Universität Zürich   **ETH Zürich**

ON DENSITIES OF EXTREME VALUE COPULAS

M.Sc. Thesis

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April 2, 2013

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Abstract

Extensive work on extreme value copulas (EVCs) can be found in the literature, but their associated densities are mainly studied in the bivariate case. This work aims to bridge the gap between bivariate and multivariate densities, as an explicit expression of the density is of utmost importance in many multivariate statistical applications. After a brief overview of copula-related theory, we present a general formula for the density of EVCs. This formula depends on the derivatives of the underlying stable tail dependence function of the copula, so we briefly recall the construction principle of the stable tail dependence function of the Smith and multivariate t distributions, see also Joe et al. (2008). A result based on Archimedean copulas with inverse generators that are regularly varying at one with tail index bigger than one is presented in Genest and Rivest (1989), and based on their representation of the stable tail dependence function, we can apply our formula for the density of a general EVC and construct new tractable EVC densities from Archimedean copulas. Under similar assumptions, we can extend the construction to nested Archimedean copulas and we are able to derive the stable tail dependence function of nested Archimedean copulas. Finally, we obtain an implicit expression for the density of the copula of the N-largest order-statistics.

À mes parents
pour leur support inconditionnel.

Acknowledgements

There are a great number of people without whom this thesis might not have been written, and to whom I am greatly indebted.

Foremost, I would like to express my sincere gratitude to my supervisors Prof.Dr.Paul Embrechts and Dr.Marius Hofert for giving me the opportunity to work with them and benefit from their experience. I cannot express how the motivation, the enthusiasm, the time and the expertise of Marius Hofert made this thesis highly enjoyable. I was extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly.

To Jennifer, I wish to offer my deepest thanks. She was and is a solid anchor on which I rely on again and again. Words can not express how grateful I am to be in her life, and how much this work was enhanced and made easier by her being in mine. My experience here has far exceeded even my own expectations due to your ongoing support.

To my parents, who have been a source of inspiration and encouragement to me throughout my life, and for the myriads of ways in which you have actively supported me in my determination to find and realize my potential. Without your support, none of this would have been possible.

I would also like to express my gratitude to the IFM2 for the financial support they have granted me through their graduate fellowship program.

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1 Introduction

This thesis will be focused on a specific class of copulas, called extreme value copulas. Being the limits of copulas of appropriately scaled componentwise maxima in independent d-variate random samples, extreme value copulas provide appropriate models for the dependence structure between so-called *rare*, or *exceptional*, or *extreme* events. On a side note, we want to mention that extreme value copulas are often a convenient choice for the modelling of data with positive lower orthant dependence.

The study of copulas and similarly standardized distribution functions can be traced back to the mid 1930s, although applications of copula theory to finance-related topics, such as risk management and derivatives pricing, is a rather modern phenomenon. The financial platform offers many situations in which one needs to consider randomness induced by various factors, and in that regard, a central issue is to describe the dependence structure between these factors. Indeed, understanding and describing dependence between random variables, the said factors, can be complicated, especially when working in higher dimensions.

In this context, copulas are appealing because, at first, they provide a great way of studying and understanding scale-free measures of dependence, and secondly, they facilitate a bottom-up approach to model building by linking individual univariate models to their joint model.

Since the year 1999, there as been a significant increase in copula-related new publications in the field of finance, where the flag example is Embrechts et al. (2002), which existed in 1998 as an ETH Zurich RiskLab Report. Due to the development of quantitative risk management, copulas are now commonly used by practicioners in the financial industry. This explosive development is mainly due to the growing presence of regulatory bodies, such as the Basel Committee, the FSA, FINMA and others imposing stronger guidelines in market practices, as well as the need for creating new financial products.

To illustrate the above claim, one could cite as an example Starica (1999), where the joint behavior of extreme returns in foreign exchange rate is investigated. On the insurance side, one can look at Cebrian et al. (2003), where extreme value copulas are applied to the SOA medical large claims database.

Section 2 provides the general framework we need throughout the thesis. Besides introducing relevant copula-related theory, emphasis is put on stable tail dependence functions and regular variation. In Section 3, we briefly present the results of Joe et al. (2008) on the Smith model and t distributions. However, the heart of the thesis is Section 4, where a general expression for the density of extreme value copulas is obtained using Faà di Bruno's Formula. Bivariate applications of our formula to Pickand's dependence functions are investigated in Section 5. In Section 6, we extend the work of Genest and Rivest (1989) by constructing tractable extreme value copula densities from Archimedean copulas satisfying certain regular variation assumptions. We continue our work in Section

7, where results from Section 6 are extended and a recursive expression for the stable tail dependence function of nested Archimedean copulas is obtained. Finally, using the Extremal Types Theorem, we derive an implicit expression for the density of the copula of the N largest order-statistics in Section 8.

2 Framework

In this section, we present the cornerstones of the theory we need in order to investigate the densities of extreme value copulas. The theory concerning copulas is quite vast, but in this section, we focus on the main results providing insights in our line of work.

2.1 Basic notation

To begin, we need to establish a standard notation that will be used throughout this work. First, we let $\boldsymbol{x}$ denote a vector $(x_1, \ldots, x_d) \in \mathbb{R}^d$, where d is always assumed to be an integer greater than or equal to 2. If not otherwise stated, all expressions such as $\boldsymbol{x} + \boldsymbol{y}$, $\boldsymbol{x} \leq \boldsymbol{y}$ and others of the sort are considered as componentwise operations. Note that in this thesis, the notation $\mathbb{R}_+$ is understood as the positive real line excluding zero.

We will reserve the symbol $\boldsymbol{X}$ to represent a random vector in $\mathbb{R}^d$, that is $(X_1, \ldots, X_d)$, where X_j denotes a random variable in $\mathbb{R}$. Such a random vector is understood to have a distribution function (H) and a survival function $(\bar{H})$, defined by

$$H(\boldsymbol{x}) = P(\boldsymbol{X} \leq \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

and

$$\bar{H}(\boldsymbol{x}) = P(\boldsymbol{X} > \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

respectively. For the vector $\boldsymbol{X}$ with distribution H as above, the marginal distribution of X_j, denoted F_j, is given by

$$F_j(x_j) = P(X_j \leq x_j) = H(\infty, \ldots, \infty, x_j, \infty, \ldots, \infty), \quad x_j \in \mathbb{R}.$$

Since we will mostly work in the general d-dimensional case, we reserve the notation $\mathcal{J}$ for the index set $\{1, \ldots, d\}$. For any set $\emptyset \neq B \subseteq \mathcal{J}$, we also introduce $\mathcal{J}_{-B}$ for the set $\mathcal{J} \backslash B$, that is the set $\mathcal{J}$ without the elements in B. In the case where B is a singleton $\{j\} \in \mathcal{J}$, we write $\mathcal{J}_{-j}$ for $\mathcal{J} \backslash \{j\}$.

As the thesis is oriented at densities of extreme value copulas, we will often need to perform high order partial differentiation operations. In order to make the formulas less cumbersome, we introduce two operators that we will use throughout the thesis, that is

$$D = \frac{\partial^d}{\partial x_d \ldots \partial x_1} \quad \text{and} \quad D_B = \frac{\partial^{|B|}}{\prod_{j \in B} \partial x_j}.$$

Also, for two positive univariate functions f and g, we write $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. If we have $f(x) \sim 0$, it is to be understood as $\lim_{x\to\infty} f(x) = 0$.

Furthermore, when considering random variables, we will be mainly concerned with their distributional properties. We then note that one can always find a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where these random variables exist as a consequence of Skorokhod's Representation Theorem. This being clarified, we will make no further mention of probability spaces in what follows.

2.2 Copulas

Before defining an extreme value copula, one needs to first define and understand what a copula is.

Definition 2.1
A *d*-dimensional *copula C* is a *d*-dimensional distribution function with standard uniform univariate margins.

The above definition is the simplest we can give, so we complement it with an equivalent definition providing more insights on copula properties.

Definition 2.2
A function $C : [0,1]^d \to [0,1]$ is a *d*-dimensional copula if and only if

(i) $C(\boldsymbol{u}) = 0$ whenever $u_j = 0$ for some $j \in \mathcal{J}$,

(ii) $C(\boldsymbol{u}) = u_j$ for all $u_j \in [0,1]$ if $u_k = 1$ for all $k \in \mathcal{J}_{-j}$,

(iii) $\Delta_{(\boldsymbol{a} \leq \boldsymbol{b}]} C = \sum_{\boldsymbol{i} \in \{0,1\}^d} (-1)^{\sum_{j=1}^{d} i_j} C(a_1^{i_1} b_1^{1-i_1}, \ldots, a_d^{i_d} b_d^{1-i_d}) \geq 0$ for $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^d$.

If (iii) holds, we say that the copula C is *d-increasing*, which means that for each non-empty hyperrectangle $(\boldsymbol{a} \leq \boldsymbol{b}]$, the C-volume $\Delta_{(\boldsymbol{a} \leq \boldsymbol{b}]} C$ is non-negative. Having a positive volume is equivalent to having a proper non-negative density, if the latter one exists.

One can observe that for any *d*-dimensional copula C for which $d \geq 3$, each *k*-dimensional margin of C is itself a *k*-dimensional copula, for $k = 2, \ldots, d-1$.

From the definitions given above, it is not yet clear why and how copulas are useful for understanding dependence between the components of a random vector. The key theorem in copula theory dates back to Sklar (1959), and explicitly links copulas as mathematical objects to their role in dependence modelling.

Theorem 2.3 (Sklar's Theorem)
For any distribution function H with margins F_j for $j \in \mathcal{J}$, there exists a copula C such

that

$$H(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d)), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{1}$$

C is uniquely determined on $\prod_{j=1}^d \operatorname{ran} F_j$ (*product range*) and is given by

$$C(\boldsymbol{u}) = H(F_1^-(u_1), \ldots, F_d^-(u_d)), \quad \boldsymbol{u} \in \prod_{j=1}^d \operatorname{ran} F_j. \tag{2}$$

Conversely, given any copula C and univariate distribution functions F_j for $j \in \mathcal{J}$, H defined by (1) is a distribution function with margins F_j for $j \in \mathcal{J}$.

Remark 2.4

Sklar's Theorem allows one to decompose any multivariate distribution function into its margins and a copula. This way, one can study multivariate distribution functions independently of the margins.

In (2), F_j^- is known as the generalised inverse of the marginal distribution F_j, defined as below.

Definition 2.5

Let $T : \mathbb{R} \to \mathbb{R}$ and be increasing. The *generalised inverse* $T^- : \mathbb{R} \to \bar{\mathbb{R}} = [-\infty, \infty]$ of T is defined as

$$T^-(u) = \inf\{x \in \mathbb{R} \mid T(x) \geq u\}$$

where $T^-(\emptyset) = \infty$.

Definition 2.5 emphasizes that flat parts of T correspond to jumps in T^-, and that jumps in T correspond to flat parts in T^-. For further properties of T^-, see Embrechts and Hofert (2012).

One might note that not all univariate distribution functions are invertible. Although, since they are non-decreasing by definition, the generalised inverse of a distribution function is always well-defined and known as the *quantile function*.

For various reasons, one might rather investigate certain probability distributions on $\mathbb{R}^d$ given their survival function $\bar{H}$ rather than their distribution function H. In that case, it is convenient to restate Theorem 2.3 in terms of survival functions.

Theorem 2.6 (Survival Sklar's Theorem)

For any survival function $\bar{H}$ with margins $\bar{F}_j$ for $j \in \mathcal{J}$, there exists a copula $\hat{C}$, referred to as the *survival copula*, such that

$$\bar{H}(\boldsymbol{x}) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{3}$$

$\hat{C}$ is uniquely determined on $\prod_{j=1}^d \operatorname{ran} \bar{F}_j$ and is given by

$$\hat{C}(\boldsymbol{u}) = \bar{H}(\bar{F}_1^-(u_1), \ldots, \bar{F}_d^-(u_d)), \quad \boldsymbol{u} \in \prod_{j=1}^d \operatorname{ran} \bar{F}_j. \tag{4}$$

Conversely, given any copula $\hat{C}$ and univariate survival functions $\bar{F}_j$ for $j \in \mathcal{J}$, $\bar{H}$ defined by (3) is a d-dimensional survival function with margins $\bar{F}_j$ for $j \in \mathcal{J}$.

It is to be noted that the generalized inverse $\bar{F}^-$ of a survival function $\bar{F}$ is defined as $\bar{F}^-(u) = F^-(1-u)$ for $u \in [0,1]$. Before going any further, we provide Figure 1 and Figure 2 for readers not familiar with the subject to visualize the behavior of copulas. Figure 2 provides a nice way of visualizing high-dimensional data in a pairwise manner, and was generated using the `splom2` function in R.
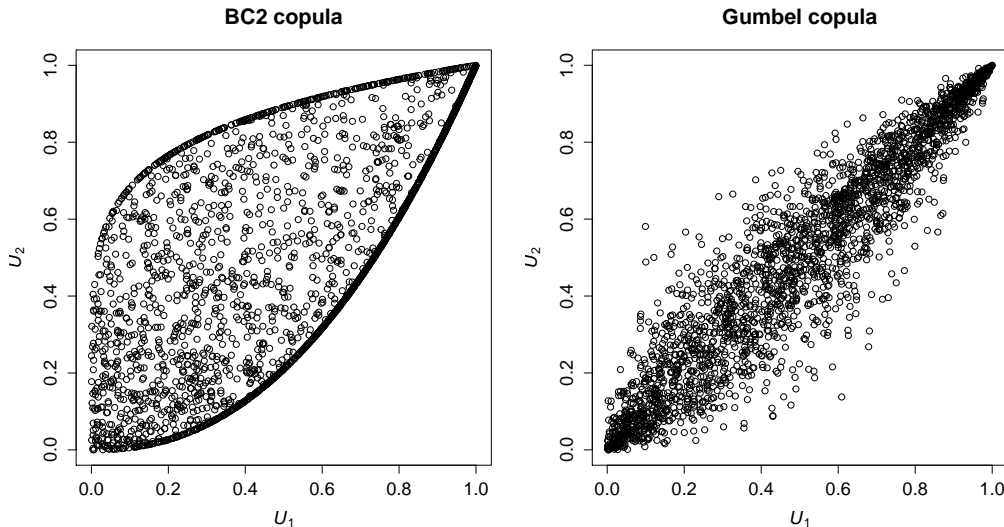
Figure 1 BC_2 copula with parameters $a = 0.9$ and $b = 0.4$ (left) generated using the algorithm presented in Mai and Scherer (2011); note the easily observable *singular component*. Gumbel copula with parameter $\theta = 5$ (right). In both samples, the sample size is $n = 2500$.

Getting this first glimpse at different copula scatterplots is interesting to see the joint behaviour of the variables, but keeping in mind that copulas are in fact distribution functions, it is relevant to be aware of the well-known bounds governing every copula. These bounds are known as the *Fréchet-Hoeffding bounds*, attributed to Fréchet (1935) and Hoeffding (1940).

Theorem 2.7 (Fréchet-Hoeffding bounds)
For any d-dimensional copula C with $d \geq 2$, it holds that

$$W(\boldsymbol{u}) = \max\left\{\sum_{j=1}^{d} u_j - d + 1, 0\right\} \leq C(\boldsymbol{u}) \leq \min_{1 \leq j \leq d}\{u_j\} = M(\boldsymbol{u}), \quad \boldsymbol{u} \in [0,1]^d,$$

where M, the *upper Fréchet-Hoeffding bound*, is a copula for any d, and W, the *lower*
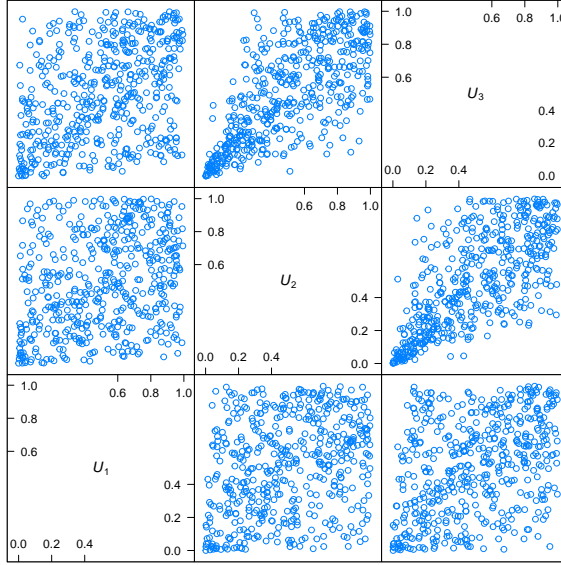
Figure 2 Three-dimensional nested Clayton copula with parameters chosen such that the Kendall's tau of the respective bivariate margins are 0.2 and 0.5. The sample size is $n = 500$.

Fréchet-Hoeffding bound, is a copula only when $d = 2$. Figure 3 explicitly shows M and W in a comparative graph.

In the next section, we introduce the notion of extreme value copulas, and present some relevant theory around the concept.

2.3 Extreme value copulas

Consider a sample of independent and identically distributed random vectors $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{id})$ with common distribution function H, margins $F_1, \ldots, F_d$, and copula C_H, under the assumption that H is continuous. To perform multivariate extreme value theory analysis, one may consider the vector of componentwise maxima, defined as

$$\boldsymbol{M}_n = (M_{n,1}, \ldots, M_{n,d}), \quad \text{where } M_{n,j} = \max_{1 \leq i \leq n} X_{ij}. \tag{5}$$

It follows by construction that the distribution of the componentwise maxima is given by an easy expression, as shown below.

$$
\begin{aligned}
H_{M_{n,j}}(x) &= P(M_{n,j} \leq x) = P(X_{1j} \leq x, \ldots, X_{nj} \leq x) = P(X_{1j} \leq x) \ldots P(X_{nj} \leq x) \\
&= F_j(x) \ldots F_j(x) \\
&= F_j^n(x)
\end{aligned}
$$
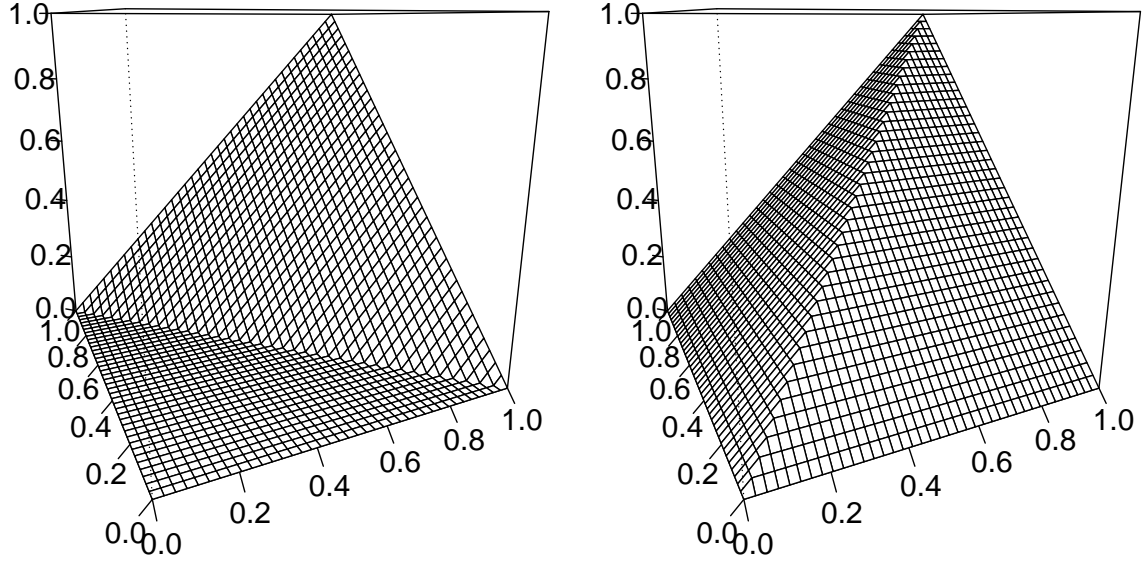
Figure 3 Lower Fréchet-Hoeffding bound W (left); upper Fréchet-Hoeffding bound M (right)

This shows that the joint distribution, as well as the marginal distributions, of $\boldsymbol{M}_n$ are given by H^n, $F_1^n, \ldots, F_d^n$ respectively. It follows that the copula of $\boldsymbol{M}_n$, which we call C_n, is given by

$$C_n(u_1, \ldots, u_d) = C_H(u_1^{1/n}, \ldots, u_d^{1/n})^n, \quad \boldsymbol{u} \in [0, 1]^d. \tag{6}$$

The family of extreme value copulas arises from (6) when the sample size n goes to infinity, in the weak sense. This limit concept is enclosed in the following definition.

Definition 2.8

A d-dimensional copula C is a d-dimensional *extreme value copula* if there exists a copula C_H such that, for $n \to \infty$,

$$C_H(u_1^{1/n}, \ldots, u_d^{1/n})^n \to C(u_1, \ldots, u_d), \quad \boldsymbol{u} \in [0, 1]^d. \tag{7}$$

In this case, we say that the copula C_H is *in the domain of attraction of C*.

In fact, the representation of extreme value copulas can be simplified using the concept of *max-stability*, as nicely explained in Segers and Gudendorf (2009).

Definition 2.9

A d-dimensional copula C is *max-stable* if it holds that

$$C(\boldsymbol{u}) = C(u_1^{1/m}, \ldots, u_d^{1/m})^m, \quad \boldsymbol{u} \in [0, 1]^d,$$

for every $m \in \mathbb{N}_0$.

From Definition 2.9, one sees that any max-stable copula is in its own domain of attraction, making it an extreme value copula from Definition 2.8. The converse can also be observed, and a very useful theorem ensues.

Theorem 2.10

A copula C is an extreme value copula if and only if it is max-stable.

By letting $u_j = e^{-x_j}$, we can take the negative logarithm on each side of (7) and by applying a linear expansion on the left-hand side, we see that (7) is equivalent to

$$\lim_{q \downarrow 0} q^{-1} \left(1 - C_H \left(1 - qx_1, \ldots, 1 - qx_d\right)\right) = -\log C(e^{-x_1}, \ldots, e^{-x_d}). \tag{8}$$

By setting the left-hand side of (8) equal to a function $\ell(\boldsymbol{x})$ for $\boldsymbol{x} \in [0, \infty)^d$ and by manipulating the equation, one obtains

$$C(e^{-x_1}, \ldots, e^{-x_d}) = \exp(-\ell(\boldsymbol{x}))$$

where the function $\ell(\boldsymbol{x})$ is referred to as the *stable tail dependence function* of C.

