Option Implied Dependence and Correlation Risk Premium

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Motivation

- **Option** markets provide valuable **information** about **future** outcomes
- Many useful **forward-looking** measures can be derived from traded option prices:
  - **Model-dependent**: Black-Scholes ATM implied volatility – old VIX (CBOE)
  - **Model-free**:
    - Volatility (MFIV) – new VIX (CBOE)
    - Skewness and kurtosis – Bakshi, Kapadia and Madan (2003); CBOE SKEW Index
    - Down- and Up- variance – Andersen and Bondarenko (2009)

Motivation

- Given a cross-section of European options with different strikes and same maturity, we can infer the whole distribution of future returns.
- Call prices are related to the **risk-neutral density (RND)** as

$$C(K) = e^{-rT} E^{Q}\left[(S_T - K)^+\right] = e^{-rT} \int_0^\infty (S_T - K)^+ f(S_T)\, dS_T,$$

where $f(S_T)$ is RND for maturity T.

- Ross (1976), Breeden and Litzenberger (1978), Banz and Miller (1978):

$$f(S_T) = e^{rT} \left.\frac{\partial^2 C(K)}{\partial K^2}\right|_{K=S_T}.$$

- Several techniques for RND estimation are available. Accurate when there are enough option strikes and they cover most of RND mass.

Motivation

- Virtually all existing model-free approaches are **univariate**, handle one asset at a time
- What about a **joint** distribution of several assets? Important for many applications:
  - Portfolio optimization
  - Multi-asset derivatives (basket options, exchange options, correlation swaps)
  - Risk management
  - Developing **forward looking** warning signals contagion/herding/systemic risk
  - FX markets
- However, a very hard problem – few **multivariate** options trade

Motivation

- In this paper, we propose a novel approach to **infer forward-looking dependence** between $d \geq 2$ assets, given
  (1) marginal distributions of assets $X_1, \ldots, X_d$, and
  (2) distribution of their weighted sum (index)
  $$S = \omega_1 X_1 + \ldots + \omega_d X_d$$
  where $\omega_1 + \ldots + \omega_d = 1$.
- The approach is **model-free** and is based on a combinatorial algorithm – **Block Rearrangement Algorithm (BRA)**
- Empirical application: the approach is implemented for the 9 industries that comprise S&P 500 Index
- Risk-neutral densities for components $X_1, \ldots, X_9$ and the index S are estimated from traded options

Existing Literature

Often rely on **model-dependent** approaches or **approximations**

- Cont and Deguest (2013) – a multi-asset model consistent with observed index options and individual stock options. See also Avellaneda and Boyer-Olson (2002), Jourdain and Sbai (2012).
- CBOE S&P 500 Implied Correlation Index

$$\rho_{cboe} = \frac{\sigma_S^2 - \sum_{i=1}^{d} \omega_i^2 \sigma_i^2}{2 \sum_{i=1}^{d-1} \sum_{j>i} \omega_i \omega_j \sigma_i \sigma_j}.$$

- Driessen, Maenhout, and Vilkov (2009) use S&P 100 options to estimate the correlation risk; see also Skintzi and Refenes (2005).
- Driessen, Maenhout, and Vilkov (2012), Buss and Vilkov (2012), Jackwerth and Vilkov (2015) – stochastic correlation models.

Rearrangement Algorithm – Basic Setup

- **Inputs:** d random variables $\mathbf{X_1} \sim \mathbf{F_1}$, $\mathbf{X_2} \sim \mathbf{F_2}$, ..., $X_d \sim F_d$.
- **Goal:** look for a dependence such that the variance of sum $S = X_1 + ... + X_d$ is minimized.
- Assume that each X_j is sampled into n **equiprobable** values, i.e., we consider the realizations $x_{ij} := F_j^{-1}(\frac{i-0.5}{n})$ and arrange them in an $n \times d$ matrix:

$$\mathbf{X} = [X_1, \dots, X_d] = \begin{bmatrix} \mathbf{x_{11}} & \mathbf{x_{12}} & \dots & x_{1d} \\ \mathbf{x_{21}} & \mathbf{x_{22}} & \dots & x_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{x_{n1}} & \mathbf{x_{n2}} & \dots & x_{nd} \end{bmatrix}$$

- Want to rearrange elements x_{ij} (by columns), such that after the rearrangement **variance of sum** S is minimized.
- This is an NP complete problem. Brute force search requires checking $(n!)^{(d-1)}$ rearrangements.

