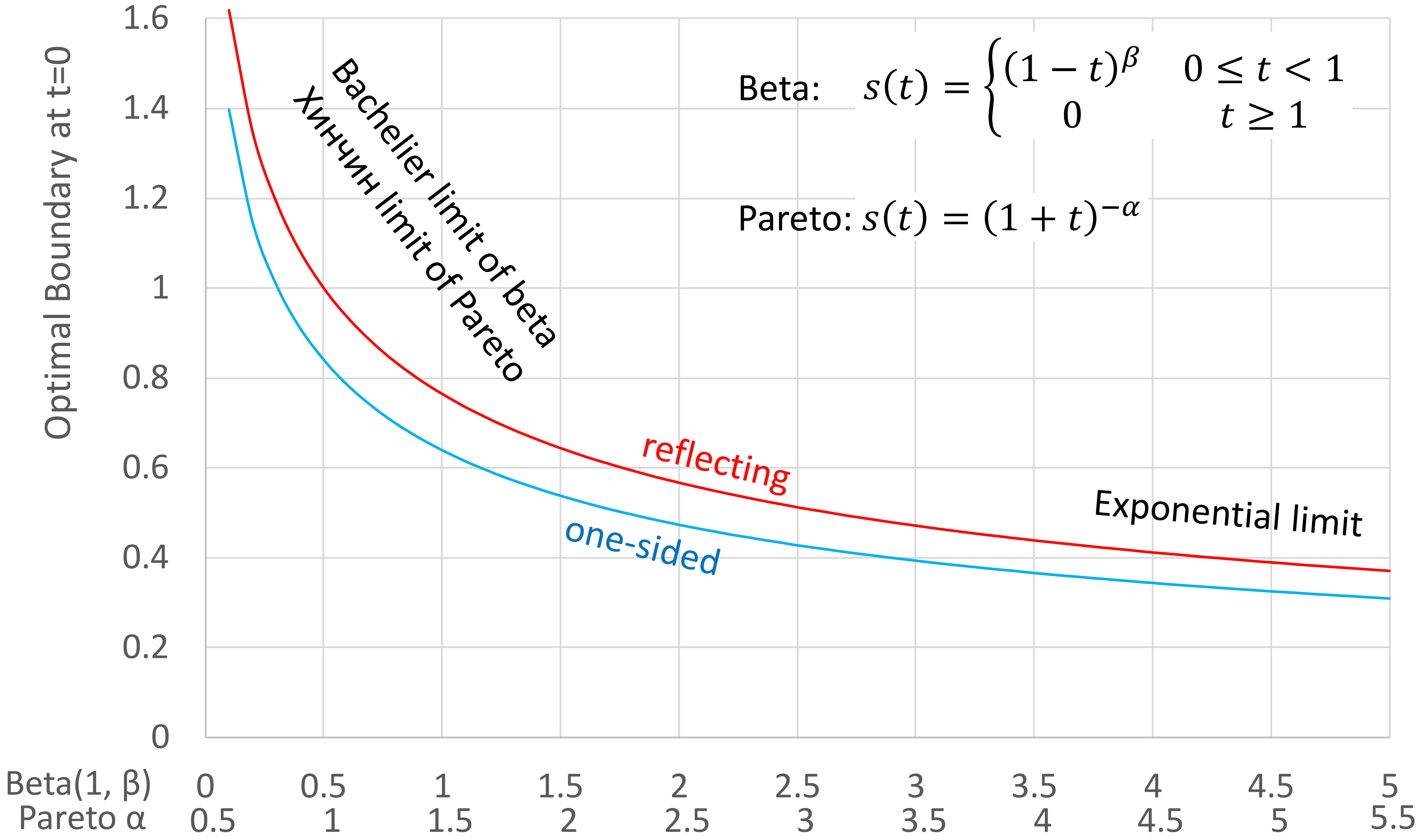

Customer Optimal Behaviour

- There are various reasons why customers might lapse a financial contract.
- Not all customers will apply backward induction or solve PDEs with movable boundary conditions.
- However, not all lapses are random either.
- For risk management purposes you want to know what is at stake if customers suddenly wise up.
- And perhaps underwrite to avoid smart customers.
- Full paper at http://ssrn.com/abstract=3213781