As the concept of *homogeneity* will become recurrent in this thesis, we first define the notion of *homogeneous function* before going any further.

Definition 2.11

A function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$f(t\boldsymbol{w}) = t^n f(\boldsymbol{w}), \quad t > 0, \quad w \in \mathbb{R}^d, \tag{9}$$

is said to be *homogeneous of order n*. For $n > 0$, we say that the function f is *positive homogeneous*.

In order to formalize the last few statements, Segers (2012) presents precise conditions under which a function ℓ properly defines an extreme value copula.

Theorem 2.12

A d-dimensional copula C is an extreme value copula if and only if

$$C(\boldsymbol{u}) = \exp(-\ell(-\log u_1, \ldots, -\log u_d)), \quad \boldsymbol{u} \in (0, 1]^d, \tag{10}$$

where the stable tail dependence function $\ell : [0, \infty)^d \to [0, \infty)$ is given by

$$\ell(\boldsymbol{x}) = \int_{\Delta_{d-1}} \max_{1 \leq j \leq d}(w_j x_j) dH(w_1, \ldots, w_d), \quad \boldsymbol{x} \in [0, \infty)^d, \tag{11}$$

for a Borel measure H on Δ_{d-1}, called the *spectral measure*, satisfying the constraints

$$\int_{\Delta_{d-1}} w_j dH(w_1, \ldots, w_d) = 1, \quad j \in \{1, \ldots, d\}. \tag{12}$$

It is worth mentionning that the stable tail dependence function obtained is convex and homogeneous of order one. Further, it satisfies the following inequalities

$$\max\{x_1, \dots, x_d\} \leq \ell(\boldsymbol{x}) \leq \sum_{i=1}^{d} x_i, \quad \boldsymbol{x} \in [0, \infty)^d.$$

However, these properties do not characterize the class of stable tail dependence functions unless $d = 2$. We suggest the reader to look at the counterexample in Beirlant et al. (2004), p.257.

By restricting the stable tail dependence function to the unit simplex, one can obtain a characterization through the use of the so called *Pickand's dependence function* $A : \Delta_{d-1} \to [1/d, 1]$ of the form

$$\ell(\boldsymbol{x}) = (x_1 + \dots + x_d)\, A(w_1, \dots, w_{d-1}), \quad w_j = \frac{x_j}{x_1 + \dots + x_d}, \quad j \in \mathcal{J}_{-d}, \qquad (13)$$

for $\boldsymbol{x} \in [0, \infty)^d \backslash \{0\}$. Note that some authors express the above A function in terms of $(w_1, \dots, w_d)$, but we suppressed the variable w_d in our representation as it is obvious that $w_d = 1 - w_1 - \dots - w_{d-1}$.

Since $\boldsymbol{u} \in (0, 1]^d \subset [0, \infty)^d \backslash \{0\}$, we can define $\tilde{w}_j = \log u_j / \sum_{j=1}^{d} \log u_j$, $j \in \mathcal{J}_{-d}$, and rewrite (10) in the following way:

$$
\begin{aligned}
C(\boldsymbol{u}) &= \exp\left\{-\ell(-\log u_1, \dots, -\log u_d)\right\} \\
&= \exp\left\{-\left(\sum_{j=1}^{d} -\log u_j\right) A\left(\frac{-\log u_1}{-\sum_{j=1}^{d} \log u_j}, \dots, \frac{-\log u_{d-1}}{-\sum_{j=1}^{d} \log u_j}\right)\right\} \\
&= \exp\left\{\left(\sum_{j=1}^{d} \log u_j\right) A\left(\frac{\log u_1}{\sum_{j=1}^{d} \log u_j}, \dots, \frac{\log u_{d-1}}{\sum_{j=1}^{d} \log u_j}\right)\right\} \\
&= \left(\prod_{j=1}^{d} u_j\right)^{A(\tilde{w}_1, \dots, \tilde{w}_{d-1})}.
\end{aligned}
$$

Even though A is convex and homogeneous of order 1, it is important to mention that the class of extreme value copulas is not defined by the class of Pickand's dependence functions A, unless $d = 2$, which is related to our prior comment on the issue.

Since bivariate extreme value copulas are frequently used in many fields and will be of interest to us in Section 5, we provide a characterization based on the use of the formula obtained above.

Theorem 2.13
A bivariate copula C is an extreme value copula if and only if

$$C(u_1, u_2) = \exp\left\{\log(u_1 u_2) A\left(\frac{\log u_1}{\log(u_1 u_2)}\right)\right\}, \quad (u_1, u_2) \in (0, 1]^2 \backslash \{(1, 1)\}, \qquad (14)$$

where $A : [0, 1] \to [1/2, 1]$ is convex and satisfies $\min\{t, 1-t\} \leq A(t) \leq 1$ for all $t \in [0, 1]$.

Figure 4 provides the reader with a visual example of three Pickand's dependence functions. It is worth noticing that the three curves are within the bounds defined in Theorem 2.13 and that the curves may not be symmetric nor smooth.
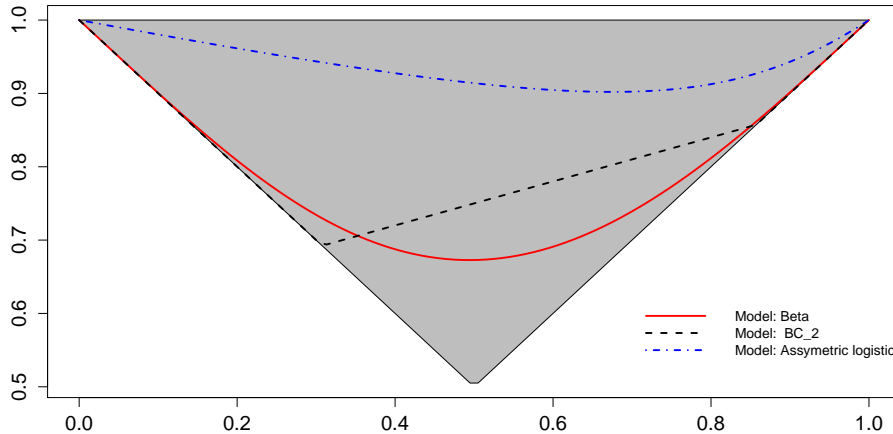
Figure 4 Different Pickand's dependence functions; red curve arises from a beta distribution with parameters $q_1 = 2$ and $q_2 = 3$, the black curve arises from a BC_2 with parameters $a = 0.9$ and $b = 0.4$ and the blue curve arises from an asymmetric logistic distribution with parameters $\psi_1 = 0.2$, $\psi_2 = 0.7$, $t = 2$. The grey area emphasizes the zone $\max\{t, 1-t\} \leq A(t) \leq 1$.

The notion of *tail dependence* describes the amount of dependence in the tail of a distribution, mainly in the bivariate case, and has been discussed profusely in financial applications related to market and credit risk, and is known to strongly influence the *Value-at-Risk* (VaR) measure. The lower tail dependence function and upper tail dependence function introduced in Jaworski (2006), Klüppelberg et al. (2008) and Joe et al. (2010) are defined as follows.

Definition 2.14
Let C be the copula of $\boldsymbol{U}$, a d-dimensional random vector with uniform margins, and let $\hat{C}$ be the survival copula of C. Then, one defines the *lower tail dependence function* as

$$\lambda_L(\boldsymbol{w}) = \lim_{q\downarrow 0} \frac{C(qw_j,\ j \in \mathcal{J})}{q}, \quad \boldsymbol{w} = (w_1, \ldots, w_d) \in \mathbb{R}_+^d, \tag{15}$$

and the *upper tail dependence function* as

$$\lambda_U(\boldsymbol{w}) = \lim_{q\downarrow 0} \frac{\hat{C}(qw_j,\ j \in \mathcal{J})}{q}, \quad \boldsymbol{w} = (w_1, \ldots, w_d) \in \mathbb{R}_+^d, \tag{16}$$

provided that the limits exist.

In the upcoming section, we address two ways to construct extreme value copulas from Representations (15) and (16), and summarize our results in a theorem. Since (15) and

(16) are very closely related, we present the developments using $\lambda_L(\boldsymbol{w})$, which are easily translatable to $\lambda_U(\boldsymbol{w})$.

Following the setup presented in Joe et al. (2008), we consider a d-dimensional copula C with continuous second-order partial derivatives. We also consider the set $\mathcal{J} = \{1, \ldots, d\}$ introduced before, along with its collection of subsets $\emptyset \neq S \subseteq \mathcal{J}$, and we denote the marginal copula corresponding to indices in S by C_S. We now introduce *marginal lower tail dependence functions* λ_S of C, $\emptyset \neq S \subseteq \mathcal{J}$, which are defined as

$$\lambda_S^L(w_j, \ j \in S) = \lim_{q \downarrow 0} \frac{C_S(qw_j, \ j \in S)}{q}, \quad \boldsymbol{w} = (w_1, \ldots, w_d) \in \mathbb{R}_+^d. \tag{17}$$

The above function is used as the primary tool in Joe et al. (2008) to investigate extreme value properties of multivariate t copulas. We now proceed by investigating a relevant property of λ_S^L, that is, homogeneity.

Proposition 2.15
The lower tail dependence functions $\lambda_S^L(\boldsymbol{w})$ are homogeneous of order one for any $\emptyset \neq S \subseteq \mathcal{J}$.

Proof. Fixing $t > 0$, and for any $\emptyset \neq S \subseteq \mathcal{J}$ we have that

$$\lambda_S^L(t\boldsymbol{w}) = \lim_{q \downarrow 0} \frac{C_S(qt\,w_j, j \in S)}{q} = t \lim_{q \downarrow 0} \frac{C_S(qt\,w_j, j \in S)}{qt} \quad = t \lim_{\tilde{q} \downarrow 0} \frac{C_S(\tilde{q}\,w_j, j \in S)}{\tilde{q}}$$
$$= t\lambda_S^L(\boldsymbol{w}),$$

where we set $\tilde{q} = qt$. Therefore, it follows that $\lambda_S^L(\boldsymbol{w})$ is a positive homogeneous function of order one for any $\emptyset \neq S \subseteq \mathcal{J}$. $\qquad\square$

Having confirmed that we have a homogeneous function, we now need to ensure that λ_S^L is continuously differentiable, as we will be interested in densities further on in this thesis. We do so by formulating the *Uniform Convergence Condition*, an assumption on the partial derivatives of the marginal copulas C_S. Furthermore, we assume that this statement holds for the rest of the thesis.

Assumption 2.16 (Uniform Convergence Condition)
Assume that any partial derivative of order $|S|$ or less of the ratios

$$\frac{C_S(qw_j, \ j \in S)}{q} \quad \text{and} \quad \frac{\hat{C}_S(qw_j, \ j \in S)}{q}$$

converges uniformly on $\mathbb{R}_+^{|S|}$ as $q \downarrow 0$, where $\emptyset \neq S \subseteq \mathcal{J}$.

Now, continuously differentiable function that are positive homogeneous are characterized by *Euler's Homogeneous Theorem*, see Wilson (1912).

Theorem 2.17 (Euler's Homogeneous Theorem)

Suppose that the function $f : \mathbb{R}_+^d \to \mathbb{R}$ is continuously differentiable, and let $\boldsymbol{a} \cdot \boldsymbol{b}$ denote the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$. Then, the function f is positive homogeneous of degree n if and only if

$$n f(\boldsymbol{x}) = \boldsymbol{x} \cdot \nabla f(\boldsymbol{x}) = \sum_{k=1}^{d} x_k \frac{\partial}{\partial x_k} f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}_+^d. \tag{18}$$

It is now possible for us to apply Theorem 2.17 to $\lambda_S^L(\boldsymbol{w})$ as we have shown it is homogeneous of order one and as it is continuously differentiable by the Uniform Convergence Condition. Doing so, it then follows that

$$\lambda_S^L(\boldsymbol{w}) = \lim_{q \downarrow 0} \sum_{k \in S} w_k \frac{\partial \lambda_S^L}{\partial w_k}. \tag{19}$$

It is important to realize that Assumption 2.16 allows us to interchange the limit and the differential operator. By construction of $\lambda_S^L(\boldsymbol{w})$, we obtain, $k \in S$,

$$\begin{aligned}
\frac{\partial \lambda_S^L}{\partial w_k} &= \frac{\partial}{\partial w_k} \left(\lim_{q \downarrow 0} \frac{C_S(q\, w_j, j \in S)}{q} \right) = \lim_{q \downarrow 0} \frac{1}{q} \frac{\partial}{\partial w_k} C_S(q\, w_j, j \in S) \\
&= \lim_{q \downarrow 0} \frac{\partial}{\partial u_k} C_S(u_j, j \in S) \Big|_{u_j = q\, w_j} \\
&= \lim_{q \downarrow 0} \frac{\partial}{\partial u_k} P(U_j \le u_j, j \in S) \big|_{u_j = q\, w_j} \\
&= \lim_{q \downarrow 0} P(U_j \le q w_j, j \in S_{-k} \,|\, U_k = q w_k).
\end{aligned}$$

The above expression for the partial derivatives of $\lambda_S^L(\boldsymbol{w})$ is thus given in terms of a conditional probability of the vector $\boldsymbol{U} \sim C$, and allows us to rewrite (19) as

$$\lambda_S^L(\boldsymbol{w}) = \lim_{q \downarrow 0} \sum_{k \in S} w_k\, P(U_j \le q w_j, j \in S_{-k} \mid U_k = q w_k), \quad \boldsymbol{w} \in \mathbb{R}_+^d, \tag{20}$$

for all $\emptyset \ne S \subseteq \mathcal{J}$.

In order to express an extreme value copula in terms of λ_S^L, we need a relationship linking λ_S^L to the lower stable tail dependence function ℓ_L. This link, which is related to the Sylvester-Poincaré sieve, is the one used by Joe et al. (2008) and can also be found in Li and Wu (2013), that is,

$$\ell_L(\boldsymbol{w}) = \lim_{q \downarrow 0} \frac{1}{q} (1 - \hat{C}(\boldsymbol{1} - q\boldsymbol{w})) = \sum_{\emptyset \ne S \subseteq \mathcal{J}} (-1)^{|S|-1} \lambda_S^L(\boldsymbol{w}). \tag{21}$$

We notice that by leveraging Representation (20) of λ_S^L, we can modify (21) and manipulate the terms in order to get

$$\ell_L(\boldsymbol{w}) = \sum_{\emptyset \ne S \subseteq \mathcal{J}} (-1)^{|S|-1} \lambda_S^L(\boldsymbol{w}) \tag{22}$$

$$= \sum_{\emptyset \neq S \subseteq \mathcal{J}} (-1)^{|S|-1} \left(\lim_{q \downarrow 0} \sum_{k \in S} w_k \, P(U_j \leq qw_j, j \in S_{-k} | \, U_k = qw_k) \right) \tag{23}$$

$$= \lim_{q \downarrow 0} \sum_{\emptyset \neq S \subseteq \mathcal{J}} (-1)^{|S|-1} \left(\sum_{k \in S} w_k \, P(U_j \leq qw_j, j \in S_{-k} | \, U_k = qw_k) \right)$$

$$= \lim_{q \downarrow 0} \sum_{k=1}^{d} w_k \left(1 - \sum_{S : |S| \geq 2} (-1)^{|S|-2} P(U_j \leq qw_j, \, j \in S_{-k} \, | \, U_k = qw_k) \right) \tag{24}$$

$$= \lim_{q \downarrow 0} \sum_{k=1}^{d} w_k \, P(U_j \geq qw_j, \, j \in \mathcal{J}_{-k} \, | \, U_k = qw_k) \tag{25}$$

$$= \lim_{q \downarrow 0} \sum_{k=1}^{d} w_k \, P(1 - U_j \leq 1 - qw_j, \, j \in \mathcal{J}_{-k} \, | \, 1 - U_k = 1 - qw_k) \tag{26}$$

$$= \lim_{q \downarrow 0} \sum_{k=1}^{d} w_k \hat{C}(1 - q\boldsymbol{w}_j, \, j \in \mathcal{J}_{-k} \, | \, 1 - qw_k) \tag{27}$$

where going from (22) to (23) is achieved by (21) and goind from (24) to (25) requires the use of the inclusion-exclusion formula. Doing so, we were able to generate two representations for ℓ_L, that is, the standard version linked to (8), and its homogeneous counterpart (27). It has to be pointed out that we are able to obtain the homogeneous representation of Joe et al. (2008) in a much simpler way, see Section 6.

The theorem below is the formalized counterpart of Theorem 2.12 for copulas known in the literature as *lower extreme value copulas* and *upper extreme value copulas*.

Theorem 2.18

Let C be the d-dimensional copula of $(U_1, \ldots, U_d)$ with standard uniform margins.

1. The lower extreme value copula of C can be expressed as

$$C^{LEV}(\boldsymbol{u}) = \exp(-\ell_L(-\log \boldsymbol{u})), \quad \boldsymbol{u} \in (0, 1]^d,$$

where

$$\ell_L(\boldsymbol{u}) = \lim_{q \downarrow 0} \sum_{k=1}^{d} w_k \hat{C}(1 - q\boldsymbol{w}_j, \, j \in \mathcal{J}_{-k} \, | \, 1 - qw_k) = \lim_{q \downarrow 0} \frac{1}{q}(1 - \hat{C}(\boldsymbol{1} - q\boldsymbol{w})).$$

2. The upper extreme value copula of C can be expressed as

$$C^{UEV}(\boldsymbol{u}) = \exp(-\ell_U(-\log \boldsymbol{u})), \quad \boldsymbol{u} \in (0, 1]^d,$$

where

$$\ell_U(\boldsymbol{u}) = \lim_{q \downarrow 0} \sum_{k=1}^{d} w_k C(1 - q\boldsymbol{w}_j, \, j \in \mathcal{J}_{-k} \, | \, 1 - qw_k) = \lim_{q \downarrow 0} \frac{1}{q}(1 - C(\boldsymbol{1} - q\boldsymbol{w})).$$

We now introduce the concept of *regular variation*, as it will be highly useful in the context of our work. In fact, most of our work in Section 6 and Section 7 will revolve around the use of regular variation properties.

2.4 Regular variation

The material contained in this section of the thesis is standard, but for readers interested in detailed discussions concerning univariate and multivariate regular variation, we recommend Resnick (2007).

Without loss of generality, assume that $\boldsymbol{X}$ is non-negative componentwise, and assume that the marginal survival functions $(\bar{F}_j)_{j \in \mathcal{J}}$ are right tail equivalent, defined as follows.

Definition 2.19
Consider the above setup. The marginal survival functions of a distribution are said to be *right tail equivalent* if

$$\frac{\bar{F}_j(x)}{\bar{F}_1(x)} = \frac{1 - F_j(x)}{1 - F_1(x)} \to 1, \quad \text{as} \quad x \to \infty,$$

for all $j \in \mathcal{J}$.

One is able to encode the extremal dependence structure of $\boldsymbol{X}$ in a so-called intensity measure ν arising from multivariate regular variation, as presented in Definition 2.20. This will be of high importance later on, as it will allow us to connect our work to the concept of VaR in Section 6.

Definition 2.20
The distribution function H of $\boldsymbol{X}$ is said to be *multivariate regularly varying* with *intensity measure ν* if

$$\lim_{t \to \infty} \frac{P(\boldsymbol{X} \in tR)}{P(X_1 > t)} = \nu(R), \quad R \subset \mathbb{R}_+^d,$$

where R is a relatively compact hyperrectangle, such that $\nu(\partial R) = 0$.

The measure ν is a Radon measure with homogeneity order $-\alpha$, that is $\nu(tR) = t^{-\alpha}\nu(R)$ where $t > 0$, for all relatively compact subsets R bounded away from the origin. In the above context, $\alpha > 0$ is known as the *tail index*.

The translation from multivariate regular variation to univariate regular variation is easily done. In general, we say that a Borel measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying at infinity with tail index $\alpha \in \mathbb{R}$* if and only if

$$f(x) = x^\alpha L(x), \quad x \geq 0,$$

where $L(t)$ is *slowly varying at infinity*, that is, a function satisfying

$$\lim_{t \to \infty} L(tx)/L(t) = 1, \quad x > 0.$$

It is standard to write RV_α for the set of regularly varying functions with tail index α, and SV, or RV_0, for the set of slowly varying functions.

Remark 2.21

Regular variation of a function f can also be defined at any point $x_0 \in \mathbb{R}$ by requiring that $f(x_0 - x^{-1})$ is regularly varying at infinity.

There are two predominant theorems that are highly important and powerful. These are used a lot in the context of extreme value theory, and even though they will not be used explicitly in this thesis, we decide to briefly present them to highlight their importance.

Theorem 2.22 (Karamata's Theorem)

Let $L \in SV$ be locally bounded on $[x_0, \infty]$, for $x_0 > 0$. Then

(i) If $\alpha < -1$, then $\int_x^\infty t^\alpha L(t)dt \sim -\frac{x^{\alpha+1}}{\alpha+1}L(x)$ as $x \to \infty$.

(ii) If $\alpha > -1$, then $\int_{x_0}^x t^\alpha L(t)dt \sim \frac{x^{\alpha+1}}{\alpha+1}L(x)$ as $x \to \infty$.

(iii) The function $x \mapsto \int_x^\infty t^{-1}L(t)dt$ is slowly varying, if finite.

Theorem 2.23 (Monotone Density Theorem)

Let $U(x) = \int_0^x u(y)dy$ where u is an ultimately monotone function. If $U(x) \sim cx^\alpha L(x)$ as $x \to \infty$ with $c \geq 0$, $\alpha \in \mathbb{R}$ and $L \in SV$, then $u(x) \sim c\alpha x^{\alpha-1}L(x)$ as $x \to \infty$.

Theorem 2.22 and Theorem 2.23 can be found in Bingham et al. (1989), along with their respective proofs.