Rearrangement Algorithm (RA)

- **Heuristic algorithm** proposed in Puccetti and Rüschendorf (2012) and Embrechts, Puccetti, and Rüschendorf (2013):

  1. For $j = 1, \ldots, d$, make the j^{th} column anti-monotonic with the sum of the other columns.
  2. If there is no improvement in var $\left(\sum_{k=1}^{d} X_k\right)$, output the current matrix $\mathbf{X}$, otherwise return to step 1.

- Step 1 ensures that the variance of the sum before rearranging X_j is larger than after rearranging $\widetilde{X}_j$

$$\text{var}\left(X_j + \sum_{k \neq j} X_k\right) \geqslant \text{var}\left(\widetilde{X}_j + \sum_{k \neq j} X_k\right).$$

Block Rearrangement Algorithm (BRA)

- When $d > 3$, the standard RA can be **improved by considering blocks**
  1. Select a random sample of n_{sim} possible partitions of the columns $\{1, 2, \ldots, d\}$ into two non-empty subsets $\{I_1, I_2\}$.
  2. For each of the n_{sim} partitions, create block matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ with corresponding row sums S_1 and S_2 and rearrange rows of $\mathbf{X}_2$ so that S_2 is anti-monotonic to S_1.
  3. If there is no improvement in var $\left(\sum_{k=1}^{d} X_k\right)$, output the current matrix $\mathbf{X}$, otherwise, return to step 1.
- When d is reasonably small $(d \leqslant 10)$, we can take $n_{sim} = 2^{d-1} - 1$, so that **all non-trivial partitions are considered.**

Inferring Dependence

- Bernard, Bondarenko, and Vanduffel, "Rearrangement Algorithm and Maximum Entropy", *Annals of Operations Research*, 2018
- Inputs: d random variables $\mathbf{X_1} \sim \mathbf{F_1}$, $X_2 \sim F_2$,..., $\mathbf{X_d} \sim \mathbf{F_d}$ and their sum $\mathbf{S} \sim \mathbf{F_S}$.
- Assume that each X_j and S are sampled into n equiprobable values, arranged in an $n \times (d+1)$ matrix:

$$\mathbf{M} = [X_1, \ldots, X_d, -S] = \begin{bmatrix} \mathbf{x_{11}} & x_{12} & \ldots & \mathbf{x_{1d}} & -\mathbf{s_1} \\ \mathbf{x_{21}} & x_{22} & \ldots & \mathbf{x_{2d}} & -\mathbf{s_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{x_{n1}} & x_{n2} & \ldots & \mathbf{x_{nd}} & -\mathbf{s_n} \end{bmatrix}.$$

- To infer the dependence among $X_1, \ldots, X_d$, apply BRA on $\mathbf{M}$.
- Ideally, the **row sums** of the rearranged matrix are all **zero** and a compatible dependence has been found. In practice, this situation does not occur and we obtain a close **approximate solution**.

Summary of Theoretical Approach

- **BRA** can be **used to infer a possible dependence** structure among d variables given their **marginal distributions** and the **distribution of the sum**
- Although there are typically many solutions, BRA finds solutions that are "close to each other" and exhibit almost maximum entropy
- As the level of discretization n increases, BRA solutions converge to the **maximum entropy**

Illustration when X_1, X_2 are $N(0,\sigma_i)$ and S is $N(0,\sigma_S)$ such that implied correlation is 0.