2.5 Measures of concordance

For various applications, see Ghoudi et al. (1998), it is often desirable to measure the degree of association between random variables by the means of a simple real number. This allows for an easier comparison between random variables.

These measures of association are mainly studied in the case where $d = 2$, although multivariate extensions can be found in the literature. It goes without a doubt that one of the most widely used measure of association is *Pearson's correlation coefficient*, given by

$$\rho = \rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)}\sqrt{\text{Var}(X_2)}},$$

where it is assumed that both X_1 and X_2 have finite second moment. See Stigler (1989) for the history of the uprising of this coefficient.

In order to avoid well-known flaws of the correlation coefficient as a measure of association (see Embrechts et al. (2002)), the two following measures have been introduced.

Definition 2.24 (Spearman's rho and Kendall's tau)

Let $X_j \sim F_j$, $j \in \{1, 2\}$, be continuously distributed random variables with bivariate

copula C. *Spearman's rho* is defined by

$$\rho_S = \rho(F_1(X_1), F_2(X_2)) = 12 \int_{[0,1]^2} uvdC(u,v) - 3,$$

which equals $\rho(F_1(X_1), F_2(X_2))$.

If (X_1', X_2') is an iid copy of (X_1, X_2), then *Kendall's tau* is defined by

$$\tau = \mathbb{E}[\mathrm{sign}\{(X_1 - X_1')(X_2 - X_2')\}] = 4 \int_{[0,1]^2} C(u,v)dC(u,v) - 1,$$

where $\mathrm{sign}(x) = \mathbb{1}(0 < x < \infty) - \mathbb{1}(-\infty < x < 0)$

3 Stable tail dependence functions of elliptical distributions

We find it important to shortly address the class of elliptical distributions as we will encounter them again in Section 6. This class is one of the most widely used in modelling, and can be defined via its stochastic representation

$$\boldsymbol{X} \overset{d}{=} \boldsymbol{\mu} + RA\boldsymbol{U},$$

where $\boldsymbol{\mu} \in \mathbb{R}^d$ is the *location vector*, R is a non-negative variable known as the *radial part*, $\boldsymbol{U} \sim U(\{\boldsymbol{x} \in \mathbb{R}^k : \|x\| = 1\})$ is independent of R, and $A \in \mathbb{R}^{d \times k}$. The *dispersion matrix* of $\boldsymbol{X}$ is $\Sigma = AA^\top$. Note that the characteristic function of an elliptical random vector $\boldsymbol{X}$ is

$$\phi_{\boldsymbol{X}}(\boldsymbol{t}) = \mathbb{E}[\exp(i\boldsymbol{t}^\top \boldsymbol{X})] = e^{i\boldsymbol{t}^\top \boldsymbol{\mu}}\psi(\boldsymbol{t}\Sigma\boldsymbol{t}),$$

where ψ is known as the *characteristic generator*.

For a vector $\boldsymbol{X}$ that is elliptically distributed, we write $\boldsymbol{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$. It is to be noted that $\boldsymbol{\mu}$ is unique, but both ψ and Σ are unique only up to a positive constant, since for any $c > 0$, $\boldsymbol{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi) = E_d(\boldsymbol{\mu}, c\Sigma, \psi(\cdot/c))$.

We want to point out that the greek letter ψ is the standard symbol used in the literature when referring to characteristic generators of elliptical distributions, and is not to be confused with Archimedean generators, which will be an important part of our work in Section 6.

3.1 Relevant properties of elliptical distributions

In this section, we start by highlighting relevant properties of elliptical distributions that make this class of distributions one of the favorites of practioners in many fields of applications.

Property 3.1

Univariate marginal distribution functions and higher-dimensional margins of an elliptically distributed vector $\boldsymbol{X}$ are themselves elliptical.

Property 3.2

Elliptical distributions are symmetric around $\boldsymbol{\mu}$. That is, for $\boldsymbol{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$, it holds that

$$P(\boldsymbol{X} > \boldsymbol{a}) = P(\boldsymbol{X} \leq 2\boldsymbol{\mu} - \boldsymbol{a}), \quad \boldsymbol{a} \in \mathbb{R}^d.$$

Property 3.3

Conditional distributions of elliptical distributions are themselves elliptical, but in general with a different generator $\tilde{\psi}$. We note that some insights on the new characteristic generator $\tilde{\psi}$ are provided in Embrechts et al. (2002).

That last property is so important that it can be formalized as a theorem, which is very well known and can be found in many books. See Kotz and Nadarajah (2004) for further details.

Theorem 3.4

For $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ where $\mathbf{X}_1 = (X_1, \ldots, X_k)$ and $\mathbf{X}_2 = (X_{k+1}, \ldots, X_d)$, $\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, it holds that

$$\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2 \sim E_k(\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, \tilde{\psi}). \tag{28}$$

Using the standard notation, we minimize the notational burden by defining the following two new variables

$$\boldsymbol{\mu}_{1,2} = \boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \quad \text{and} \quad \Sigma_{11,2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

As pointed out in Joe et al. (2010), $\lim_{q\downarrow 0} P(U_j \geq qw_j, \ j \in \mathcal{J}_{-k} \mid U_k = qw_k)$ can be expressed as $H_k(w_j/w_k, j \in \mathcal{J}_{-k})$, for $1 \leq k \leq d$, where H_k is a $(d-1)$-dimensional subdistribution. Here, the interesting fact is that the limit disappears.

Through the use of Properties 3.1, 3.2, 3.3, Theorem 3.4 and the above comment, Joe et al. (2008) are able to obtain a closed-form expression for the stable tail dependence functions of the Smith and the multivariate t distributions. The simplicity of these expressions, which will be discussed below, arise from the fact that elliptical distributions are closed under various operations.

3.2 Multivariate t distribution and Smith model

The normal distribution is part of the elliptical distribution class, and is considered to be the flag model of the family. For multiple reasons, the multivariate normal distribution is very appealing in finance. Indeed, under the simplifying assumptions of Black & Scholes,

one could consider a basket of assets for which every individual components' returns behave in a normal way. Within the same class, the t distribution is very interesting due to its heavy tails, meaning that it is more prone to producing values that fall far from its mean. This last property explains the uprising of its use in mathematical finance.

Student's t density function   **Normal density function**
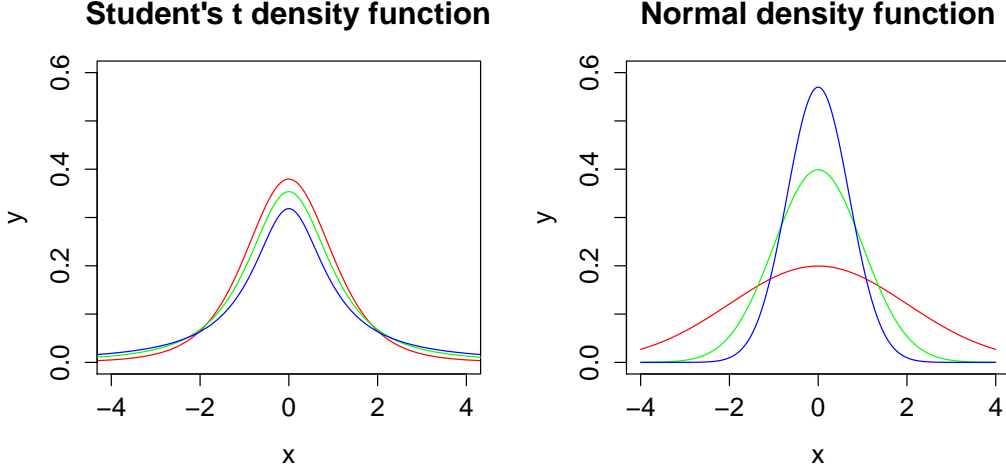
Figure 5 Tail comparison of the Student's t density with $\nu_{\text{red}} = 5$, $\nu_{\text{green}} = 2$, $\nu_{\text{blue}} = 1$ (left) and the Normal density with $\sigma_{\text{red}} = 0.7$, $\sigma_{\text{green}} = 1$, $\sigma_{\text{blue}} = 2$ (right).

As explicitly shown in Joe et al. (2008), the stable tail dependence function of the joint model of $\boldsymbol{X} = (X_1, \ldots, X_d) \sim T_{d,v,\Sigma}$ for v and $\Sigma = (\rho_{ij})$ its degrees of freedom and dispersion matrix, respectively, where $F_j = T_v$ for $j \in \mathcal{J}$, is given by

$$\ell(w_1, \ldots, w_d) = \sum_{k=1}^{d} w_k \, T_{d-1,v+1,R_k} \left(\sqrt{\frac{v+1}{1-\rho_{jk}^2}} \left[(w_j/w_k)^{-\frac{1}{v}} - \rho_{jk} \right], \, j \in \mathcal{J}_{-k} \right), \quad (29)$$

where

$$R_k = \begin{pmatrix} 1 & \cdots & \rho_{1,k-1;k} & \rho_{1,k+1;k} & \cdots & \rho_{1,d;k} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1,k-1;k} & \cdots & 1 & \rho_{k-1,k+1;k} & \cdots & \rho_{k-1,d;k} \\ \rho_{1,k+1;k} & \cdots & \rho_{k-1,k+1;k} & 1 & \cdots & \rho_{k+1,d;k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{1,d;k} & \cdots & \rho_{k-1,d;k} & \rho_{k+1,d;k} & \cdots & 1 \end{pmatrix},$$

where $\rho_{j,i;k} = \frac{\rho_{ji} - \rho_{jk}\rho_{ik}}{\sqrt{1-\rho_{ji}^2}\sqrt{1-\rho_{ik}^2}}$ for $j \neq k$, $i \neq k \in \mathcal{J}$.

Having an expression for the t extreme value copula, Joe et al. (2008) leverage on the result of Hüsler and Reiss (1989) by letting $\nu \to \infty$ with $\rho_{ij} \to 1$, and under the restriction that

$$\rho_{ij}(\nu) = 1 - 2\theta_{ij}^2/\nu,$$

18

it was shown in Joe et al. (2008) that the stable tail dependence function of the Smith model is given by

$$\ell(\boldsymbol{w}) = \sum_{k=1}^{d} w_k \, \Phi_{d-1,R_k} \left(\theta_{jk}^{-1} + \frac{\theta_{jk}}{2} \log \left(\frac{w_k}{w_j} \right), \; j \in \mathcal{J}_{-k} \right), \tag{30}$$

where R_k is the correlation matrix whose (i, k)-th entry is given by

$$\rho_{ik} = \frac{1/\theta_{ik}^2 + 1/\theta_{jk}^2 - 1/\theta_{ij}^2}{2/(\theta_{ik}\theta_{jk})}$$

for $i, j \in I \backslash \{k\}$. More will be adressed on Hüsler and Reiss (1989) in Section 5, and in Section 6, we will comment on how one can use the results of Joe et al. (2008) and Joe et al. (2010) to obtain the extreme value copula densities arising from their representation.

4 The function for densities of extreme value copulas

In this section, we leverage the copula theory that was presented in Section 2. Our goal here is to design and present a methodology to obtain a general formula for the density of any d-dimensional extreme value copula. This is of utmost importance for statistical applications such as estimating copula parameters.

The above goal is achieved by relying on Representation (10) of extreme value copulas, that is,

$$C(\boldsymbol{u}) = \exp(-\ell(-\log u_1, \ldots, -\log u_d)), \quad \boldsymbol{u} \in (0,1]^d,$$

and by introducing *Faà di Bruno's Formula*.

4.1 Faà di Bruno's Formula

One formula that proves to be very useful in the context of our work is the one for the d-th derivative of a composition of two functions f and g involving the differential operator D. The formula is named after the mathematician Faà di Bruno, and can be found in various forms, although it originally dates back to the work of Arbogast (1800). We refer the reader to Craik (2005) for information and developments on these various forms, as we focus on the use of one specific form of the formula that proves to be consistent with our work setup.

The version of Faà di Bruno's Formula we use holds regardless of whether the d variables are all distinct, all identical, or partitioned into several indistinguishable variables, and states that

$$\frac{\partial^d}{\partial x_d \ldots \partial x_1} f(g(\boldsymbol{x})) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(\boldsymbol{x})) \cdot \prod_{B \in \pi} \frac{\partial^{|B|} g(\boldsymbol{x})}{\prod_{j \in B} \partial x_j}$$

$$= \sum_{\pi \in \Pi} f^{(|\pi|)}(g(\boldsymbol{x})) \cdot \prod_{B \in \pi} D_B \, g(\boldsymbol{x}) \qquad (31)$$

where π runs through the set Π of all partitions of the set $\{1, \dots, d\}$ and $B \in \pi$ means that B runs through the list of all elements of the partition π. Note that $|\pi|$ denotes the number of sets in π.

Here, $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$. One might want to note that a further generalization of this formula considers the case where $g(\boldsymbol{x})$ is a vector valued function, and is due to Ma (2009).

In order to avoid any confusion, we will finish this section of the thesis by properly defining the notion of a partition of a set. This definition will also help us further on when performing combinatorial manipulations.

Definition 4.1
Consider the set $P = \{P_1, \dots, P_n\}$. P is a *partition* of the set Ω if and only if $P_i \neq \emptyset$ for all $i \in \{1, \dots, n\}$, $\bigcup_{i=1}^n P_i = \Omega$ and $P_j \cap P_k = \emptyset$ for all $j \neq k \in \{1, \dots, n\}$.

4.2 General formula for the density of an extreme value copula

In Section 2, we have shown that it is possible to express a multivariate extreme value copula C in the form of (10), namely

$$C(\boldsymbol{u}) = \exp\{-\ell(-\log u_1, \dots, -\log u_d)\}, \quad \boldsymbol{u} \in (0, 1]^d.$$

The key here is to note that C can be expressed as a composition of functions of the form $C(\boldsymbol{u}) = f(g(\boldsymbol{u}))$, where we have

$$f(x) = \exp\{-x\}$$

and

$$g(\boldsymbol{u}) = \ell(-\log u_1, \dots, -\log u_d).$$

Therefore, using Faà di Bruno's Formula (31) and letting $\boldsymbol{x} = -\log \boldsymbol{u}$, we obtain an expression for the density of an extreme value copula in the following way

$$c(\boldsymbol{u}) = D \, C(u_1, \dots, u_d) = Df(g(\boldsymbol{u})) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(\boldsymbol{u})) \cdot \prod_{B \in \pi} D_B g(\boldsymbol{u})$$

$$= \sum_{\pi \in \Pi} (-1)^{|\pi|} f(g(\boldsymbol{u})) \cdot \prod_{B \in \pi} \left(D_B \ell(\boldsymbol{x}) \cdot \prod_{j \in B} \frac{\partial x_j}{\partial u_j} \right),$$

$$= f(g(\boldsymbol{u})) \sum_{\pi \in \Pi} (-1)^{|\pi|} \cdot \prod_{B \in \pi} \left(D_B \ell(\boldsymbol{x}) \cdot \prod_{j \in B} -\frac{1}{u_j} \right)$$

$$= C(\boldsymbol{u}) \sum_{\pi \in \Pi} (-1)^{|\pi|} \cdot \left(\prod_{B \in \pi} \prod_{j \in B} -\frac{1}{u_j} \right) \prod_{B \in \pi} D_B \ell(\boldsymbol{x})$$

$$= C(\boldsymbol{u}) \sum_{\pi \in \Pi} (-1)^{|\pi|} \cdot \left(\prod_{j=1}^{d} -\frac{1}{u_j} \right) \prod_{B \in \pi} D_B \ell(\boldsymbol{x})$$

$$= \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \sum_{\pi \in \Pi} (-1)^{|\pi|+d} \prod_{B \in \pi} D_B \ell(\boldsymbol{x}) \tag{32}$$

$$= \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \sum_{m=1}^{d} \sum_{\pi:|\pi|=m} (-1)^{m+d} \prod_{B \in \pi} D_B \ell(\boldsymbol{x}). \tag{33}$$

Formula (33) is computationally intensive, as the total number of partitions of an n-element set is the Bell number B_n, where the first Bell number is $B_0 = 1$, and further numbers are defined via the recursion

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

In order to maintain a nice order amongst the set $\{\pi \in \Pi\}$, we chose to rewrite (32) into (33) using the relationship $\sum_{\pi \in \Pi} = \sum_{\pi:|\pi|=m}$, as it provides us with a more tracktable result. This will become especially important for implementation purposes.

So, by relying on the stable tail dependence function representation of extreme value copulas, and by using Faà di Bruno's Formula, we were able to provide a general formula for the density of extreme value copulas, which we summarize in the theorem below.

Theorem 4.2 (Main Theorem)
If the derivatives of ℓ exist and are well-defined, the density of a d-dimensional extreme value copula C is of the form

$$c(\boldsymbol{u}) = \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \sum_{m=1}^{d} (-1)^{d-m} \sum_{\pi:|\pi|=m} \prod_{B \in \pi} D_B \ell(\boldsymbol{x})|_{\boldsymbol{x}=-\log(\boldsymbol{u})}, \tag{34}$$

where $\boldsymbol{u} \in (0,1)^d$, $\Pi(\boldsymbol{u})$ is the independence copula, and

(i) π runs through the set Π of all partitions of the set $\mathcal{J} = \{1, \dots, d\}$,

(ii) $B \in \pi$ means that B runs through the list of all elements of the partition π.

We apply Theorem 4.2 in Sections 5, 6 and 7. In the next section, we rely on the Pickand's dependence function representation of bivariate extreme value copulas.

5 Applications in the bivariate case

A lot of work can be found on bivariate extreme value copulas, see Capéraà et al. (1997) and Genest et al. (2011). Therefore, this section of the thesis aims at providing the

reader with an application of Theorem 4.2 in the bivariate case. This will allow us to better understand and visualize the objects we are working with, and we will be able to leverage the insights from this section and use them as stepping stones for further developments in Sections 6 and 7.

Before we go any further, the following observation must be made. Consider any pair (X_1, X_2) having a bivariate extreme value distribution and marginal distributions F_1 and F_2. By defining the mapping

$$T(x) = -1/\ln x, \quad x \in (0, \infty],$$

we see that $T(x)$ is strictly increasing for $x \in (0, 1]$. Using the fact that copulas are invariant under strictly increasing mappings, it follows that the pair (X_1, X_2) has the same copula as the pair $(-1/\ln F_1(X_1), -1/\ln F_2(X_2))$.

Using the above in combination with Theorems 2.3 and 2.13, we get that the distribution of the modified pair can be written as

$$H(x, y) = \exp\left[-\left(\frac{1}{x} + \frac{1}{y}\right) A\left(\frac{y}{x + y}\right)\right]. \tag{35}$$

5.1 Bivariate density

We know from Equation (14) that in the bivariate case, it holds that

$$\ell(u_1, u_2) = (u_1 + u_2) A\left(\frac{u_1}{u_1 + u_2}\right).$$

Now, it trivially follows that the only two partitions of $\Pi = \{1, 2\}$ are $\pi_1 = \{\{1\}, \{2\}\}$ and $\pi_2 = \{\{1, 2\}\}$. Letting $q = \frac{u_1}{u_1 + u_2}$, we have the following partial derivatives:

$$\frac{\partial q}{\partial u_1} = \frac{u_2}{(u_1 + u_2)^2}, \quad \frac{\partial q}{\partial u_2} = -\frac{u_1}{(u_1 + u_2)^2}.$$

As mentionned before, the key elements arising from the application of Faà di Bruno's Formula are the building blocks given by $D_B \ell(\boldsymbol{w})$. Letting $\boldsymbol{u} = (u_1, u_2)$, we are now able to compute these blocks with the help of the partial derivatives found above.

For $B = \{1\}$, we have

$$\frac{\partial \ell(\boldsymbol{u})}{\partial u_1} = A(q) + (u_1 + u_2) \frac{\partial A(q)}{\partial q} \frac{\partial q}{\partial u_1} = A(q) + (1 - q) A'(q).$$

For $B = \{2\}$, we have

$$\frac{\partial \ell(\boldsymbol{u})}{\partial u_2} = A(q) + (u_1 + u_2) \frac{\partial A(q)}{\partial q} \frac{\partial q}{\partial u_2} = A(q) - q A'(q).$$

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For $B = \{1, 2\}$, we have

$$\frac{\partial^2 \ell(\boldsymbol{u})}{\partial u_1 \partial u_2} = -A''(q)(1-q)\frac{u_1}{(u_1 + u_2)^2}.$$

Now, with the above building blocks at hand, one can apply Theorem 4.2 for $d = 2$ and obtain the expression for any bivariate extreme value copula with Pickand's dependence function A.

$$c(u_1, u_2) = \frac{C(u_1, u_2)}{u_1 u_2}\left[\left(\frac{\partial \ell(\boldsymbol{x})}{\partial x_1}\frac{\partial \ell(\boldsymbol{x})}{\partial x_2}\right) - \frac{\partial^2 \ell(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]_{\boldsymbol{x} = (-\log u_1, -\log u_2)} \tag{36}$$

In order to use Formula (36), one has to explicitly know the Pickand's dependence function associated to the bivariate copula being considered. Pickand's dependence functions, introduced by Pickands (1981) have been extensively covered in modern literature, see Bücher et al. (2011), and are used in many applications. Therefore, we present in the next section a general method to compute it for various bivariate models.

5.2 Computing the Pickand's dependence function of a bivariate copula

Pickand's dependence functions arise in various contexts when studying copulas, whether it is when trying to compute Kendall's tau, or when creating a scheme such as the Ghoudi-Khoudraji-Rivest algorithm, presented in Ghoudi et al. (1998). In multiple cases, one can compute the function A using the technique we will present below.

To present the method, we show the steps that need to be performed by applying our technique to an example model, which we have chosen to be the bivariate beta distribution. This example will serve as a guideline for any other models the reader might consider, and we will simply state our results for other famous bivariate models without showing our calculations.

5.2.1 Bivariate beta distribution

The beta distribution is due to Tawn and Coles (1994), and it can be written as follows

$$H(x, y) = \exp\left[-\frac{1}{x}\left\{1 - B\left(q_1 + 1, q_2; \frac{x\, q_1}{x\, q_1 + y\, q_2}\right)\right\} - \frac{1}{y}\left\{B\left(q_1, q_2 + 1; \frac{x\, q_1}{x\, q_1 + y\, q_2}\right)\right\}\right], \tag{37}$$

where $q_1, q_2 > 0$, $x, y \in [0, 1]$ and $B(\alpha, \beta; \rho) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\rho w^{\alpha-1}(1-w)^{\beta-1}dw$, which is a normalized incomplete beta distribution. It is interesting to know that the bivariate beta distribution is widely used in the modeling of hydrological variables, see Murshed et al. (2012).

By equating (37) with (35), and taking the logarithm on each side, we can divide each side of the equality by $-\left(\frac{1}{x} + \frac{1}{y}\right)$, and after simple algebraic manipulations, obtain

$$A\left(\frac{y}{x+y}\right) = \frac{y}{x+y}\left\{1 - B\left(q_1 + 1, q_2; \frac{\frac{x}{x+y}\, q_1}{\frac{x}{x+y}\, q_1 + \frac{y}{x+y}\, q_2}\right)\right\}$$
$$+ \frac{x}{x+y}\left\{B\left(q_1, q_2 + 1; \frac{\frac{x}{x+y}\, q_1}{\frac{x}{x+y}\, q_1 + \frac{y}{x+y}\, q_2}\right)\right\}.$$

Now, letting $q = \frac{y}{x+y}$, it follows that $1 - q = \frac{x}{x+y}$, and we finally obtain

$$A(q) = q\left\{1 - B\left(q_1 + 1, q_2; \frac{(1-q)\, q_1}{(q_2 - q_1)q + q_1}\right)\right\}$$
$$+ (1-q)\left\{B\left(q_1, q_2 + 1; \frac{(1-q)\, q_1}{(q_2 - q_1)q + q_1}\right)\right\}.$$

Summarizing the three major steps of the procedure used above, we infer the following steps to recover the Pickand's dependence function A from a given bivariate model.

1. Equate the bivariate distribution H with the Pickand's dependence function Representation (35).

2. Take the logarithm on each side of the obtained equality and perform the necessary algebraic manipulations.

3. Set $q = \frac{y}{x+y}$ and simplify the equation.

With the above function A, one is now able to use Formula (36), as it is not difficult to compute A' and A'' using standard calculus. By using Formula (36) and any function A obtained through the above procedure, we can write a compact R code generating various plots of bivariate extreme value copula densities.

5.2.2 Bivariate asymmetric logistic distribution

The bivariate asymmetric logistic extreme value distribution is due to Tawn (1988) and is frequently found in the survival analysis literature. The model is given by

$$H_{\psi_1,\psi_2}(x,y) = \exp\left\{-\frac{(1-\psi_1)}{x} - \frac{(1-\psi_2)}{y} - \left[\left(\frac{\psi_1}{x}\right)^\theta + \left(\frac{\psi_2}{y}\right)^\theta\right]^{1/\theta}\right\},$$

where $0 \le \psi_1, \psi_2 \le 1$ and $\theta > 1$. Using our procedure described above, we find that its associated Pickands dependence function is given by

$$A(q) = (\psi_2 - \psi_1)q - \psi_2 + 1 + \left[(q\,\psi_1)^\theta + (\psi_2(1-q))^\theta\right]^{1/\theta}.$$
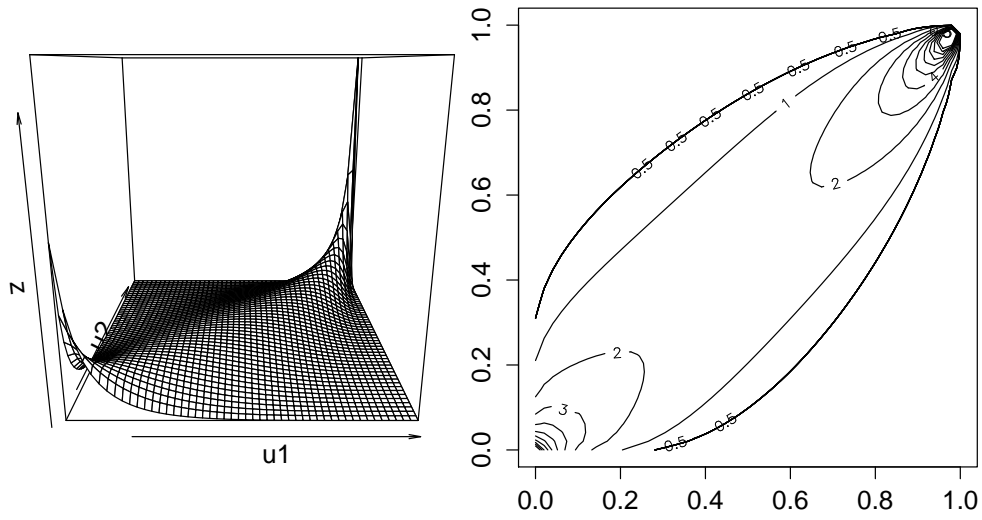
Figure 6 Density (left) and contour plot (right) of a bivariate beta copula with beta parameters $q_1 = 2, q_2 = 5$
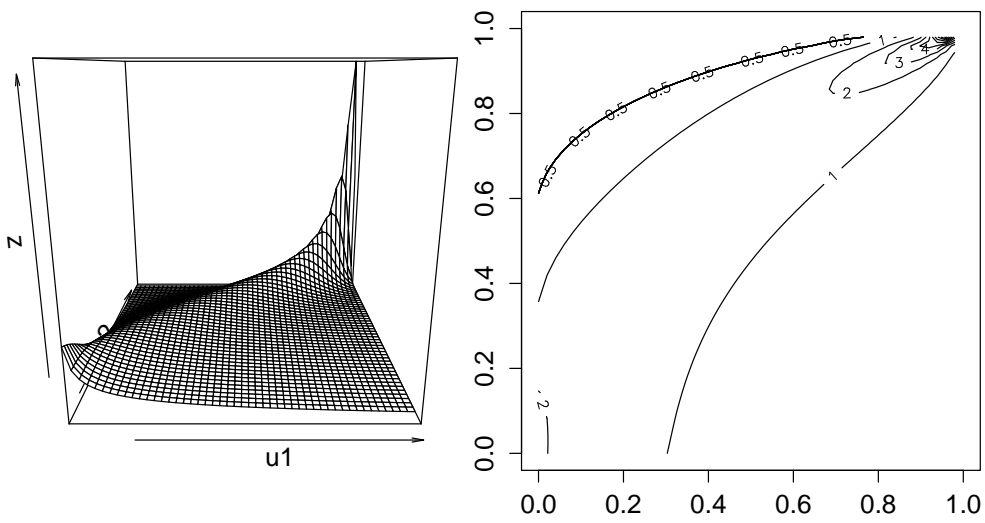
Figure 7 Density (left) and contour plot (right) of a bivariate asymmetric logistic copula with asymmetry parameters $\psi_1 = 0.3, \psi_2 = 0.7$ and $\theta = 3$

5.2.3 Bivariate asymmetric negative logistic distribution

The model for the bivariate asymmetric negative logistic distribution is given by

$$H_{\psi_1,\psi_2}(x,y) = \exp\left\{ -\frac{1}{x} - \frac{1}{y} - \left[\left(\frac{\psi_1}{x}\right)^\theta + \left(\frac{\psi_2}{y}\right)^\theta \right]^{1/\theta} \right\},$$

where $0 \leq \psi_1, \psi_2 \leq 1$ and $\theta < 0$. An application of the presented procedure yields that the Pickands dependence function of the model is defined by

$$A(q) = 1 - \left[(q\,\psi_1)^\theta + (\psi_2(1-q))^\theta \right]^{1/\theta}.$$

This model is similar in structure to the one introduced by Tawn when $\psi_1 = \psi_2 = 1$, giving a symmetric version of the family. The limiting cases $\theta \uparrow 0$ and $\theta \to -\infty$ respectively generate the commonotonicity and independence models.
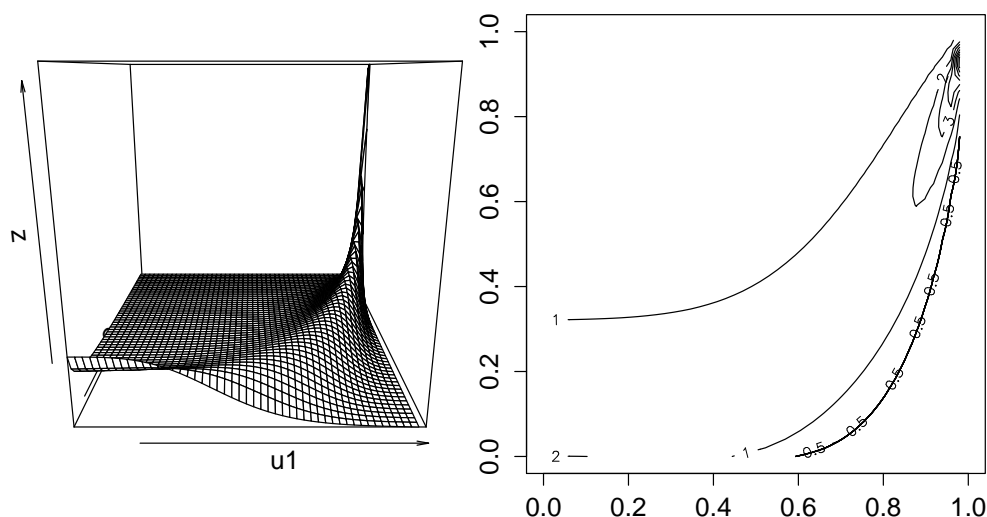
Figure 8 Density (left) and contour plot (right) of a bivariate asymmetric negative logistic copula with asymmetry parameters $\psi_1 = 0.8, \psi_2 = 0.2$ and $\theta = 3$

5.2.4 Hüsler-Reiss distribution

The Hüsler-Reiss model, see Hüsler and Reiss (1989), is used in various applications beyond the field of finance, such as the modelling of spatial variation of extreme storms, see Smith (1990). The model arises as Hüsler and Reiss (1989) let the correlation coefficient ρ of the bivariate Gaussian copula be dependent on the sample size n, that is, $\rho = \rho_n$, such that $\rho_n \to 1$ as $n \to \infty$. If

$$(1 - \rho_n)\log_n \to \lambda^2 \in [0, \infty],$$

then the bivariate model is given by

$$H(x,y) = \exp\left\{-\frac{1}{y}\Phi\left(s\left(\frac{x}{x+y}\right)\right) - \frac{1}{x}\Phi\left(a - s\left(\frac{x}{x+y}\right)\right)\right\},$$

where $s(w) = \frac{\{a^2 + 2\log(w) - 2\log(1-w)\}}{2a}$, $a = 2\lambda$ and Φ is a $N(0,1)$ distribution function. Its associated Pickands dependence function is

$$A(q) = (1-q)\,\Phi\left(\frac{a}{2} + \log\left(\frac{1-q}{q}\right)\frac{1}{a}\right) + q\,\Phi\left(\frac{a}{2} + \log\left(\frac{q}{1-q}\right)\frac{1}{a}\right).$$
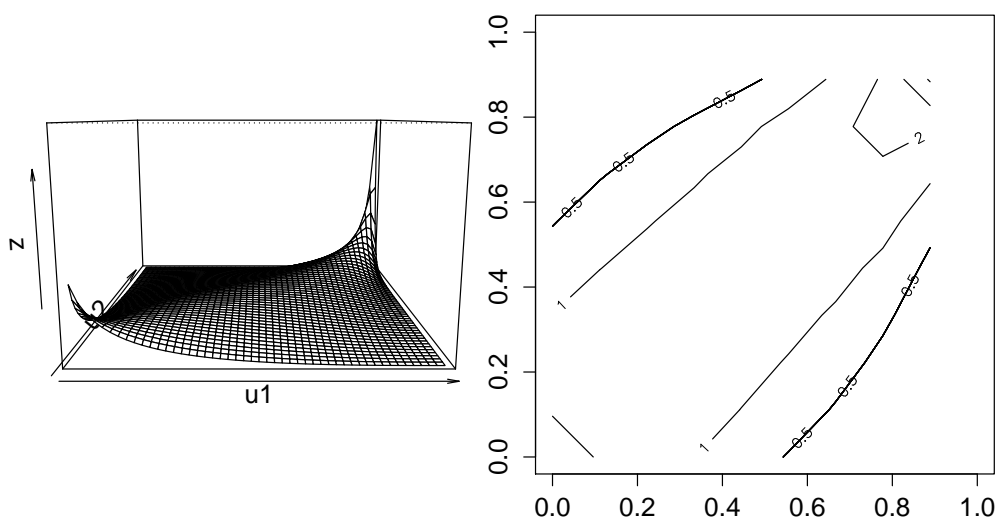
Figure 9 Density (left) and contour plot (right) of a bivariate Hüsler-Reiss copula with $\lambda = 2$

5.2.5 Bivariate circular distribution

This distribution is due to Coles and Walshaw (1994), but it arises from the previous work of Smith on the topology of circles. Coles and Walshaw used the circular distribution for directional modelling of extreme wind speeds. The model is given by

$$H(x,y) = \exp\left\{-\frac{\int_B f_0(w,\theta,\xi)dw}{x} - \frac{\int_{\hat{B}} f_0(\hat{w},-\theta,\xi)d\hat{w}}{y}\right\}$$

where $\theta_2 \geq \theta_1$, $\theta_2 - \theta_1 \leq \pi$ and $\theta = (\theta_1 - \theta_2)/2$. Furthermore,

$$f_0(w,\theta,\xi) = \frac{1}{2\,\pi\,I_0(\xi)}\exp\{\xi\cos(w-\theta)\},$$

and

$$B = \left\{ w \in (0, 2\pi] \ : \ \sin(w) > \frac{\log(w) - \log(1-w)}{2 \pi \ sin(\theta)} \right\}, \quad \hat{B} = (0, 2\pi] \backslash B.$$

Its associated Pickands dependence function is

$$A(q) = q \int_B f_0(w, \theta, \xi) dw + (1-q) \int_{\hat{B}} f_0(\hat{w}, -\theta, \xi) d\hat{w}.$$

5.2.6 Asymmetric mixed model

The asymmetric mixed distribution was introduced by Tawn (1988). It is comprehensively discussed in Klüppelberg and May (2006), where a full characterization of polynomial models by means of their dependence function is given. The model for the bivariate asymmetric mixed model distribution is given by

$$H(x, y) = \exp \left\{ - \left(\frac{1}{x} + \frac{1}{y} \right) + \frac{(2\beta + \alpha)x + (\beta + \alpha)y}{(x + y)^2} \right\},$$

where $\beta \geq 0$, $\beta + 2\alpha \leq 1$ and $\beta + 3\alpha \geq 0$.

It is interesting to note that the strength of dependence increases when α is fixed and β increases, and that complete dependence cannot be obtained. Independence is obtained when both parameters are zero. Its associated Pickands dependence function given by

$$A(q) = \alpha q^3 + \beta w^2 - (\alpha + \beta)q + 1.$$

Figure 10 Density (left) and contour plot (right) of a bivariate mixed model copula with asymmetry parameters $\alpha = 0.4, \beta = 0.1$

6 Constructing extreme value copula densities from given copulas

It is important to mention that not every copula C will have well-defined lower or upper stable tail dependence functions. Focusing on copulas for which such functions exist, we realize thro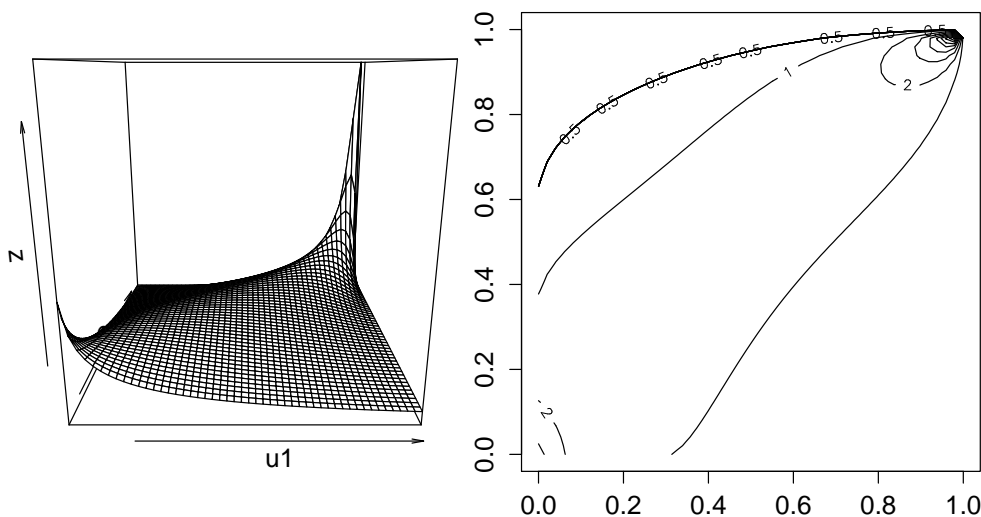ugh an application of Theorem 4.2 that we are now in the position to obtain the formula for the density of an extreme value copula constructed from any initial copula C with well-defined ℓ_L or ℓ_U.

We proceed as follows: Given a copula C, we compute its stable tail dependence function ℓ, if well-defined, using Theorem 2.18 and then we apply Theorem 4.2 using as inputs C and its associated ℓ. The approach of Joe et al. (2008) can be seen as a special case of the procedure based on C being the EVC of the multivariate t or the Smith model. The approach of Genest and Rivest (1989) for C being Archimedean under certain regular variation assumptions, which will be discussed later, will lead to an explicit form of new EVC density constructed from Archimedean copulas.

Before going any further, we first note a nice simplifying result arising from radial symmetry

6.1 Radially symmetric copulas

The observation of interest is that if the initial copula C is radially symmetric, we can obtain a nice relationship between ℓ_L and ℓ_U. We first introduce the concept of radial symmetry for vectors and copulas, see Nelsen (2006) p.36.

Definition 6.1
A random vector $\boldsymbol{X}$ is called radially symmetric about $\boldsymbol{a}$ if $\boldsymbol{X} - \boldsymbol{a} \overset{d}{=} \boldsymbol{a} - \boldsymbol{X}$.

Proposition 6.2
Let $\boldsymbol{X} \sim H$ with continous margins F_j for $j \in \mathcal{J}$, copula C and survival copula $\hat{C}$. If X_j is symmetric around a_j for $j \in \mathcal{J}$, then $\boldsymbol{X}$ is radially symmetric about $\boldsymbol{a}$ if and only if $C = \hat{C}$.

Proof. See Nelsen (2006), p.37. The proof addresses the bivariate case, but can easily be extended to higher dimensions. □

By Proposition 6.2, a copula is radially symmetric if and only if $C = \hat{C}$. In this case, the lower and upper stable tail dependence functions are equal, that is,

$$\ell_L(\boldsymbol{w}) = \ell_U(\boldsymbol{w}). \tag{38}$$

This statement might not be surprising, but we deem it important to highlight as various commonly used copulas are radially symmetric, such as elliptical copulas. Another

trivial example would be the independence copula $\Pi(\boldsymbol{u})$, as well as the Fréchet-Hoeffding bounds M and W.

Having just mentioned elliptical copulas as flag examples of radially symmetric copulas, we deem it appropriate to follow-up on the work of Joe et al. (2008) and Joe et al. (2010) presented in Section 3. By relying on the homogeneous representation of the stable tail dependence function of a copula, they have obtained explicit representations of ℓ for both the Smith model and the multivariate t distribution.

With their expression for ℓ, we can apply Theorem 4.2, where the only non trivial terms are $D_B\ell(\boldsymbol{w})$. Investigating these terms by applying the chain rule, we see that in the case of the Smith model, we have

$$
D_B\ell(\boldsymbol{w}) = \sum_{k=1}^{d} w_k \, D_B\Phi_{d-1,R_k}\left(\delta_{jk}^{-1} + \frac{\delta_{jk}}{2}\log\left(\frac{w_k}{w_j}\right) \; j \in \mathcal{J}_{-k}\right)
$$
$$
+ \sum_{k \in B} D_{B\setminus\{k\}}\Phi_{d-1,R_k}\left(\delta_{jk}^{-1} + \frac{\delta_{jk}}{2}\log\left(\frac{w_k}{w_j}\right) \; j \in \mathcal{J}_{-k}\right),
$$

and in the case of the multivariate t, we have

$$
D_B\ell(\boldsymbol{w}) = \sum_{k=1}^{d} w_k \, D_B T_{d-1,\upsilon+1,R_k}\left(\sqrt{\frac{\upsilon+1}{1-\rho_{jk}^2}}\left[(w_j/w_k)^{-\frac{1}{\upsilon}} - \rho_{jk}\right], \; j \in \mathcal{J}_{-k}\right)
$$
$$
+ \sum_{k \in B} D_{B\setminus\{k\}}T_{d-1,\upsilon+1,R_k}\left(\sqrt{\frac{\upsilon+1}{1-\rho_{jk}^2}}\left[(w_j/w_k)^{-\frac{1}{\upsilon}} - \rho_{jk}\right], \; j \in \mathcal{J}_{-k}\right).
$$

The above expressions are fairly non-trivial to compute, but highly similar in structure, as they were based on the homogeneous form of the stable tail dependence function. What we conclude from the above is that this form is interesting to investigate EVCs arising from certain models, as you can leverage knowledge of conditional distributions, but they are not useful in the context of our work, that is, the application of Theorem 4.2.

6.2 General copulas

As we will need to access the partial derivatives of the ℓ_L and ℓ_U functions generated by a copula C, we start by investigating the limiting behaviour of $\lim_{q\downarrow 0}(1 - C(\mathbf{1} - q\boldsymbol{w}))/q$ before performing differentiation.

Remark 6.3
In the case where one has knowledge about the initial copula C, and remembering that we are working under the Uniform Convergence Condition, it is tempting to say that

$$
D_B\ell_U(\boldsymbol{w}) = D_B\left(\lim_{q\downarrow 0}\frac{1 - C(\mathbf{1} - q\boldsymbol{w})}{q}\right) = -\lim_{q\downarrow 0}\frac{1}{q}D_B C(\mathbf{1} - q\boldsymbol{w}), \tag{39}
$$

but the reader must be warned. Interchanging the limit and the differential operator early in the process will result in a loss of information, as the impact of $1/q$ as $q \downarrow 0$ is drastic. We must take the limit first, and then perform differentiation.