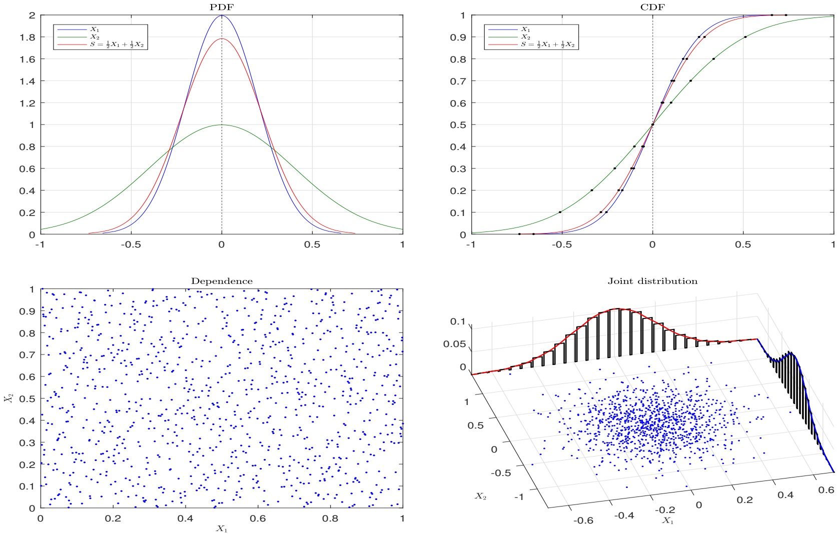


Illustration when X_1, X_2 are $N(0,\sigma_i)$ and S is $N(0,\sigma_S)$ such that implied correlation is 0.97.

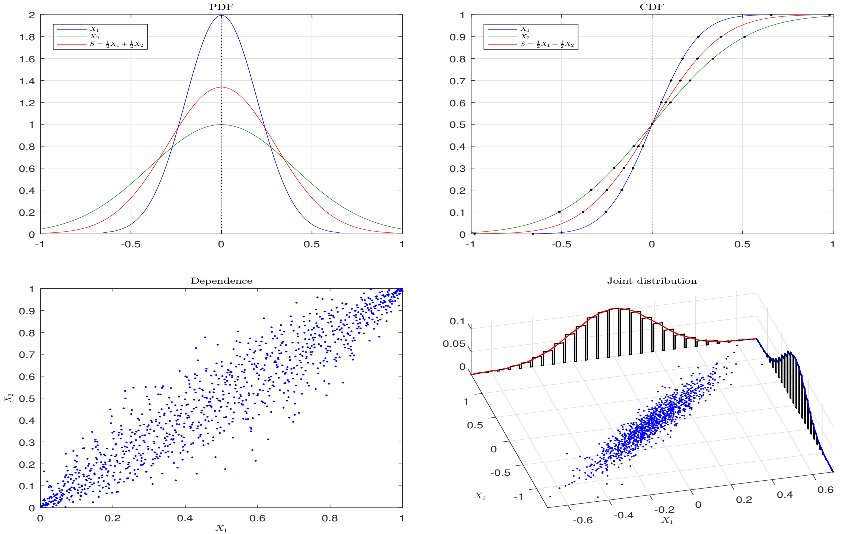
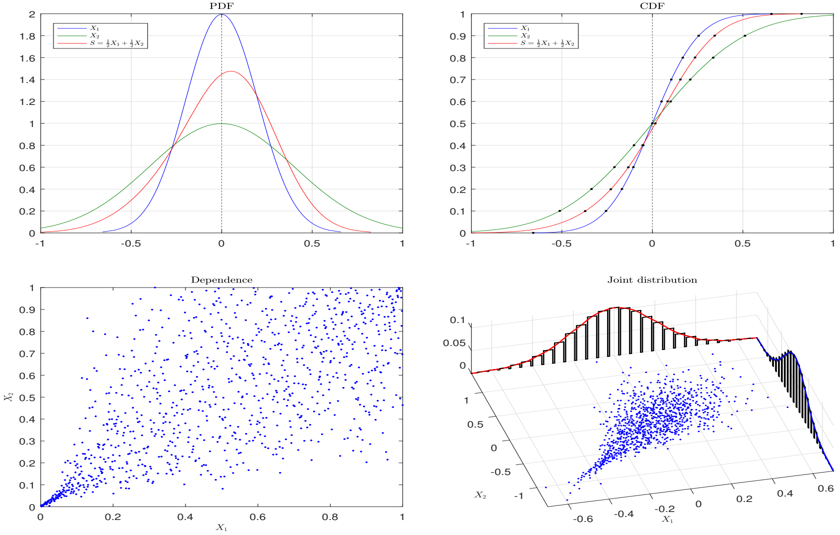


Illustration when X_1, X_2 are $N(0,\sigma_i)$ and S is skewed.