Taking Remark 6.3 into consideration, we realize that a combination of L'Hospital's rule and the multivariate chain rule could provide us with a better expression, that is,

$$D_B\left(\lim_{q\downarrow 0}\frac{1-C(\mathbf{1}-q\boldsymbol{w})}{q}\right) \overset{H}{=} D_B\left(-\lim_{q\downarrow 0}\frac{\partial C(\mathbf{1}-q\boldsymbol{w})}{\partial q}\right) = D_B \lim_{q\downarrow 0}\nabla C|_{\mathbf{1}-q\boldsymbol{w}}^{\top} \cdot \boldsymbol{w},$$

where $\nabla C|_{\mathbf{1}-q\boldsymbol{w}}$ is the gradient of C evaluated at $\mathbf{1} - q\boldsymbol{w}$. However, the latter is neither more simple nor more explicit than our starting point, and in fact, we recognize the above expression to be the homegenous form of the stable dependence function! Basically, we have recovered the expression derived in Joe et al. (2008), but with much less work.

Now, one has to realize that we are looking at properties in the far tails of the copula, which links directly to the concept of regular variation we introduced in Section 2. Therefore, we look into classes of copulas that can exhibit nice regular variation properties, such as the class of *Archimedean copulas*. It is to be noted that from this section onwards, we will use $\ell := \ell_U$ to shorten the notation.

6.3 Archimedean copulas

A specific class of copulas that is of high interest is the class of Archimedean copulas. They are interesting to us as the family contains several well-known members, see Table 1, and its explicit construction makes it convenient to work with. In order to present Archimedean copulas, we use the definition provided in McNeil and Nešlehová (2009).

Definition 6.4
A non-increasing and continuous function $\psi : [0, \infty] \to [0, 1]$ such that $\psi(0) = 1$, $\psi(\infty) = \lim_{x\to\infty}\psi(x) = 0$ and ψ is strictly decreasing on $[0, \inf\{x : \psi(x) = 0\})$ is called an *Archimedean generator* with inverse $\psi^{-1} : (0, 1] \to [0, \infty)$ where $\psi^{-1}(0) = \inf\{x : \psi(x) = 0\}$ by convention. A (d-dimensional) copula C is called *Archimedean* if it permits the representation

$$C(\boldsymbol{u}) = \psi(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d)) = \psi(t(\boldsymbol{u})), \quad \boldsymbol{u} \in [0, 1]^d, \tag{40}$$

where

$$t(\boldsymbol{u}) = \sum_{j=1}^{d}\psi^{-1}(u_j).$$

A generator is said to be *completely monotone* if $(-1)^k\psi^{(k)}(t) \geq 0$ for all $k \in \mathbb{N}_0$, $t \in (0, \infty)$, and we denote the set of all completely monotone Archimedean generators by Ψ_∞.

Remark 6.5

Many authors define Archimedean copulas in terms of $\phi = \psi^{-1}$, and refer to ϕ as the generator. Although it appears to be a simple matter of convention, Definition 6.4 turns out to be more natural for studying the structure of Archimedean copulas.

Family	Parameter	Generator $\psi(t)$	Inverse $\psi^{-1}(t)$
Gumbel	$\theta \in [1, \infty)$	$\exp(-t^{1/\theta})$	$(-\log(t))^\theta$
Clayton	$\theta \in (0, \infty)$	$(1 + t)^{-1/\theta}$	$t^{-\theta} - 1$
AMH	$\theta \in [0, 1)$	$(1 - \theta)(e^t - \theta)^{-1}$	$\log(1 - \theta(1 - t)) - \log(t)$
Joe	$\theta \in [1, \infty)$	$1 - (1 - e^{-t})^{1/\theta}$	$-\log(1 - (1 - t)^\theta)$
Frank	$\theta \in (0, \infty)$	$-\log(1 - (1 - e^{-\theta})e^{-t})/\theta$	$-\log(1 - e^{-\theta t}) + \log(1 - e^{-\theta})$

Table 1 Well-known one-parameter Archimedean generators ψ with corresponding inverse generators ψ^{-1}.

6.3.1 Foundations for our work

As previously mentioned, we will be interested in Archimedean copulas satisfying certain regular variation properties. This has been discussed in Genest and Rivest (1989), on which we base all the work performed further in this thesis. It is to be noted that from now on, the notation $\psi^{-1}(1 - t) \in RV_\alpha$ is to be interpreted as ψ^{-1} is regularly varying at 1 with tail index α. The next proposition is at the core of the work presented by Genest and Rivest (1989).

Proposition 6.6

Let $C = \psi(t(\boldsymbol{u}))$ be an Archimedean copula where the inverse generator $\psi^{-1}(1-t) \in RV_\alpha$ for $\alpha > 1$. Then, the stable tail dependence function ℓ arising from C is given by

$$\ell(\boldsymbol{w}) = (\sum_{j=1}^{d} w_j^\alpha)^{1/\alpha}.$$

The above can be understood as such; distributions whose copula is an Archimedean copula with Representation (40) and inverse generator satisfying $\psi^{-1}(1 - t) \in RV_\alpha$ for $\alpha > 1$ belongs to the MDA of the Gumbel distribution.

In order to better understand Proposition 6.6, we present a proposition commonly used in the literature, see Larsson and Nešlehová (2011). We prove this proposition thoroughly on our own, as it will be a key for our further developments in Section 7.

Proposition 6.7

Let ψ be an Archimedean generator. Assume that $\psi^{-1}(1 - q) \in RV_\alpha$ for $\alpha > 1$, that is

$$\lim_{q \downarrow 0} \frac{\psi^{-1}(1 - qw)}{\psi^{-1}(1 - q)} = w^\alpha.$$

Then it follows that

$$\lim_{q\downarrow 0} \frac{1 - \psi(qw)}{1 - \psi(q)} = w^{1/\alpha}.$$

Proof. The inverse of $f(w) = w^\alpha$ is $f^{-1}(w) = w^{1/\alpha}$ and both functions are continuous and monotone. Now, we define the function $f_q(w) = \frac{\psi^{-1}(1-qw)}{\psi^{-1}(1-q)}$. By the properties of Archimedean generators, we know that $f_q(w)$ is continuous in both q and w, and monotone in w. By construction, it follows that

$$\lim_{q\downarrow 0} f_q(w) = f(w) = w^\alpha.$$

An application of Theorem 1.5.12 in Bingham et al. (1989) implies that

$$\lim_{q\downarrow 0} f_q^{-1}(w) = f^{-1}(w) = w^{1/\alpha}.$$

The inverse of $f_q^{-1}(w)$ can be obtained via,

$$w = \frac{\psi^{-1}(1 - qf_q^{-1}(w))}{\psi^{-1}(1-q)} \quad \Leftrightarrow \quad \psi(w(\psi^{-1}(1-q))) = 1 - qf_q^{-1}(w)$$

$$\Leftrightarrow \quad f_q^{-1}(w) = \frac{1 - \psi(w(\psi^{-1}(1-q)))}{q}.$$

We now have that

$$w^{1/\alpha} = \lim_{q\downarrow 0} f_q^{-1}(w) = \lim_{q\downarrow 0} \frac{1 - \psi(w(\psi^{-1}(1-q)))}{q} = \lim_{q\uparrow 1} \frac{1 - \psi(w(\psi^{-1}(q)))}{1 - q},$$

and since $q \uparrow 1$ behaves similarly as $\psi(q)$ when $q \downarrow 0$, we let $q := \psi(q)$, to finally obtain

$$w^{1/\alpha} = \lim_{q\downarrow 0} \frac{1 - \psi(qw)}{1 - \psi(q)},$$

which proves the claim we were interested in. $\qquad\square$

To investigate the tail index α of an inverse generator ψ^{-1} that is regularly varying at one, we suggest applying the definition of regular variation as presented in Remark 2.21 and subsequently using l'Hospital's rule with respect to q. For example, model 4.2.14 in Nelsen (2006) has $\psi^{-1}(t) = (t^{-1/\theta} - 1)^\theta$ for $\theta \in [1, \infty)$ and yields

$$\lim_{q\downarrow 0} \frac{\psi^{-1}(1 - qw)}{\psi^{-1}(1 - q)} = \lim_{q\downarrow 0} \frac{((1-qw)^{-1/\theta} - 1)^\theta}{((1-q)^{-1/\theta} - 1)^\theta} = \left(\lim_{q\downarrow 0} \frac{(1-qw)^{-1/\theta} - 1}{(1-q)^{-1/\theta} - 1} \right)^\theta \overset{H}{=} w^\theta,$$

so $\alpha = \theta \in [1, \infty)$, which confirms that this copula is applicable in the framework of Genest and Rivest (1989).

Note that one can construct and sample new Archimedean copulas by considering certain generator transforms to construct new generators $\tilde{\psi}$ from generators $\psi \in \Psi_\infty$, see Hofert (2010). One such transformation leads to *tilted generators*, that is,

$$\tilde{\psi}(t) = \frac{\psi(t+h)}{\psi(h)}, \quad h \geq 0,$$

see Hofert (2010) p.104. Another transformation, which is of high interest to us, leads to *tilted outer power generators*, that is,

$$\tilde{\psi}(t) = \psi\{(c^\theta + t)^{1/\theta} - c\}, \tag{41}$$

for $c > 0$, $\theta \in [1, \infty)$, see Hofert (2010) p.111. The latter transformation yields a proper generator because the function $(c^\theta + t)^{1/\theta} - c$ is non-negative and has a completely monotone derivative. The composition of two completely monotone functions is itself monotone, so $\tilde{\psi} \in \Psi_\infty$. In the case where $c = 0$, this transformation then yields the commonly called *outer power* transformation. From (41), one can easily obtain that

$$\tilde{\psi}^{-1}(t) = (\psi^{-1}(t) + c)^\theta - c^\theta, \quad t \in (0, 1].$$

The latter structure is interesting to us, because when $c = 0$, we have that

$$\tilde{\psi}^{-1}(t) = (\psi^{-1}(t))^\theta, \quad t \in (0, 1],$$

which means that if $\psi^{-1}(1 - t) \in RV_\alpha$, it follows that for transformation (41) with $c = 0$,

$$\lim_{q \downarrow 0} \frac{\tilde{\psi}^{-1}(1 - qw)}{\tilde{\psi}^{-1}(1 - q)} = \left(\lim_{q \downarrow 0} \frac{\psi^{-1}(1 - qw)}{\psi^{-1}(1 - q)} \right)^\theta = w^{\alpha\theta},$$

which implies that $\tilde{\psi}^{-1}$ is regularly varying at one with tail index $\tilde{\alpha} = \alpha\theta$, that is, $\tilde{\psi}^{-1}(1 - t) \in RV_{\tilde{\alpha}}$. In order to be able to apply the result of Genest and Rivest (1989) to the outer power generator $\tilde{\psi}$, we simply need to have $\alpha > 1/\theta$ to ensure that $\tilde{\alpha} > 1$.

This is particularly interesting as many generators from Table 4.1 of Nelsen (2006) (see p.116) have inverse generators regularly varying at one with tail index exactly equal to one. The outer power transform discussed above allows us to obtain a higher tail index for these generators, making them applicable in the framework of Genest and Rivest (1989).

By combining (8) and Theorem 2.18 from Section 2, we have shown that we can represent the stable tail dependence function as

$$\lim_{q \downarrow 0} \frac{1}{q}(1 - C(\mathbf{1} - q\boldsymbol{w})) = \ell(\boldsymbol{w}),$$

and by combining the above with the result of Proposition 6.6, it follows that for any Archimedean copula C with inverse generator $\psi^{-1}(1 - t) \in RV_\alpha$ for $\alpha > 1$, we have that

$$\lim_{q \downarrow 0} \frac{1}{q}(1 - C(\mathbf{1} - q\boldsymbol{w})) = \ell(\boldsymbol{w}) = (\sum_{j=1}^{d} w_j^\alpha)^{1/\alpha}, \tag{42}$$

where the right-hand side of (42) follows from Genest and Rivest (1989) and has a simple structure. Therefore, one can use an iteration technique to apply the differential operator D_B, and verify that for any set $\emptyset \neq B \subseteq \mathcal{J}$, we have

$$
\begin{aligned}
D_B \ell(\boldsymbol{w}) &= \alpha w_{j_1}^{\alpha-1} \left(\frac{\partial^{|B|-1}}{\prod_{j \in B \setminus \{j_1\}} \partial w_j} (w_1^\alpha + \ldots + w_d^\alpha)^{1/\alpha-1} \right) \frac{1}{\alpha} \\
&= \alpha^2 w_{j_1}^{\alpha-1} w_{j_2}^{\alpha-1} \left(\frac{\partial^{|B|-2}}{\prod_{j \in B \setminus \{j_1,j_2\}} \partial w_j} (w_1^\alpha + \ldots + w_d^\alpha)^{1/\alpha-2} \right) \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1 \right) \\
&= \ldots \\
&= \alpha^{|B|} (\prod_{j \in B} w_j)^{\alpha-1} (\sum_{j=1}^d w_j^\alpha)^{1/\alpha-|B|} \prod_{b=0}^{|B|-1} \left(\frac{1}{\alpha} - b \right) \\
&= \alpha^{|B|} (\prod_{j \in B} w_j)^{\alpha-1} (\sum_{j=1}^d w_j^\alpha)^{1/\alpha-|B|} (1/\alpha)_{|B|}, \qquad (43)
\end{aligned}
$$

where $(1/\alpha)_{|B|} = \prod_{b=0}^{|B|-1} (\frac{1}{\alpha} - b)$ is the *falling factorial function* evaluated at $x = 1/\alpha$. Formula (43) is very convenient to manipulate, and we notice that for every partition π of $\{1, \ldots, d\}$, it follows that

$$
\begin{aligned}
\prod_{B \in \pi} D_B \ell(\boldsymbol{w}) &= \prod_{B \in \pi} \alpha^{|B|} \prod_{B \in \pi} (\prod_{j \in B} w_j)^{\alpha-1} \prod_{B \in \pi} (\sum_{j=1}^d w_j^\alpha)^{1/\alpha-|B|} \prod_{B \in \pi} (1/\alpha)_{|B|} \\
&= \alpha^d (\prod_{j=1}^d w_j)^{\alpha-1} (\sum_{j=1}^d w_j^\alpha)^{\sum_{B \in \pi} (\frac{1}{\alpha} - |B|)} \prod_{B \in \pi} (1/\alpha)_{|B|} \\
&= \alpha^d (\prod_{j=1}^d w_j)^{\alpha-1} (\sum_{j=1}^d w_j^\alpha)^{|\pi|/\alpha-d} \prod_{B \in \pi} (1/\alpha)_{|B|}. \qquad (44)
\end{aligned}
$$

By plugging formula (44) in Theorem 4.2, we obtain a formula for the density c^* of an extreme value copula constructed from any Archimedean copula C with inverse generator ψ^{-1} being regularly varying at one with tail index $\alpha > 1$. The expression of c^* is as follows

$$
c^*(\boldsymbol{u}) = \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \sum_{m=1}^d (-1)^{d-m} \alpha^d (\prod_{j=1}^d w_j)^{\alpha-1} (\sum_{j=1}^d w_j^\alpha)^{m/\alpha-d} \sum_{\pi:|\pi|=m} \prod_{B \in \pi} (1/\alpha)_{|B|}, \qquad (45)
$$

but we may obtain a more convenient version of the above expression. First, we define

$$
a_{d,m}^\alpha(\boldsymbol{w}) = \alpha^d (\prod_{j=1}^d w_j)^{\alpha-1} (\sum_{j=1}^d w_j^\alpha)^{m/\alpha-d}.
$$

Then, we investigate the rightmost term in (45). Let $\boldsymbol{i} = (i_1, \ldots, i_d) \in \mathbb{N}_0^d$ and define $\mathcal{P}_{d,m}$ as

$$
\mathcal{P}_{d,m} = \left\{ \boldsymbol{i} \in \mathbb{N}_0^{d-m+1} : \sum_{j=1}^{d-m+1} i_j = m, \quad \sum_{j=1}^{d-m+1} j i_j = d \right\}.
$$

Thinking of i_j as the number of blocks in a given partition π with $|\pi| = m$ for which $|B| = j$, $B \in \pi$, it follows that we can write

$$\sum_{\pi : |\pi| = m} \prod_{B \in \pi} (1/\alpha)_{|B|} = \sum_{i \in \mathcal{P}_{d,m}} \prod_{j=1}^{d-m+1} (1/\alpha)_j^{i_j},$$

as by construction, $m \leq d$ immediately implies $i_j = 0$ for all $j > d - m + 1$.

In our case, the order of the summands in $\mathcal{P}_{d,m}$ is irrelevant, so identifying the *tuples* i composing $\mathcal{P}_{d,m}$ directly relates to solving a problem known in combinatorial mathematics as the *partition problem of the positive integer $d - m + 1$*; we briefly remark that in the case the order of the summands is relevant, the problem is referred to as the *composition of a positive integer d*.

For every $1 \leq m \leq d$ and possible tuple $i \in \mathbb{N}_0^{d-m+1}$ satisfying the partition of the integer $d - m + 1$ problem, there are exactly

$$\frac{\binom{d}{i_1, \dots, i_{d-m+1}}}{\prod_{j=1}^{d-m+1} j!^{i_j}} = \frac{d!}{\prod_{j=1}^{d-m+1} i_j! \, j!^{i_j}}$$

tuples having the same composing elements $i_1, \dots, i_{d-m+1}$. The reason we divide the multinomial coefficient by $\prod_{j=1}^{d-m+1} j!^{i_j}$ is to account for the fact that permutations within a given tuple should not be counted multiple times.

We provide the reader with Example 6.8 to illustrate the concept explained above, as the problem is best understood with a visual example.

Example 6.8
Consider the problem for $m = 4$. Then, solving the partition of 4 problem as presented in its full generality yields the following,

	$m = 1$	$m = 2$	$m = 2$	$m = 3$	$m = 4$
$|B_1|$	4	3	2	2	1
$|B_2|$	0	1	2	1	1
$|B_3|$	0	0	0	1	1
$|B_4|$	0	0	0	0	1

For example, consider $m = 3$. There are exactly $\binom{4}{2,1,1}/2! = 15$ ways to partition a set of size 4 into two sets of size 1 and one set of size 2. We divided by $(2!)$ because there are exactly two blocks of size one in the partition. To confirm our reasoning, we split possible allocations along the number ways to partition d objects into m non-empty subsets, which is equal to the *Stirling number of the second kind* denoted $S(d, m)$, and sum over all m.

$$\underbrace{\binom{4}{4}}_{= S(4,1)} + \underbrace{\binom{4}{3,1}}_{= S(4,2)} + \underbrace{\frac{\binom{4}{2,2}}{2!} + \frac{\binom{4}{2,1,1}}{2!}}_{= S(4,3)} + \underbrace{\frac{\binom{4}{1,1,1,1}}{4!}}_{S(4,4)} = 15,$$

This shows that our counting technique is correct.

The partition problem of $d - m + 1$ can be thought of as allocating d distinguishable balls into m indistinguishable bins, where the number of balls per bin is fixed.

So, we have shown that

$$\sum_{\pi:|\pi|=m} \prod_{B\in\pi} (1/\alpha)_{|B|} = \sum_{\mathcal{P}_{d,m}} \frac{d!}{\prod_{j=1}^{d-m+1} i_j!} \prod_{j=1}^{d-m+1} \left(\frac{(1/\alpha)_j}{j!^{i_j}}\right)^{i_j}.$$

The above formula is useful, as we recognize it to be exactly the so-called *Bell polynomial* $B_{d,m}(x_1, \ldots, x_{d-m+1})$ evaluated at $(x_1, \ldots, x_{d-m+1}) = ((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1})$, that is,

$$\sum_{\mathcal{P}_{d,m}} \frac{d!}{\prod_{j=1}^{d-m+1} i_j!} \prod_{j=1}^{d-m+1} \left(\frac{(1/\alpha)_j}{j!^{i_j}}\right)^{i_j} = B_{d,m}((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1}). \qquad (46)$$

Leveraging on the first item of Proposition 3.2 presented in Hofert and Pham (2013), we have that for all $m \in \{0, \ldots, d\}$,

$$B_{d,m}((x)_1 y^{x-1}, \ldots, (x)_{d-m+1} y^{x-(d-m+1)}) = y^{xm-d} \sum_{l=m}^{d} s(d,l)S(l,m)x^l,$$

where $s(d,l)$ is the *Stirling number of the first kind.* From this statement, we see that letting $y = 1$ and $x = 1/\alpha$ in the above yields the right-hand side of (46). With this explicit and quantifiable expression of the Bell polynomial, we are now in a position to formulate the theorem below.

Theorem 6.9
Let $C(\boldsymbol{u}) = \psi(t(\boldsymbol{u}))$ be an Archimedean copula, and assume that the inverse generator $\psi^{-1}(1-t) \in RV_\alpha$ with tail index $\alpha > 1$. Then,

$$c^*(\boldsymbol{u}) = \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \sum_{m=1}^{d} (-1)^{m-d} a_{d,m}^\alpha(-\log \boldsymbol{u}) B_{d,m}((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1}), \quad \boldsymbol{u} \in (0,1)^d,$$

$$(47)$$

describes the density of a d-dimensional extreme value copula.

Summarizing, we obtained (47), which is the density of an extreme value copula arising from an Archimedean copula that is not an extreme value copula itself, but falls in the MDA of the Gumbel. We now tackle the numerical implementation of Formula (47).

6.3.2 Numerical evaluation

For statistical inference, one aims at using the likelihood function in order to parametrically estimate the parameter vector $\boldsymbol{\theta} \subseteq \Theta \in \mathbb{R}^d$ of the considered model. For many

applications, it is more convenient to work in terms of the log-likelihood function, as it will achieve its maximum value at the same points as the likelihood function itself due to the monotonicity of the logarithm function.

As such, one often sees theoretical uses of the log-densities, but when dealing with numerical implementation, one can not always numerically evaluate the density and take the logarithm afterwards. Hence, we suggest an alternative method to access the log-density.

The first thing one wants to do is to look at the signs of the constituants of our equation. One can easily show that $\text{sign}(C(\boldsymbol{u})/\Pi(\boldsymbol{u})) = 1$. For each $j \in \{1, \ldots, d\}$, we have $u_j \in (0,1)$, and it follows that $-\log u_j > 0$. Moreover, $\alpha^d > 0$ since $\alpha > 1$, and it follows that $\text{sign } a_{d,m}^\alpha(-\log \boldsymbol{u}) = 1$.

Since the remaining Bell polynomial term involves multiple falling factorial functions evaluated at $1/\alpha$ for $\alpha > 1$, there is potential for the remaining term to be negative. Recalling the initial structure of the falling factorial function, it is clear that $\text{sign}(1/\alpha)_{|B|} = (-1)^{|B|-1}$ for $\alpha > 1$, and so for $\pi \in \Pi$ such that $|\pi| = m$, we get

$$\text{sign} \prod_{B \in \pi} (1/\alpha)_{|B|} = (-1)^{\sum_{i=1}^{m} |B_i|-1} = (-1)^{d-m}.$$

From here, it becomes obvious that

$$\text{sign } B_{d,m}((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1}) = \text{sign} \sum_{\pi:|\pi|=m} \prod_{B \in \pi} (1/\alpha)_{|B|} = (-1)^{d-m}.$$

This implies that by moving the $(-1)^{d-m}$ term as follows,

$$c^*(\boldsymbol{u}) = \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \sum_{m=1}^{d} a_{d,m}^\alpha(\boldsymbol{w}) \Big((-1)^{d-m} B_{d,m}((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1}) \Big),$$

all the terms in the above are positive, and we can access the log-density of our newly constructed density via

$$\log c^*(\boldsymbol{u}) = \log \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} + \log \Big(\sum_{m=1}^{d} a_{d,m}^\alpha(\boldsymbol{w})(-1)^{d-m} B_{d,m}((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1}) \Big).$$

Since taking the logarithm of the first term is trivial, we focus on the latter one. We start by defining

$$x_m = \log(a_{d,m}^\alpha(\boldsymbol{w})) + \log \Big((-1)^{d-m} B_{d,m}((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1}) \Big),$$

and we let $x_{\max} = \max_{1 \leq m \leq d} \{x_m\}$. Now, we note that

$$\log \Big(\sum_{m=1}^{d} a_{d,m}^\alpha(\boldsymbol{w})(-1)^{d-m} B_{d,m}((1/\alpha)_1, \ldots, (1/\alpha)_{d-m+1}) \Big) = \log \sum_{m=1}^{d} \exp(x_m)$$

$$= x_{\max} + \log \sum_{m=1}^{d} \exp(x_m - x_{\max}),$$

and since all the summands in the latter sum are in the interval (0,1], the corresponding logarithm can easily be computed.

6.3.3 Testing the code with the Gumbel-Hougaard copula

It is pretty straightforward to realize that if the copula C being passed in Theorem 6.9 is the Gumbel copula, which is defined by generator and inverse generator as presented in Table 1, then $c^*(\boldsymbol{u})$ is in fact the density of the Gumbel-Hougaard copula.

The `copula` package in `R`, which is the go-to standard in mathematical applications related to copulas, is already equipped with an algorithm for the multidimensional density of the Gumbel copula. Therefore, we use it as a benchmark to test our code by measuring the absolute error of our algorithm against the `copula` result, for various combinations of d and θ, over a sample of 200 random points in $(0, 1)^d$.
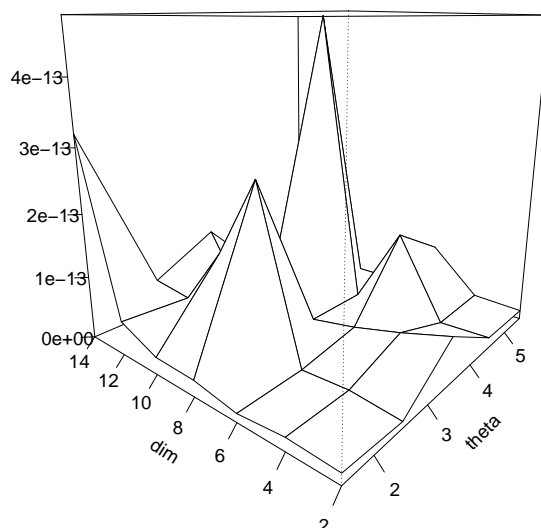
Figure 11 Absolute error of our algorithm against the `copula` result for various combinations of d and θ, over a sample of size 200.

The above values, which are of the order of `.Machine$double.eps` on our computer, confirm that our algorithm works properly in all dimensions, and therefore, we can now be confident that we obtain relevant density and log-density values for models other than the Gumbel up to family-related issues. We provide our code in Appendix B.

Remark 6.10

It is interesting to notice from Table 1 that when $\theta = 1$, the resulting copula is the independence copula, and when θ tends to infinity, the copula tends to the comonotonicity copula. This means that the Gumbel-Hougaard copula interpolates between independence and perfect positive dependence through the variation of θ.

Figure 12 EVC density constructed by passing an Archimedean copula with generator $\psi(t) = (1 + x^{1/\theta})^{-1}$ with $\theta = 2.3$ in Theorem 6.9, see 4.2.12 in Nelsen (2006), p.116

6.3.4 Special case: Galambos copula

A famous copula model is the Galambos copula, which we denote C_{Glb}, also known as the Negative Logistic copula. In fact, this copula is the survival copula of the Gumbel-Hougaard copula studied in the previous section, denoted here by C_{Gu}.

For this specific case, we do not rely on the technique we developed in the prior sections, and rather rely on a trick linking the copula to its survival copula, the Sylvester-Poincaré sieve. Indeed, using the latter sieve, one can write

$$C_{\mathrm{Glb}}(\boldsymbol{u}) = \hat{C}_{\mathrm{Gu}}(\boldsymbol{u}) = \sum_{\pi \subseteq \mathcal{J}} (-1)^{|\pi|} C_{\mathrm{Gu}}\left((1 - u_1)^{\mathbb{1}_{\pi(1)}}, \dots, (1 - u_d)^{\mathbb{1}_{\pi(d)}}\right),$$

where, for $j \in \mathcal{J}$, we define

$$\mathbb{1}_{\pi(j)} = \begin{cases} 1 & \text{if } j \in \pi, \\ 0 & \text{if } j \notin \pi. \end{cases}$$
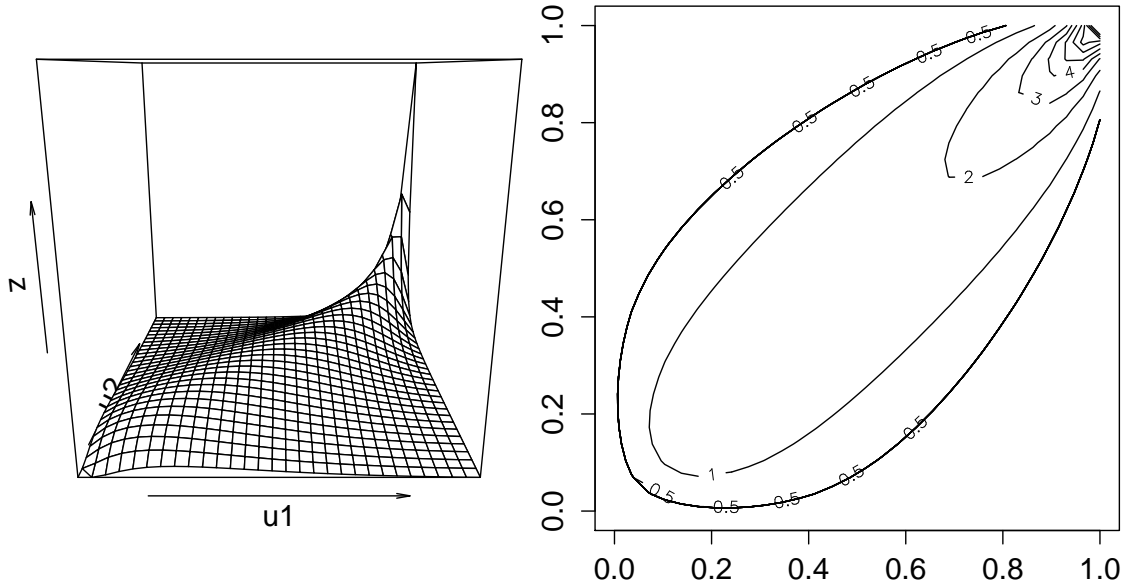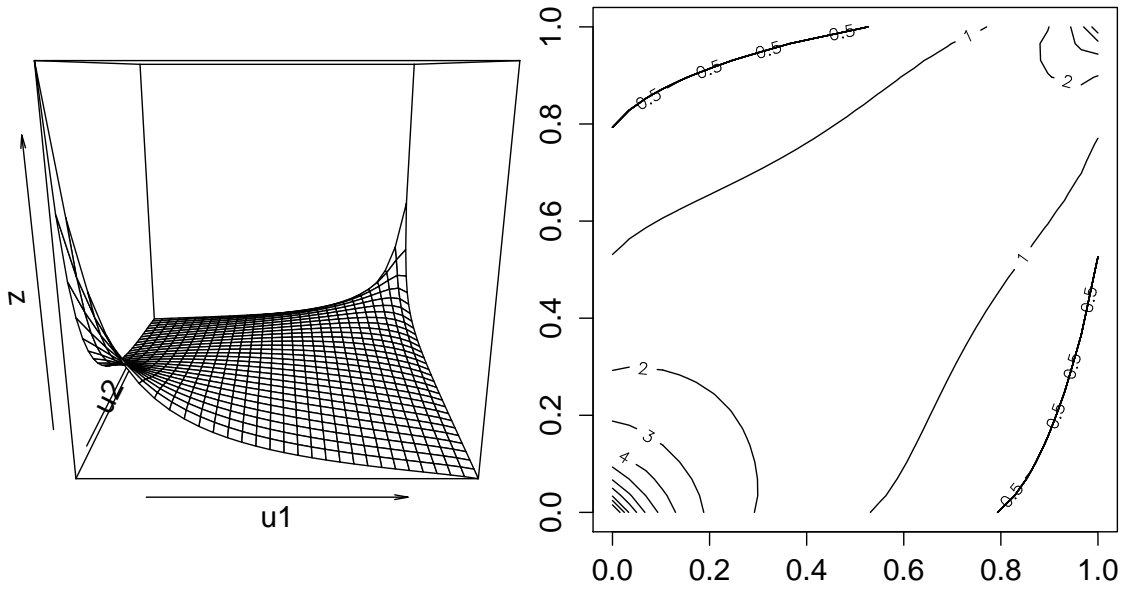
40

Figure 13 EVC density constructed by passing an Archimedean copula with generator
$\psi(t) = (1 + x^{1/\theta})^{-\theta}$ with $\theta = 7/2$ in Theorem 6.9, see 4.2.14 in Nelsen (2006),
p.116

Using the fact that differentiation is a linear operator, one gets

$$c_{\text{Glb}}(\boldsymbol{u}) = D\, C_{\text{Glb}}(\boldsymbol{u}) = \sum_{\pi \subseteq \mathcal{J}} (-1)^{|\pi|} D\, C_{Gu}\left((1-u_1)^{\mathbb{1}_{\pi(1)}}, \ldots, (1-u_d)^{\mathbb{1}_{\pi(d)}}\right),$$

and since the only summand that will not vanish under the differential operator D is
the one where $\pi = \{1, \ldots, d\}$, we get

$$
\begin{aligned}
&= (-1)^d D\, C_{\text{Gu}}\left((1-u_1), \ldots, (1-u_d)\right) \\
&= (-1)^{2d} c_{\text{Gu}}(1-u_1, \ldots, 1-u_d) \\
&= c_{\text{Gu}}(\boldsymbol{1} - \boldsymbol{u})
\end{aligned}
$$

So, the density of the Galambos copula is fully described by the density of the Gumbel-
Hougaard copula, which we can obtain from Theorem 6.9. A proper application of our
code provides the density obtained numerically.

6.3.5 The bridge to Value-at-Risk

There exists a strong link between our previous work and the concept of Value-at-Risk
(VaR), as tail dependence is one of the main drivers of the latter measure. One can find
many publications on aggregated dependent losses, we refer to Barbe et al. (2006) as an
example, since this concept is one of high importance in the financial industry.

The asymptotic analysis of VaR for aggregated dependent losses mainly consists in evaluating the integral of the *upper tail density*

$$\Gamma(\boldsymbol{w}) = D\left(\lim_{q\downarrow 0} \frac{\hat{C}(q\boldsymbol{w})}{q}\right), \quad \boldsymbol{w} \in \mathbb{R}^d,$$

of a copula C over some upper subset in $\mathbb{R}^d$. In the context of our work, we have that

$$\lim_{q\downarrow 0} \frac{1 - C(1 - q\boldsymbol{w})}{q} = \lim_{q\downarrow 0} \frac{1 - \sum_{\pi\subseteq\mathcal{J}}(-1)^{|\pi|}\hat{C}\left((qu_1)^{\mathbb{1}_{\pi(1)}}, \ldots, (qu_d)^{\mathbb{1}_{\pi(d)}}\right)}{q}$$

$$= \lim_{q\downarrow 0} \frac{\sum_{\emptyset\neq\pi\subseteq\mathcal{J}}(-1)^{|\pi|-1}\hat{C}\left((qu_1)^{\mathbb{1}_{\pi(1)}}, \ldots, (qu_d)^{\mathbb{1}_{\pi(d)}}\right)}{q}.$$

Using the above manipulations, and assuming that C is Archimedean with inverse generator $\psi^{-1}(1-t) \in RV_\alpha$ with $\alpha > 1$, we can leverage on our prior work and obtain

$$\Gamma(\boldsymbol{w}) = D\left(\lim_{q\downarrow 0} \frac{\hat{C}(q\boldsymbol{w})}{q}\right) = (-1)^{d-1}D_d\left(\lim_{q\downarrow 0} \frac{1 - C(1 - q\boldsymbol{w})}{q}\right) = (-1)^{d-1}a_{d,1}^\alpha(\boldsymbol{w})(1/\alpha)_d$$

$$= (\prod_{j=1}^d w_j)^{\alpha-1}(\sum_{j=1}^d w_j^\alpha)^{1/\alpha-d}\prod_{j=2}^d((j-1)\alpha - 1). \tag{48}$$

Now, consider a multivariate regularly varying loss vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ with upper tail density (48), joint distribution H and tail equivalent continuous margins $F_1, \ldots, F_d$. Moreover, consider any componentwise order-preserving norm $||\cdot||$ in $\mathbb{R}_+^d$. Using the multivariate regular variation property of $\boldsymbol{X}$, we then get

$$\lim_{t\to\infty} \frac{P(||\boldsymbol{X}|| > t)}{\bar{F}_1(t)} = \lim_{t\to\infty} \frac{P(\boldsymbol{X} \in tW)}{P(X_1 > t)} = \nu(W), \tag{49}$$

where $W = \{w_j \geq 0, j \in \mathcal{J} : ||\boldsymbol{w}|| > 1\}$. Li and Wu (2013) provide the following theorem linking the intensity measure ν to the regular variation of the margins F_j, the upper tail density Γ of the underlying copula, and the norm $||\cdot||$.

Theorem 6.11
Let H be a distribution with tail equivalent margins $F_1, \ldots, F_d$. If the marginal densities, assumed to exist, are regularly varying with tail index $\beta + 1$, $\beta > 0$, and the copula C has upper tail density Γ and satisfies the Uniform Convergence Condition, then H is multivariate regularly varying with tail density Γ_H such that

$$\Gamma_H(\boldsymbol{w}) = \beta^d\Gamma(w_1^{-\beta}, \ldots, w_d^{-\beta})\prod_{j=1}^d w_j^{-\beta-1}.$$

Using a theorem from de Haan and Resnick (1987), it follows that $\nu(W) = \int_W \Gamma_H(\boldsymbol{w})d\boldsymbol{w}$. Therefore, we have that

$$\nu(W) = \beta^d\int_W \Gamma(w_1^{-\beta}, \ldots, w_d^{-\beta})\prod_{j=1}^d w_j^{-\beta-1}d\boldsymbol{w}$$

$$= \beta^d \prod_{j=2}^{d} ((j-1)\alpha - 1) \int_W (\prod_{j=1}^{d} w_j)^{-\alpha\beta - 1} (\sum_{j=1}^{d} w_j^{-\alpha\beta})^{1/\alpha - d} d\boldsymbol{w}$$

Corollary 2.4 in the seminal paper by Embrechts et al. (2009) shows that we can go from (49) to the following expression

$$\lim_{p\uparrow 1} \frac{\mathrm{VaR}_p(||\boldsymbol{X}||)}{\mathrm{VaR}_p(X_1)} = \nu(W)^{1/\beta},$$

where $||\boldsymbol{X}|| = \sum_{j=1}^{d} X_j$. This means that in the context of our work, we have insights on the asymptotics of VaR for aggregated dependent losses $\boldsymbol{X}$ with Archimedean copula such that the inverse generator $\psi^{-1}(1-t) \in RV_\alpha$ with $\alpha > 1$.

In this context, it is worth mentioning that Theorem 2.5 from Embrechts et al. (2009) is applicable, and so for all $\beta > 1$, there exists a $p_0 > 0$ such that for all $p_0 < p < 1$, it holds that

$$\mathrm{VaR}_p(\sum_{j=1}^{d} X_j) < \sum_{j=1}^{d} \mathrm{VaR}_p(X_j),$$

and for all $0 < \beta < 1$, there exists a $p_1 > 0$ such that for all $p_1 < p < 1$, it holds that

$$\mathrm{VaR}_p(\sum_{j=1}^{d} X_j) > \sum_{j=1}^{d} \mathrm{VaR}_p(X_j).$$

7 Extensions to nested Archimedean copulas

Recently, there has been a lot of interest in multivariate hierarchical models, that is, models capturing different dependencies between and within different groups of random variables. For example, Puzanova (2011) introduces a hierarchical model of tail dependent asset returns for measuring portfolio credit risk, where the degrees of dependence between and within sub-portfolios are controlled through the use of nested Archimedean copulas.

The goal of this section is to leverage the results we obtained in Section 6 and extend it to the class of nested Archimedean copulas (NACs).

As the name mentions, the basic notion behing NACs is to nest Archimedean copulas. At each level of the NAC, the idea is to aggregate copulas from the previous level. A d-dimensional copula C is called a NAC if it is an Archimedean copula with arguments possibly replaced by other NACs.

As presented in Hofert (2011), if C is given recursively by (40) for $d = 2$ and, up to permutation of the arguments, for $d \geq 3$, by

$$C(u_1, \ldots, u_d; \psi_0, \ldots, \psi_{d-2}) = \psi_0(\psi_0^{-1}(u_1) + \psi_0^{-1}(C(u_2, \ldots, u_d; \psi_1, \ldots, \psi_{d-2}))), \quad (50)$$

then C is called *fully nested Archimedean copula* with $d - 1$ *nesting levels.* Otherwise, C is called *partially nested Archimedean copula.* Both fully nested Archimedean copula and partially nested Archimedean copula are summarized as NACs.

In order for (50) to be a copula, McNeil (2008) introduces the *sufficient nesting condition* stating that $\psi_i^{-1} \circ \psi_j$ is completely monotone for all nodes appearing in a NAC, where i is understood as the parent node, and j is the child node, borrowing language from graph theory.

We start by considering a partially nested Archimedean copula with 2 nesting levels and d_0 groups. The representation of such a copula is

$$C(\boldsymbol{u}) = C_0(C_1(\boldsymbol{u}_1; \psi_1), \ldots, C_{d_0}(\boldsymbol{u}_{d_0}; \psi_{d_0}); \psi_0), \quad \boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{d_0}), \qquad (51)$$

where d_0 denotes the dimension of C_0 and each copula C_s, $s \in \{0, \ldots, d_0\}$ is Archimedean with generator ψ_s. It is obvious that one can rewrite the above equation in a more explicit way, as follows,

$$C(\boldsymbol{u}) = \psi_0(\sum_{s=1}^{d_0} \psi_0^{-1}(C_s(\boldsymbol{u}_s))) = \psi_0(\sum_{s=1}^{d_0} \psi_0^{-1}(\psi_s(\sum_{l=1}^{d_s} \psi_s^{-1}(u_{sl})))).$$
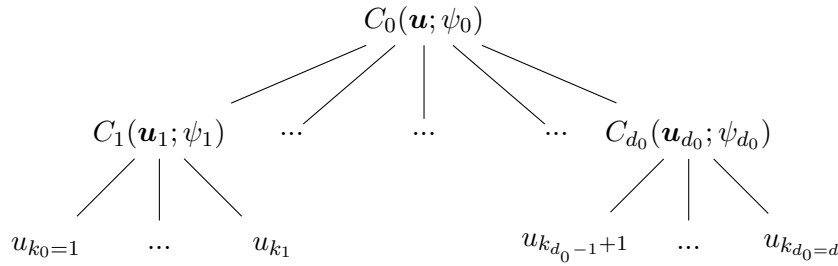
Figure 14 Tree decomposition of a d-dimensional partially nested Archimedean copula with 2 nesting levels and d_0 groups.

7.1 Derivation of the stable tail dependence function of nested Archimedean copulas

Leveraging on Proposition 6.7 and the tricks we used in its proof, we obtain the following theorem.