Empirical Application – S&P 500 Sectors

- SPDR ETFs, S&P 500 Index and its 9 sectors:

Description	Ticker	Abbreviation
SPDR S&P 500 ETF Trust	SPY	spx
Consumer Discretionary Sector SPDR Fund	XLY	cdi
Consumer Staples Sector SPDR Fund	XLP	cst
Energy Sector SPDR Fund	XLE	ene
Financial Sector SPDR Fund	XLF	fin
Health Care Sector SPDR Fund	XLV	hea
Industrial Sector SPDR Fund	XLI	ind
Materials Sector SPDR Fund	XLB	mat
Technology Sector SPDR Fund	XLK	tec
Utilities Sector SPDR Fund	XLU	uti

- **9 sectors** that do not overlap and that cover entire S&P 500
- Daily option data from CBOE
- Sample: 04/2007 - 09/2017

S&P 500 Sectors

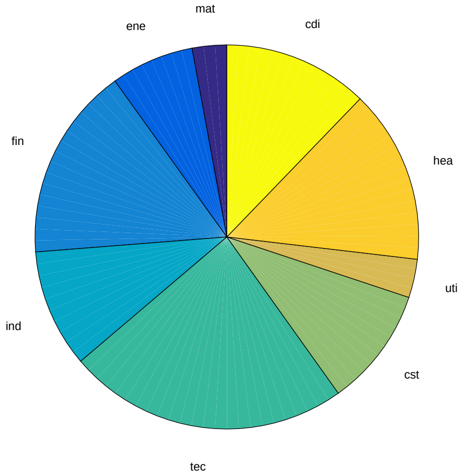


Figure: Sector weights in September 2016.

S&P 500 Sectors

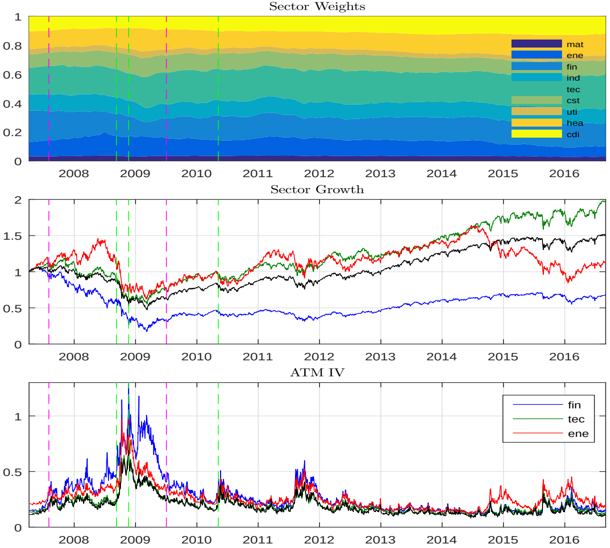


Figure: Sector weights over time. Pink vertical lines indicate Financial crisis. Green vertical lines: 08-Sep-08, 20-Nov-08, and 06-May-10.

Implementation Details

- Daily frequency, τ is at least 30 days, or closest available
- Estimate RNDs for S and each X_j from traded options on SPY and $d = 9$ Sector ETFs
- Estimate RNDs nonparametrically with **Positive Convolution Approximation (PCA)**, Bondarenko (2003)
- Discretize each distribution into $n = 1000$ equiprobable returns and arrange them in $n \times (d+1)$ matrix:

$$\mathbf{M} = [X_1, \ldots, X_d, -S] = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1d} & -s_1 \\ x_{21} & x_{22} & \ldots & x_{2d} & -s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nd} & -s_n \end{bmatrix}.$$

- Apply BRA on matrix $\mathbf{M}$ to infer dependence structure

Implementation Details

- Compute **various dependence statistics**:
  - **Pairwise correlations** and their value-weighted average
  - **Correlations conditional** on various events $\rho(R_i, R_j \,|\, Scenario)$, which can depend on the aggregate market or other factors:
    - localized or "corridor" correlation: $Scenario = \{q_1 \leq R_S \leq q_2\}$ for some quantiles q_1, q_2
    - **Down and Up correlations**: Let R_S^M be the median of R_S

$$\rho_{i,S}^{d,\mathbb{Q}} = \text{corr}^{\mathbb{Q}}(R_i, R_S \,|\, R_S \leq R_S^M)$$

$$\rho_{i,S}^{u,\mathbb{Q}} = \text{corr}^{\mathbb{Q}}(R_i, R_S \,|\, R_S > R_S^M),$$