Theorem 7.1

Let C be a d-dimensional two-level partially nested Archimedean copula of the form of (51), and assume that for $s \in \{0, \ldots, d_0\}$, the inverse generator $\psi_s^{-1}(1-t) \in RV_{\alpha_s}$ where $\alpha_s > 1$. Then, the stable tail dependence function of the copula C is given by

$$\ell(\boldsymbol{w}) = (\sum_{s=1}^{S}(\sum_{l=1}^{d_s} w_{sl}^{\alpha_s})^{\alpha_0/\alpha_s})^{1/\alpha_0}.$$

Proof. We start by looking at the limit version of the result from Genest and Rivest (1989) stated earlier, that is, for each $s \in \{0, \ldots, d_0\}$, we have

$$\lim_{q \downarrow 0} \frac{1 - C_s(\mathbf{1} - q\boldsymbol{w}_s)}{q} = \Big(\sum_{l=1}^{d_s} w_{sl}^{\alpha_s}\Big)^{1/\alpha_s}.$$

Since $\alpha_s > 1$ and $\boldsymbol{w}_s \in \mathbb{R}_s^+$, it follows that for every $s \in \{1, \ldots, d_0\}$, there exists a small enough q_s and $\delta_s > 0$ such that for all $0 < q < q_s$, we have

$$\Big(\sum_{l=1}^{d_s}(w_{sl} - \delta_s)^{\alpha_s}\Big)^{1/\alpha_s} \leq \lim_{q \downarrow 0} \frac{1 - C_s(\mathbf{1} - q\boldsymbol{w}_s)}{q} \leq \Big(\sum_{l=1}^{d_s}(w_{sl} + \delta_s)^{\alpha_s}\Big)^{1/\alpha_s}. \tag{52}$$

Now, from the monotonicy property of Archimedean generators, it holds that for every $a < b$ such that $a, b \in (0, 1]$,

$$1 - \psi_0\Big(\sum_{s=1}^{d_0} \psi_0^{-1}(1 - a)\Big) < 1 - \psi_0\Big(\sum_{s=1}^{d_0} \psi_0^{-1}(1 - b)\Big).$$

If we let q be such that $0 < q < \min_{s=1,\ldots,d_0}\{q_s\}$, and define $\ell_q(\boldsymbol{w}) = \frac{1 - C(\mathbf{1} - q\boldsymbol{w})}{q}$, we get

$$
\begin{aligned}
\limsup_{q \downarrow 0} \ell_q(\boldsymbol{w}) &= \limsup_{q \downarrow 0} \frac{1}{q}\Big(1 - \psi_0\Big(\sum_{s=1}^{d_0} \psi_0^{-1}(C_s(\mathbf{1} - q\boldsymbol{w}_s))\Big)\Big) \\
&= \limsup_{q \downarrow 0} \frac{1}{q}\Big(1 - \psi_0\Big(\sum_{s=1}^{d_0} \psi_0^{-1}\Big(1 - \frac{1 - C_s(\mathbf{1} - q\boldsymbol{w}_s)}{q}q\Big)\Big)\Big) \\
&\leq \limsup_{q \downarrow 0} \frac{1}{q}\Big(1 - \psi_0\Big(\sum_{s=1}^{d_0} \psi_0^{-1}\big(1 - \big(\sum_{l=1}^{d_s}(w_{sl} + \delta_s)^{\alpha_s}\big)^{1/\alpha_s}q\big)\Big)\Big) \\
&= \limsup_{q \downarrow 0} \frac{1}{q}\left(1 - \psi_0\left(\sum_{s=1}^{d_0} \frac{\psi_0^{-1}\Big(1 - (\sum_{l=1}^{d_s}(w_{sl} + \delta_s)^{\alpha_s})^{1/\alpha_s}q\Big)}{\psi_0^{-1}(1 - q)}\right)\psi_0^{-1}(1 - q)\right).
\end{aligned}
$$

Relying on the fact that $\psi_0^{-1}(1 - t) \in RV_{\alpha_0}$ for $\alpha_0 > 1$, there exists a q_0 and $\delta_0 > 0$ such that for all $0 < q < q_0$,

$$(x - \delta_0)^{\alpha_0} \leq \frac{\psi_0^{-1}(1 - xq)}{\psi_0^{-1}(1 - q)} \leq (x + \delta_0)^{\alpha_0}.$$

If $q_0 > \min_{s=1,\ldots,d_0}\{q_s\}$, we set $q_0 = \min_{s=1,\ldots,d_0}\{q_s\}$ Then, it follows that

$$\limsup_{q \downarrow 0} \ell_q(\boldsymbol{w}) \leq \limsup_{q \downarrow 0} \frac{1}{q}\Big(1 - \psi_0\Big(\sum_{s=1}^{d_0}\big(((\sum_{l=1}^{d_s}(w_{sl} + \delta_s)^{\alpha_s})^{1/\alpha_s} + \delta_0)^{\alpha_0}q\big)\Big)\psi_0^{-1}(1 - q)\Big).$$

Substituting $q := 1 - \psi_0(q)$ in the above equation, we get

$$\limsup_{q \downarrow 0} \ell_q(\boldsymbol{w}) \leq \limsup_{q \downarrow 0} \frac{1 - \psi_0\left(\sum_{s=1}^{d_0}((\sum_{l=1}^{d_s}(w_{sl} + \delta_s)^{\alpha_s})^{1/\alpha_s} + \delta_0)^{\alpha_0}q\right)}{1 - \psi_0(q)}.$$

Relying on the use of Proposition 6.7, we get that

$$\limsup_{q \downarrow 0} \ell_q(\boldsymbol{w}) \leq (\sum_{s=1}^{d_0}((\sum_{l=1}^{d_s}(w_{sl} + \delta_s)^{\alpha_s})^{1/\alpha_s} + \delta_0)^{\alpha_0})^{1/\alpha_0}.$$

Using similar arguments, we are able to obtain a lower bound for the lim inf, so we finally obtain

$$(\sum_{s=1}^{d_0}((\sum_{l=1}^{d_s}(w_{sl}-\delta_s)^{\alpha_s})^{1/\alpha_s}-\delta_0)^{\alpha_0})^{1/\alpha_0} \leq \lim_{q \downarrow 0} \ell_q(\boldsymbol{w}) \leq (\sum_{s=1}^{d_0}((\sum_{l=1}^{d_s}(w_{sl}+\delta_s)^{\alpha_s})^{1/\alpha_s}+\delta_0)^{\alpha_0})^{1/\alpha_0}.$$

Since δ_s is arbitrary for all $s \in \{0, \ldots, d_0\}$, an application of the Squeeze Lemma finalizes the proof of the claim. $\qquad\square$

Theorem 7.1 is only applicable to partially nested Archimedean copulas with two nesting levels, but we realize we can leverage knowledge from the steps performed in its proof to extend the scope of our results to any NAC structure by creating a recursive expression for the stable tail dependence function representation.

Borrowing concepts from the field of graph theory, we can think of a NAC with L nesting levels as a *rooted tree* with *depth L*, where it is understood that the root copula C_0 has depth 0. We want to point out that in R, the root copula has depth equal to one.

Let us define N_l to be the number of copulas present at depth $l \in \{0, \ldots, L-1\}$, and M_l to be the number of final nodes, that is, a node of the form u_j which is a component of $\boldsymbol{u}$. Then, in the spirit of Theorem 7.1, we start at depth $L-1$, where for each copula $C_{L-1,L-1_j}$ such that $j \in \{1, \ldots, N_{L-1}\}$, we can find a $q_{L-1,L-1_j}$ and $\epsilon_{L-1,L-1_j} > 0$ such that by selecting $q_{L-1} = \min_{1 \leq j \leq N_{L-1}}\{q_{L-1,L-1_j}\}$, we can apply a step as in (52). From there, we repeatedly apply steps as in our proof of Theorem 7.1 working our way updward towards the root, building a sequence $(q_l)_{l=0}^{L-1}$ and a sequence of sequences $((\delta_{l,l_j})_{j=1}^{N_l})_{l=0}^{L-1}$. Then by selecting $q = \min_{0 \leq l \leq L-1}\{q_l\}$, we can find the lim inf and lim sup bounds and proceed by applying the Squeeze Lemma to obtain

$$\ell(\boldsymbol{w}) = (\sum_{j=1}^{N_1} \ell_{1,j}^{\alpha_0} + \sum_{m=1}^{M_1} w_{1i_m}^{\alpha_0})^{1/\alpha_0}, \tag{53}$$

where $\ell_{1,j}$ is the stable tail dependence function stemming from the tree rooted at $C_{1,j}$ and w_{1i_m} are the final nodes of the first level. This notation might not be the prettiest nor the most convenient, but it provides us with a recursive tool to express the stable tail dependence function of any NAC.

Example 7.2

In order to understand (53), we provide an example. Consider a nine-dimensional partially nested copula C (see Figure 15 for the tree representation) of the form

$$C(\boldsymbol{u}) = C_0(u_1, u_2, u_3, C_1(u_4, u_5, u_6, u_7, C_2(u_8, u_9; \psi_2); \psi_1); \psi_0), \quad \boldsymbol{u} \in (0,1)^9,$$

and assume that the inverse generators $\psi_s^{-1}(1-t) \in RV_{\alpha_s}$ for $\alpha_s > 1$ where $s \in \{0,1,2\}$. Then, using the recursive expression (53), we get that the stable tail dependence function of C is

$$\ell(\boldsymbol{w}) = \ell_0(w_1, w_2, w_3, \ell_1(w_4, w_5, w_6, w_7, \ell_2(w_8, w_9))),$$

or in its fully explicit form,

$$\ell(\boldsymbol{w}) = (w_1^{\alpha_0} + w_2^{\alpha_0} + w_3^{\alpha_0} + (w_4^{\alpha_1} + w_5^{\alpha_1} + w_6^{\alpha_1} + w_7^{\alpha_1} + (w_8^{\alpha_2} + w_9^{\alpha_2})^{\alpha_1/\alpha_2})^{\alpha_0/\alpha_1})^{1/\alpha_0}.$$
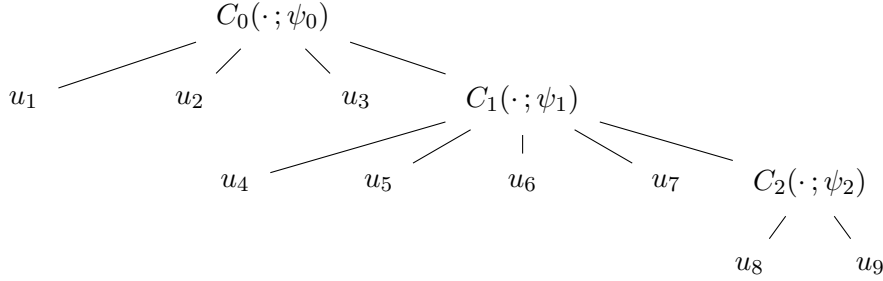
Figure 15 Tree structure of the nine-dimensional partially nested Archimedean copula C described in Example 7.2.

7.2 Extreme value copula densities constructed from partially nested Archimedean copulas

Note that the stable tail dependence function obtained in Theorem 7.1, by construction, the one of a nested Gumbel copula with d_0 groups and two nesting levels where $\theta_s = \alpha_s$. Therefore, an application of Theorem 2.12 yields

$$C(\boldsymbol{u}) = \exp(-\ell(-\log \boldsymbol{u})) = \exp((\sum_{s=1}^{d_0}(\sum_{l=1}^{d_s} -\log u_{sl}^{\alpha_s})^{\alpha_0/\alpha_s})^{1/\alpha_0}),$$

where $C(\boldsymbol{u})$ is as (51). In item 2 of Remark 4.1 in Hofert and Pham (2013), the authors show that the density of a nested Gumbel copula with d_0 groups and two nesting levels can be expressed as

$$c(\boldsymbol{u}) = \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \left((-1)^d \sum_{k=d_0}^{d} b_{\boldsymbol{d},k}^{d_0}(\boldsymbol{t}(\boldsymbol{u}))(\sum_{j=1}^{k}(-t(\boldsymbol{u})^{1/\theta_0})^j s_{kj}(1/\theta_0)) \right) \prod_{s=1}^{d_0} \theta_s^{d_s}(\prod_{j=1}^{d_s} -\log u_{sj})^{\theta_s - 1},$$

where $\mathring{\psi}_{0s} = \psi_0^{-1} \circ \psi_s$ and

$$t(\boldsymbol{u}) = \psi_0^{-1}(C(\boldsymbol{u})),$$
$$\boldsymbol{t}(\boldsymbol{u}) = (t_1(\boldsymbol{u}_1), \dots, t_{d_0}(\boldsymbol{u}_{d_0})),$$
$$a_{s,nk}(t) = B_{n,k}(\mathring{\psi}_{0s}'(t), \dots, \mathring{\psi}_{0s}^{(n-k+1)}(t)),$$
$$b_{\boldsymbol{d},k}^{d_0} = \sum_{j \in \mathcal{Q}_{\boldsymbol{d},k}^{d_0}} \prod_{s=1}^{d_0} a_{s,d_s j_s}(t_s(\boldsymbol{u}_s)),$$

with $\boldsymbol{d} = (d_1, \dots, d_{d_0})$ and

$$\mathcal{Q}_{\boldsymbol{d},k}^{d_0} = \left\{ \boldsymbol{j} \in \mathbb{N}^{d_0} : \sum_{s=1}^{d_0} j_s = k, j_s \leq d_s, \ s \in \{1, \dots, d_0\} \right\}.$$

Equating their expression with the one we obtained from Theorem 4.2, it follows that we must have

$$\sum_{m=1}^{d} (-1)^{d-m} \sum_{\pi: |\pi|=m} \prod_{B \in \pi} D_B \ell(\boldsymbol{w}) =$$
$$\left((-1)^d \sum_{k=d_0}^{d} b_{\boldsymbol{d},k}^{d_0}(\boldsymbol{t}(\boldsymbol{u}))(\sum_{j=1}^{k}(-t(\boldsymbol{u})^{1/\theta_0})^j s_{kj}(1/\theta_0)) \right) \prod_{s=1}^{d_0} \theta_s^{d_s}(\prod_{j=1}^{d_s} w_{sj})^{\theta_s-1}$$

whenever the function ℓ is given by $\ell(\boldsymbol{w}) = (\sum_{s=1}^{d_0}(\sum_{l=1}^{d_s} w_{sl}^{\theta_s})^{\theta_0/\theta_s})^{1/\theta_0}$. So, leveraging on the work of Hofert and Pham (2013), we are able to formulate the following result.

Theorem 7.3

Let C be a d-dimensional partially nested Archimedean copula with two nesting levels and d_0 groups of the form of (51), and assume that for $s \in \{0, \dots, d_0\}$, the inverse generator $\psi_s^{-1}(1-t) \in RV_{\alpha_s}$ where $\alpha_s > 1$ for all s. Then,

$$c^*(\boldsymbol{u}) = \frac{C(\boldsymbol{u})}{\Pi(\boldsymbol{u})} \left((-1)^d \sum_{k=d_0}^{d} b_{\boldsymbol{d},k}^{d_0}(\boldsymbol{t}(\boldsymbol{u}))(\sum_{j=1}^{k}(-t(\boldsymbol{u})^{1/\alpha_0})^j s_{kj}(1/\alpha_0)) \right)$$
$$\cdot \prod_{s=1}^{d_0} \alpha_s^{d_s}(\prod_{j=1}^{d_s} -\log u_{sj})^{\alpha_s-1}$$

where $C(\boldsymbol{u})$ is (51), describes the density of a d-dimensional extreme value copula.

8 Copulas of extreme-value distributions

For this section, we extend the work of Mendes and Sanfins (2007) to the multivariate case by heavily relying on the so-called Extremal Types Theorem, which can be found

in Embrechts et al. (1997). For any fixed $N \geq 2$ and any $n \geq N$, let $M_{1,n}, \ldots M_{N,n}$ be the N largest order statistics of an iid sample of size n for which $X_i \sim F$ is independent of n. If for sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, the random variables $a_n M_{1,n} + b_n$ converge in distribution, then the random vector

$$(a_n M_{1,n} + b_n, \ldots, a_n M_{N,n} + b_n) \tag{54}$$

also converges in distribution, where the limit distribution is parametrized by (ξ, μ, σ) with $\mu, \xi \in (-\infty, \infty)$ and $\sigma > 0$. For any $j \geq 1$, it follows that the j-th marginal distribution is given by

$$F_j(x) = \begin{cases} 0 & , \quad x < \mu - \sigma/\xi, \ \xi > 0, \\ \exp(-\Lambda(x)) \sum_{k=0}^{j-1} \frac{\Lambda(x)^k}{k!}, & \xi\left(\frac{x-\mu}{\sigma}\right) > -1 \text{ for } \xi \neq 0 \text{ or } x \in \mathbb{R} \text{ for } \xi = 0, \\ 1 & , \quad x > \mu - \sigma/\xi, \ \xi < 0, \end{cases} \tag{55}$$

where

$$\Lambda(x) = \begin{cases} (1 + \xi\left(\frac{x-\mu}{\sigma}\right))^{-1/\xi}, & \xi \neq 0, \\ \exp(-\frac{x-\mu}{\sigma}) & , \quad \xi = 0. \end{cases}$$

Distributions $F_j(x)$ as above are known as *Generalized Extreme Value distributions* (GEV).

We spare the details of our calculations to the reader, but for any $j \geq 1$, $F_j(x)$ can easily be differentiated through the use of the product rule, and then simplified using a telescoping sum argument. Doing so, we obtain its density $f_j(x)$, with explicit representation

$$f_j(x) = \begin{cases} -\exp(-\Lambda(x))\Lambda'(x)\frac{\Lambda(x)^{j-1}}{(j-1)!}, & \xi\left(\frac{x-\mu}{\sigma}\right) - 1 \text{ for } \xi \neq 0 \text{ or } x \in \mathbb{R} \text{ for } \xi = 0, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Previous work by Smith (1986) shows from the Extremal Types Theorem that the joint density h_N of a limiting extreme value distribution for normalized sums of the N largest order statistics of a sequence as in (54) is given by

$$h_N(x_1, \ldots, x_N) = \begin{cases} (-1)^N \exp(-\Lambda(x_N)) \prod_{i=1}^{N} \Lambda'(x_i), & (x_1, \ldots, x_N) \in \Omega_{\mu, \sigma, \xi} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

where the set $\Omega_{\mu, \sigma, \xi}$ is defined as

$$\Omega_{\mu, \sigma, \xi} = \begin{cases} \{(x_1, \ldots, x_N) \in \mathbb{R}^N \ x_1 > \ldots > x_N > \mu - \sigma/\xi\}, & \xi > 0, \\ \mathbb{R}^N & , \quad \xi = 0, \\ \{(x_1, \ldots, x_N) \in \mathbb{R}^N \ \mu - \sigma/\xi > x_1 > \ldots > x_N\}, & \xi < 0. \end{cases}$$

A distribution with density h_N as above is referred to as a *Multivariate Generalized Extreme Value distribution* (MGEV).
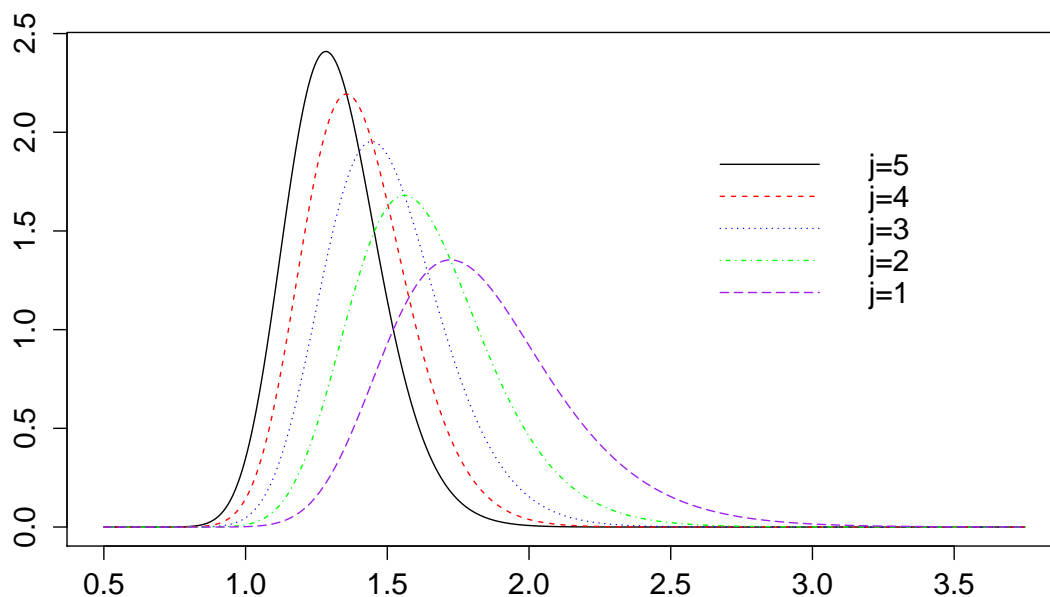
Figure 16 Densities of the first five largest order-statistics, that is, $j = 1, \ldots, 5$ with parametrization $\mu = 2$, $\sigma = 0.4$ and $\xi = 0$, without loss of generality

8.1 Copula densities arising from Multivariate Generalized Extreme Value distributions

As we just saw, the Extremal Types Theorem provides us with plenty of information on the joint distribution and marginal distributions of the model, along with their respective densities. By applying the d-dimensional differential operator to (2), we obtain a general formula copula densities via

$$c(\boldsymbol{u}) = \frac{h(F_1^-(u_1), \ldots, F_d^-(u_d))}{f_1(F_1^-(u_1)) \ldots f_d(F_d^-(u_d))}, \quad \boldsymbol{u} \in \prod_{j=1}^d \operatorname{ran} F_j. \tag{56}$$

Inserting the expressions of h_N and f_j, $j \in \{1, \ldots, N\}$ in (56), we obtain a general formula for the density of a copula corresponding to a multivariate extreme value distribution. After algebraic manipulations and various simplifications, we get

$$c(u_1, \ldots, u_N) = \frac{(N-1)!}{\Lambda(F_N^-(u_N))^{N-1}} \prod_{j=1}^{N-1} \frac{(j-1)!}{\exp\{-\Lambda(F_j^-(u_j))\} \Lambda(F_j^-(u_j))^{j-1}} \tag{57}$$

Now, the only non-explicit components in (57) are the inverses of the marginal distribution functions. The structure of the functions $F_j(x)$ for $j \in \{1, \ldots, N\}$ does not allow us to describe their respective inverses explicitly in all generality. Therefore, we investigate a way to obtain these inverses implictly.

Note that $F_j(x_j) = F_{\text{Poi}(\Lambda(x_j))}(j-1)$ for a fixed x_j, where $F_{\text{Poi}(\lambda)}(j)$ describes the distribution function of the $\text{Poi}(\lambda)$ evaluated at j. It is to be noted that $\Lambda(x_j) > 0$, so our application of the Poisson distribution is valid. Luckily for us, this quantile function is already implemented in R as `qpois` and we can use it to evaluate F_j^-, $j \in \{1, \ldots, N\}$, which we need as input for (57).