  - Also **Spearman's rho** – not affected by changes in marginal distributions (not sensitive to changes in volatility)

$$\text{Spearman's rho}(R_i, R_j) = \rho(F_i(R_i), F_j(R_j))$$

  - Other **tail indices**

Selective Date: 08-Sep-2008

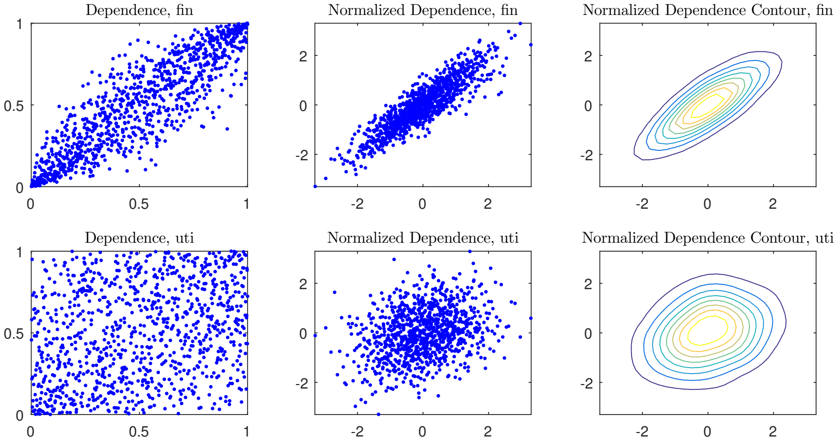


Figure: Implied Dependence.

Selective Date: 08-Sep-2008

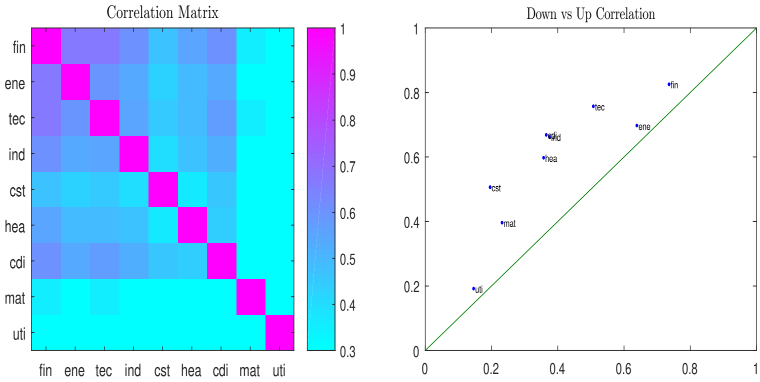


Figure: Implied Correlations.

Selective Date: 20-Nov-2008

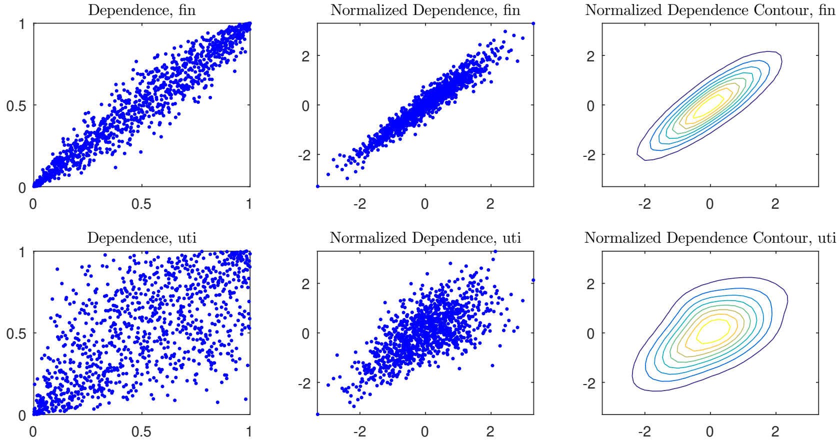


Figure: Implied Dependence.

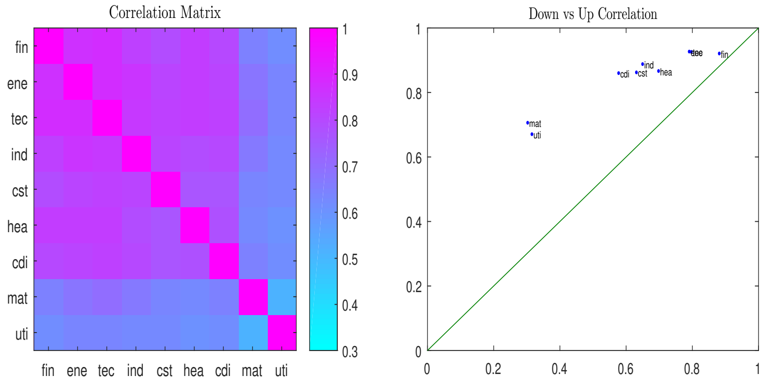


Figure: Implied Correlations.