8.2 A side note for $j = 1$ and $j = 2$

In this section, we consider two specific cases, that is, $j = 1$ and $j = 2$, and we focus on the terms in (57) containing F_1^- and F_2^-. The first inverse is in fact a proper inverse, so we write, F_1^{-1}, and we can compute it to be

$$F_1^{-1}(y) = \begin{cases} \frac{\sigma}{\xi}[(-\log y)^{-\xi} - 1] + \mu, & \xi \neq 0, y \in (0,1], \\ -\sigma \log(-\log y) + \mu & , & \xi = 0, y \in (0,1]. \end{cases}$$

Using the expression of $F_2(x)$ as given in (55), it follows that

$$u = F_2 \circ F_2^-(u) = \exp(-\Lambda(F_2^-(u)))(1 + \Lambda(F_2^-(u))), \quad u \in (0,1),$$

and by defining $\Upsilon_2(u) = \exp(-\Lambda(F_2^-(u)))$, the above can be rewritten as

$$u = \Upsilon_2(u)(1 - \log \Upsilon_2(u)), \quad u \in (0,1). \tag{58}$$

Because of the presence of the logarithms, which are intricately connected to exponential functions, we leverage on the structure of (58) to obtain a nice solution for $\Upsilon_2(u)$, that is,

$$u = \Upsilon_2(u)(1 - \log \Upsilon_2(u)), \quad \Leftrightarrow \quad -\frac{u}{\Upsilon_2(u)} + 1 = \log \Upsilon_2(u),$$
$$\Leftrightarrow \quad -\frac{u}{\Upsilon_2(u)} \exp(-u/\Upsilon_2(u)) = -u/e,$$
$$\Leftrightarrow \quad -u/\Upsilon_2(u) = W(-u/e),$$
$$\Leftrightarrow \quad \Upsilon_2(u) = -u/W(-u/e).$$

where W is the so-called *Lambert W function*. The Lambert W function defines the set of branches of the inverse relation of the function $f(w) = w \exp(w)$ where w is any complex number. In other words, the defining equation for $W(z)$ is

$$z = W(z)e^{W(z)}, \quad z \in \mathbb{C}.$$

The history of the Lambert W function is, in our opinion, one of great interest, and we invite the reader to look into it, as we will not cover it in the thesis. As the Lambert W function is part of the `gsl` package in R, the obtained expression for $\Upsilon_2(u)$ can be implemented as

```
## Implementation of Upsilon_2(u)
upsilon_2 <- function(x) -x/(lambert_W0(-x/exp(1)))
```

We can now insert the obtained expressions for $j = 1$ and $j = 2$ into (57), as opposed to relying on the quantile function `qpois` as suggested before. However, it has to be noted that for $j \geq 3$, reliance on `qpois` will be necessary for numerical evaluations.

9 Conclusion

After presenting an overview of the important theory related to copulas, EVCs, and regular variation, we investigated the construction of the stable tail dependence functions of the Smith and multivariate t distributions, as presented in Joe et al. (2008). At the core of our work, however, was the derivation of an explicit formula for the density of EVCs though the use of Faà di Bruno's Formula. Our main theorem requires knowledge of the stable tail dependence function of the initial copula, so we leveraged the work of Genest and Rivest (1989) to create new tractable EVC densities from Archimedean copulas with generator inverses satisfying certain regular variation assumptions. Doing so, we connected our work on Archimedean copulas to the additive properties of VaR as presented in Embrechts et al. (2009). We implemented our results in `R` and provided an efficient log-density implementation for inference-based statistical applications. Moreover, a recursive expression for the stable tail dependence function of nested Archimedean copulas was obtained, and the EVC density arising from partially nested Archimedean copulas with generator inverses satisfying certain regular variation assumptions was derived, based on the work of Hofert and Pham (2013). Finally, we derived an implicit expression for the density of the limiting copula of the N largest order-statistics as an extension of the work of Mendes and Sanfins (2007).

Further research could be conducted to see if one can imbed the obtained expression of the stable tail dependence function of nested Archimedean copulas in the framework of Theorem 2.5 as presented in Embrechts et al. (2009).

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A Implementation of various Pickand's dependence functions and their first and second order derivatives

```r
##' GUMBEL PICKANDS' DEPENDENCE FUNCTION AND DERIVATIVES
##' w: in [0,1]
##' th.: theta parameter of Gumbel
##' assy1,assy2: assymetry parameters
##' Functions below return A, A' and A'' for the Gumbel model
##' Author: Gabriel Doyon
AfuncGU ← function(w,th.,assy1,assy2)  (assy2-assy1)*w-assy2+1+((assy1*w)^(th
    .)+(assy2*(1-w))^(th.))^(1/th.)
AfuncDGU ← function(w,th.,assy1,assy2)  ((th.* assy1* (w* assy1)^(th.-1)-th. *
    assy2 *((1-w)* assy2)^(th.-1))* ((w *assy1)^th.+((1-w) *assy2)^th.)^(1/th
    .-1))/th.-assy1+assy2
AfuncDDGU ← function(w,th.,assy1,assy2)  ((th.-1)* (w *assy1)^th.* (assy2-w*
    assy2)^th. *((w *assy1)^th.+(assy2-w* assy2)^th.)^(1/th.-2))/((w-1)^2* w^2)
```

```r
##' GALAMBOS PICKANDS' DEPENDENCE FUNCTION AND DERIVATIVES
##' t: in [0,1]
##' th.: theta parameter of Galambos
##' assy1,assy2: assymetry parameters
##' Functions below return A, A' and A'' for the Galambos model
##' Author: Gabriel Doyon
AfuncGA ← function(t,th.,assy1,assy2)  1-((assy1*t)^(-th.)+(assy2*(1-t))^(-th
    .))^(-1/th.)
AfuncDGA ← function(t,th.,assy1,assy2)  ((th.*assy2 *((1-t)* assy2)^(-th.-1)-
    th.* assy1 *(t* assy1)^(-th.-1)) *((t *assy1)^(-th.)+((1-t)* assy2)^(-th.))
    ^(-1/th.-1))/th.
AfuncDDGA ← function(t,th.,assy1,assy2)  ((-(-th.-1)* th. *assy1^2 *(t* assy1)
    ^(-th.-2)-(-th.-1) *th. *assy2^2 *((1-t)* assy2)^(-th.-2))* ((t* assy1)^(-
    th.)+((1-t)* assy2)^(-th.))^(-1/th.-1))/th.+((-1/th.-1) *(th. *assy2 *((1-t
    )* assy2)^(-th.-1)-th.* assy1 *(t* assy1)^(-th.-1))^2* ((t* assy1)^(-th.)
    +((1-t)* assy2)^(-th.))^(-1/th.-2))/th.
```

```r
##' MIXED MODEL PICKANDS' DEPENDENCE FUNCTION AND DERIVATIVES
##' t: in [0,1]
##' assy1,assy2: assymetry parameters
##' Functions below return A, A' and A'' for the Mixed model
##' Author: Gabriel Doyon
AfuncMM ← function(t,assy1,assy2) assy1*t^3 + assy2*t^2 - (assy1+assy2)*t + 1
AfuncDMM ← function(t,assy1,assy2) 3*assy1*t^2 + 2*assy2*t - (assy1+assy2)
AfuncDDMM ← function(t,assy1,assy2) 6*assy1*t + 2*assy2
```

A Implementation of various Pickand's dependence functions and their first and second order derivatives

```
1   ##' BETA MODEL PICKANDS' DEPENDENCE FUNCTION AND DERIVATIVES
2   ##' t: in [0,1]
3   ##' q1,q2: beta distribution parameters
4   ##' Functions below return A, A' and A'' for the Mixed model
5   ##' Author: Gabriel Doyon
6   AfuncBE ←function(t,q1,q2)  t*(1-BETA1(t,q1,q2))+(1-t)*BETA2(t,q1,q2)
7   AfuncBED ←function(t,q1,q2)  (1-BETA1(t,q1,q2))-BETA2(t,q1,q2)+(1-t)*BETA2.
        diff(t,q1,q2)-t*BETA1.diff(t,q1,q2)
8   AfuncBEDD ←function(t,q1,q2)  -2*(BETA1.diff(t,q1,q2)+BETA2.diff(t,q1,q2))-t*
        BETA1.ddiff(t,q1,q2)+(1-t)*BETA2.ddiff(t,q1,q2)

10  ## Required auxiliary functions for beta model
11  h ← function(t,q1,q2)  ((1-t)*q1)/(t*(q2-q1)+q1)
12  h.diff ← function(t,q1,q2)  -q1/(t*(q2-q1)+q1)-(1-t)*q1*(q2-q1)/(t*(q2-q1)+q1)
        ^2
13  h.ddiff ← function(t,q1,q2)  2*q1*(q2-q1)/(t*(q2-q1)+q1)^2+(2*(1-t))*q1*(q2-q1
        )^2/(t*(q2-q1)+q1)^3

15  BETA1 ← function(t,q1,q2)  pbeta(h(t,q1,q2),q1+1,q2)
16  BETA2 ← function(t,q1,q2)  pbeta(h(t,q1,q2),q1,q2+1)
17  BETA1.diff ← function(t,q1,q2)  dbeta(h(t,q1,q2),q1+1,q2)*h.diff(t,q1,q2)
18  BETA2.diff ← function(t,q1,q2)  dbeta(h(t,q1,q2),q1,q2+1)*h.diff(t,q1,q2)
19  BETA1.ddiff ← function(t,q1,q2){
20    (1/beta(q1+1,q2))*((h.diff(t,q1,q2))^2 * (q1*(h(t,q1,q2))^(q1-1)*(1-h(t,q1,q2
        ))^(q2-1)
21    - (q2-1)*(h(t,q1,q2))^(q1)*(1-h(t,q1,q2))^(q2-2)) + h.ddiff(t,q1,q2)*(h(t,q1
        ,q2))^q1 * (1-h(t,q1,q2))^(q2-1))
22  }
23  BETA2.ddiff ← function(t,q1,q2){
24    (1/beta(q1,q2+1))*((h.diff(t,q1,q2))^2 * ((q1-1)*(h(t,q1,q2))^(q1-2)*(1-h(t,
        q1,q2))^(q2) - (q2)*
25    (h(t,q1,q2))^(q1-1)*(1-h(t,q1,q2))^(q2-1)) + h.ddiff(t,q1,q2)*(h(t,q1,q2))^(
        q1-1)*(1-h(t,q1,q2))^(q2))
26  }
```

```
1   ##' HUSLER REISS MODEL PICKANDS' DEPENDENCE FUNCTION AND DERIVATIVES
2   ##' t: in [0,1]
3   ##' lambda: in [0,infty]
4   ##' Author: Gabriel Doyon

6   A.HR ← function(t,lambda)  (1-t)*pnorm(f(t,lambda)) + t*pnorm(g(t,lambda))
7   A.HR.diff ← function(t,lambda){
8    -pnorm(f(t,lambda)) + pnorm(g(t,lambda)) + t*dnorm(g(t,lambda))*g.diff(t,
        lambda)
9     +(1-t)*dnorm(f(t,lambda))*f.diff(t,lambda)
10  }
11  A.HR.ddiff ← function(t,lambda){
12    -2*dnorm(f(t,lambda))*f.diff(t,lambda)+2*dnorm(g(t,lambda))*g.diff(t,lambda)+
        t*(phi.diff(g(t,lambda))*(g.diff(t,lambda))^2 + dnorm(g(t,lambda))*g.
```

```
        ddiff(t,lambda))+(1-t)*(phi.diff(f(t,lambda))*(f.diff(t,lambda))^2 +dnorm
        (f(t,lambda))*f.ddiff(t,lambda))
13  }
14
15  ## Required auxiliary functions for Husler-Reiss model
16  f ← function(t,lambda)  lambda+log((1-t)/t)*(1/(2*lambda))
17  f.diff ← function(t,lambda)  -(1/(2*lambda))*(1/(t*(1-t)))
18  f.ddiff ← function(t,lambda)  (1/(2*lambda))*((1-2*t)/(t^2*(1-t)^2))
19
20  g ← function(t,lambda)  lambda-log((1-t)/(t))*(1/(2*lambda))
21  g.diff ← function(t,lambda)  (1/(2*lambda))*(1/(t*(1-t)))
22  g.ddiff ← function(t,lambda)  (1/(2*lambda))*((2*t-1)/(t^2*(1-t)^2))
23
24  phi.diff ← function(t)  -dnorm(t)*t
```

B Code for constructing extreme value copula densities from Archimedean copulas

```r
## 'Copyright (C) 2013 Gabriel Doyon'


#Setup #######################################################################
set.seed(1)
## necessary packages
require(copula)
require(partitions)
require(matrixStats)


#Auxiliary functions #########################################################


##' @title Compute the middle term of the expression
##' @param cube: hypercube representation of (0,1]^d
##' @param d: dimension
##' @param tailIndex: from the inverse archimedean generator regularly varying
##    with alpha=tailIndex
##' @return the term identified as a_{d,m}^alpha in Theorem
##' @author Gabriel Doyon
middle.term <- function(cube,d,tailIndex){
  w <- -log(cube) #modifiy initial vector in -log form
  componentwise.product <- matrix(apply(w,1,prod)^(tailIndex-1),nrow=nrow(w))
  componentwise.sum <- matrix(apply(w^tailIndex,1,sum),nrow=nrow(w))
  comp.sum.exp <- matrix(nrow=nrow(w),ncol=d)
  for(m in 1:d){comp.sum.exp[,m]<-componentwise.sum^(m/tailIndex - d)}
  output <-  tailIndex^d *comp.sum.exp*componentwise.product[,]
  return(output)
}


##' @title Computes the falling factorial of x with parameter B
##' @param x: argument of the falling factorial
##' @param B: falling factorial computed until (x-(B-1))
##' @return The evaluated falling factorial for parameters x and B.
##' @author Gabriel Doyon
falling.factorial <- function(x,B){
  output <-1
  for(i in 1:B){output <- output * x
                x <- x - 1}
  return(output)
}


##' @title Computes the Bell polynomial value
##' @param d: desired dimension
##' @param m: subdimension of Bell polynomial
```

```r
45  ##' @param tail.Index: tail index of inverse generator
46  ##' @return The B_{d,m} evaluated along the falling factorial sequence
47  ##' @author Gabriel Doyon
48  combinatorialArgs ← function(d,m,tail.Index){
49    ## Case m=1
50    if(m==1){return((-1)^(m+d)*falling.factorial(1/tail.Index,d))}
51    ## Cases 1<m≤ d
52    else{
53      #Obtaining and modifying the matrix of restricted partitions
54      r.parts ← restrictedparts(d,m)
55      focused.parts← r.parts[,r.parts[nrow(r.parts),] ≠ 0] #removing the
            columns ending with 0
56      if(!is.matrix(focused.parts)){focused.parts ← matrix(focused.parts)} #
            ensure we have matrices as output
57
58      #Computing the multinomial coefficient of the columnwise allocation
59      num ← factorial(d)/colProds(factorial(focused.parts)) #will act as our
            numerator
60
61      #Computing the number possible permutations for our adjustment
62      denom ← matrix(nrow=d,ncol=ncol(focused.parts))
63      for(j in 1:d){ denom[j,]= colSums(focused.parts == j)}
64      denom ← colProds(factorial(denom)) #will act as our denominator
65
66      #Number of possible ways
67      factors ← matrix(num/denom)
68
69      falling.factorial.matrix ← matrix(mapply(falling.factorial,1/tail.Index,
            focused.parts),nrow=m)
70      aggregation ← matrix(colProds(falling.factorial.matrix))
71
72      return((-1)^(m+d)*sum(factors*aggregation))
73    }
74  }
75
76
77  #Main function ############################################################
78
79
80  ##' @title Computes the d-dimensional density and log-density of an EVC created
          via the initial selected AC model
81  ##' @param d: desired dimension
82  ##' @param theta: desired dimension
83  ##' @param model: desired initial AC model
84  ##' @param mesh: divides the interval [0,1] in sizes 1/mesh (allows user for
          selection of fine or coarse grid)
85  ##' @param npoints: number of random points in the hypercube to be considered (
          recommended for d3)
86  ##' @param c is a predefined hypercube passed by the user
```

```
87   ##' @return result$dens: density,  result$log.dens: log-density, result$cube:
        points in hypercube
88   ##' @author Gabriel Doyon
89   EVCD.from.Arch ← function(d, theta, mesh=NULL, model = c("Clayton","Gumbel","2
        ","12","14","15","21"),c=NULL, npoints=NULL,...)
90   {
91
92     ## variable instantiation
93     tail.Index ← NULL
94     psi ← NULL
95     psi.inverse ← NULL
96     Acop ← NULL
97     cube ← c
98
99     ## input checks
100    stopifnot(d>1) #require at least d=2
101    model ← match.arg(model)
102    if(is.null(npoints) && is.null(mesh)) return(print("You need to supply either
           a 'mesh' or a 'npoints' parameter"))
103
104    ## model switch
105    switch(model,
106           "Clayton"={
107             ## model specific parameters
108             stopifnot(theta>0)
109             tailIndex ← 1
110             psi ← function(x) (1+x)^(-1/theta)
111             psi.inverse ← function(x) x^(-theta)-1
112             Acop ← function(x) (psi(sum(psi.inverse(x))))
113             print("Selected model: Clayton")
114           },
115           "Gumbel"={
116             ## model specific parameters
117             stopifnot(theta>1)
118             tailIndex ← theta
119             psi ← function(x) exp(-x^(1/theta))
120             psi.inverse ← function(x) (-log(x))^theta
121             Acop ← function(x) (psi(sum(psi.inverse(x))))
122             print("Selected model: Gumbel")
123           },
124           "2"={
125             ## model specific parameters
126             stopifnot(theta>1)
127             tailIndex ← theta
128             psi ← function(x) 1-x^(1/theta)
129             psi.inverse ← function(x) (1-x)^theta
130             Acop ← function(x) max(psi(sum(psi.inverse(x))),0)
131             print("Selected model: Nelsen 4.2.2")
132           },
```

```r
133          "12"={
134            ## model specific parameters
135            stopifnot(theta>1)
136            tailIndex ← theta
137            psi ← function(x) (1+x^(1/theta))^(-1)
138            psi.inverse ← function(x) (1/x-1)^theta
139            Acop ← function(x) max(psi(sum(psi.inverse(x))),0)
140            print("Selected model: Nelsen 4.2.12")
141          },
142          "14"={
143            ## model specific parameters
144            stopifnot(theta>1)
145            tailIndex ← theta
146            psi ← function(x) (1+x^(1/theta))^(-theta)
147            psi.inverse ← function(x) (x^(-1/theta)-1)^theta
148            Acop ← function(x) psi(sum(psi.inverse(x)))
149            print("Selected model: Nelsen 4.2.14")
150          },
151          "15"={
152            ## model specific parameters
153            stopifnot(theta>1)
154            tailIndex ← theta
155            psi ← function(x) (1-x^(1/theta))^(theta)
156            psi.inverse ← function(x) psi(x)
157            Acop ← function(x) max(psi(sum(psi.inverse(x)))^(1/theta),0)^theta
158            print("Selected model: Nelsen 4.2.15")
159          },
160          "21"={
161            ## model specific parameters
162            stopifnot(theta>1,d==2)
163            tailIndex ← theta
164            fct ← function(x) (1-(1-x)^theta)^(1/theta)
165            Acop ← function(x) 1-(1-max(sum(fct(x))-1,0)^(theta))^(1/theta)
166            print("Selected model: Nelsen 4.2.21. Only implemented for d=2")
167          },
168  {
169    return(print("No model selected"))
170  }
171    )
172
173    ## Hypercube generation & instantiation of necessary data-holding variables
174    if(is.null(cube)){
175      if(is.null(npoints)==FALSE){cube ← matrix(data = runif(npoints*d,min
            =0.0001,max=0.9999), nrow = npoints, ncol = d)}
176      else{cube ← do.call(expand.grid,replicate(d, seq_len(mesh)/(mesh+1),
            simplify=FALSE))}
177    }
178    dEVC ← matrix(nrow=nrow(cube),ncol=1)
179    log.dEVC ← matrix(nrow=nrow(cube),ncol=1)
```

```r
180
181
182     ## creates function C(u)/Pi(u)    ("C.o.I" stands for Copula over Independence
        )
183     C.o.I ← function(u) (Acop(u))/prod(u)
184
185     ## density formula
186     comb.arg ←  matrix(mapply(combinatorialArgs, d, 1:d, tailIndex),nrow=1)  #
        computes combinatorial args
187     adm ← middle.term(cube,d,tailIndex)
188     mult.mat ←  apply(t(t(adm)*comb.arg[,]),1,sum) # elementwise multiplication
        of...
189                             #...the middle term matrix by the comb. args. and
                                summing along rows
190
191     first.vec ← apply(cube,1,C.o.I)
192     dEVC ← first.vec * mult.mat
193
194     ## log density formula
195     repmat ← do.call("rbind", rep(list(comb.arg), nrow(cube)))
196     xm ← log(repmat) + log(adm) #log(0)=-Inf
197     xmax ← matrix(apply(xm,1,max),byrow=T)
198     exp_xm.minus.xmax ← exp(xm - do.call("cbind", rep(list(xmax),d))) #exp(xm-
        xmax)
199     log.dEVC← (xmax + log(apply(exp_xm.minus.xmax,1,sum))) + log(first.vec)
200
201     ## grapical processing for d=2
202     if(d==2 && is.null(npoints)){
203       par(mfcol=c(1,2), mar=c(1,1,1,1), oma=c(1,1,0,1))
204       persp(x=(1:mesh)/(mesh+1),y=(1:mesh)/(mesh+1),matrix(dEVC,nrow=mesh),xlab="
          u1", ylab="u2",zlab = "z")
205       contour(matrix(dEVC,nrow=mesh),levels=cbind(0.5,1:10))
206     }
207
208     ## function returns both the density and the log density
209     return(list(dens=dEVC,logDens=log.dEVC,cube=cube))
210 }
```