Up and down average pairwise correlations

From **option prices**, we estimate:

$$\rho_{i,j}^{g,\mathbb{Q}} = \mathrm{corr}^{\mathbb{Q}}(R_i, R_j)$$

$$\rho_{i,j}^{d,\mathbb{Q}} = \mathrm{corr}^{\mathbb{Q}}(R_i, R_j \mid R_S \leq R_S^M)$$

and

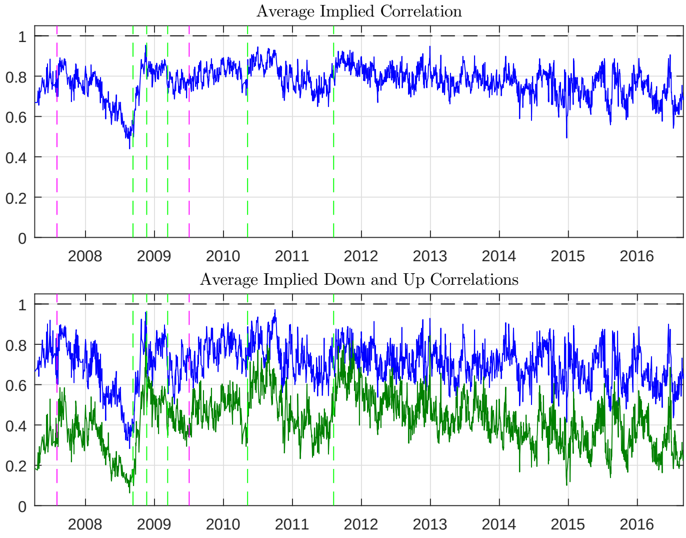
$$\rho_{i,j}^{u,\mathbb{Q}} = \mathrm{corr}^{\mathbb{Q}}(R_i, R_j \mid R_S > R_S^M),$$

We then average

$$\rho^{x,\mathbb{Q}} = \frac{\sum_{i<j} \pi_i \pi_j \rho_{i,j}^{x,\mathbb{Q}}}{\sum_{i<j} \pi_i \pi_j},$$

with $\pi_i = \omega_i \sigma_i$

Implied Correlation



Up and down correlation risk premia

From **option prices**, we estimate:

$$\rho_{i,j}^{d,\mathbb{Q}} = \text{corr}^{\mathbb{Q}}(R_i, R_j \,|\, R_S \leq R_S^M)$$

and

$$\rho_{i,j}^{u,\mathbb{Q}} = \text{corr}^{\mathbb{Q}}(R_i, R_j \,|\, R_S > R_S^M),$$

From corresponding **stock prices** daily returns

$$\rho_{i,j}^{d,\mathbb{P}} = \text{corr}^{\mathbb{P}}(R_i, R_j \,|\, R_S \leq R_S^M)$$

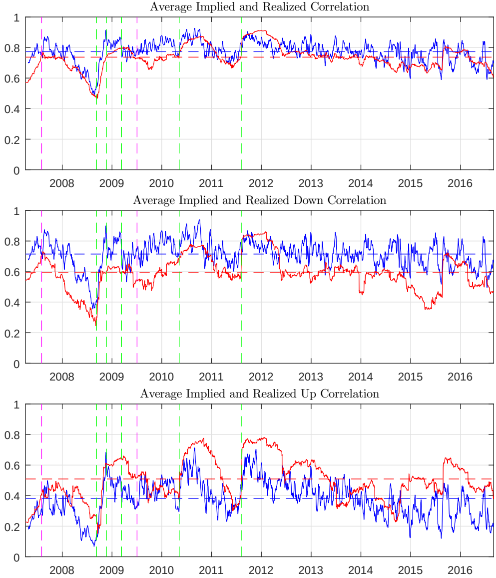
and

$$\rho_{i,j}^{u,\mathbb{P}} = \text{corr}^{\mathbb{P}}(R_i, R_j \,|\, R_S > R_S^M),$$

Correlation risk premium (global, up and down):

$$\rho_{i,j}^{g,\mathbb{P}} - \rho_{i,j}^{g,\mathbb{Q}}, \quad \rho_{i,j}^{u,\mathbb{P}} - \rho_{i,j}^{u,\mathbb{Q}}, \quad \rho_{i,j}^{d,\mathbb{P}} - \rho_{i,j}^{d,\mathbb{Q}}$$

Implied and Realized Correlation



Results

What we observe

$$\rho_{i,j}^{u,\mathbb{Q}} < \rho_{i,j}^{u,\mathbb{P}} < \rho_{i,j}^{d,\mathbb{P}} < \rho_{i,j}^{d,\mathbb{Q}}$$

Asymmetry under $\mathbb{P}$ was observed in the literature: Longin and Solnik (JOF 2001), Ang and Bekaert (RFS 2002), Hong, Tu and Zhou (RFS 2007), Jondeau (CSDA, 2016)... higher correlations in "bear markets"

Under $\mathbb{Q}$, this asymmetry is **amplified** and we give evidence that this asymmetry in the correlations comes from an **asymmetry in the dependence** and **not** from properties of the **marginal** distributions.

Margins or Dependence?

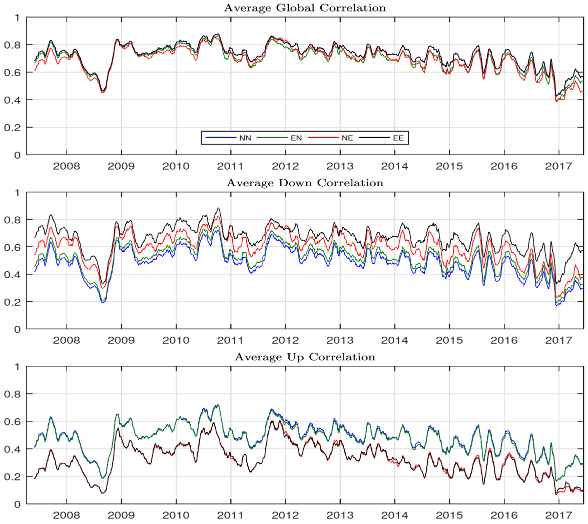


Figure 4.10: **Implied Correlations**. Average implied global, down, and up correlations are computed for the four cases (NN, EN, NE, EE), where the first letter denotes the type of margins (Normal or Empirical) and the second letter denotes the type of the copula (Normal or Empirical). Statistics are plotted as 1-month moving averages.

Additional Elements To Be Found in the Paper

- Implied dependence is **non-Gaussian**, **time-varying**, and **asymmetric**

- Global Correlation Risk Premium **disappears** when computed with Spearman's Rho, whereas the Down (resp. Up) Correlation Risk Premium stays significantly negative (resp. positive)

- **Alternative semi-parametric approach** to our model-free approach to model the joint distribution of assets in the risk-neutral world:
  - Fit **margins** with **model-free** approach
  - Fit **dependence** using a two-parameter **Skewed Normal Copula**

  Model sufficiently flexible to re-obtain the results on the global, down and up correlation risk premia

Conclusions (1/2)

- A **novel algorithm** to infer the dependence among variables given their marginal distributions and distribution of the sum
- Consistent with **maximum entropy**. This is a desirable property: a dependence with lower entropy would mean that we use information that we do not possess
- **Application** to S&P 500 Sector options:
  - Implied dependence is **non-Gaussian**, time-varying, and asymmetric
  - **Down correlation is larger than Up correlation**
  - Correlation risk premium: **Down** (strongly negative), **Up** (positive), **Global** (negative)
  - Parsimonious multivariate model with a two-parameter copula
  - Evidence of extreme events / left tail dependence
  - **Correlation indices** (down, up), improving on CBOE index

Conclusions (2/2)

A number of potential **applications**:

- Identify **properties of a "good" multivariate model** to reproduce option prices available in the market (such as stochastic correlation, asymmetry between average up and down correlation, etc).
- A new approach to price any **path-independent multivariate derivatives** (basket options and correlation swaps). Joint work with Oleg Bondarenko and Steven Vanduffel.
- Detection of **arbitrage** opportunities – Dispersion arbitrage
- Disentangle modelling of **volatility** (margins) and of the **dependence** (copula)
- New forward-looking indicators of **contagion/tail risk**
- Covariance matrix estimation / Optimal **portfolio** construction