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QUADRATIC RISK MINIMIZATION WITH TERMINAL WEALTH CONSTRAINTS

An Application to Definied Contribution Pension Schemes

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE ETH

in

Mathematics

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Submission date, place:

05.08.2018, Edinburgh

Abstract

In order to obtain sufficient funds at retirement, plan members of a defined contribution pension scheme invest their wealth in a portfolio of assets. This thesis examines the target-based optimization strategy as an alternative to the popular lifecycle strategies dominating the market. By using either the dynamic programming approach or the martingale approach, the optimization problem is solved in different markets under the additional constraint of a non-negative wealth process. As the investment horizon is usually very long, the inflation and the salary risk are taken into account and an index-linked bond is added to the market. In order to compare the different investment strategies, a performance methodology is introduced and the results are illustrated in numerical examples.

Acknowledgment

I would like to express my profound gratitude to Prof. Dr. Mario V. Wüthrich for supervising this master thesis and enabling me to write most of my thesis abroad. Similar thanks go to Dr. Catherine Donnelly, for accepting me as a visiting scholar at Heriot-Watt University in Edinburgh. The quality and the coherence of this work was made possible by the guidance and advice of both my supervisors. I would also like to thank Ayse Arik, Andrés Barajas, Thomas Bernhardt, Oytun Haçarız and Michael Zaugg for their constant support and for allowing me an insight in the life and work as a researcher.

A grant from the Swiss-European Mobility Program is also gratefully acknowledged

Finally, I thank my family and friends for their confidence and constant support during my time at ETH Zürich.

Contents

Chapter 1

Introduction

Pension plans and their underlying investment strategies are of huge importance for plan members on defined contribution (DC) plans, as their wealth at retirement is directly linked to the performance of the underlying strategy. In recent years, different variants of a lifecycle strategy have gained favor in the UK pension market. These strategies alter the asset allocations depending on age and years to retirement, but usually do so independently of the asset price. Typically, these retirement funds initially have a high allocation to stocks, but move towards less volatile assets as the retirement date approaches. These lifecycle strategies have many desirable properties, e.g. reducing the volatility of the wealth outcomes and allowing to plan ahead more securely than for other strategies, see e.g. [Blake et al., 2001].

On the other hand, these benefits come at the price of giving up substantial upside potential and might perform very poorly depending on the performance of the asset. Various recent works have focused on studying this asset allocation problem in more mathematical terms, leading to the study of stochastic optimal control and stochastic optimization problems. Two main approaches are of note in this regard, the first being the maximization of the expected utility of the terminal wealth, studied among others in [Battocchio and Menoncin, 2004], [Cairns et al., 2006] and more recently in [Donnelly et al., 2015]. In this approach, the investor is characterized by some utility function which is maximized by the optimal strategy. The second approach is the mean-variance portfolio optimization problem studied in [Bielecki et al., 2005], [Yao et al., 2013], [Wu et al., 2015] and [Menoncin and Vigna, 2017]. Instead of characterizing the investor by some utility function, the variance of the portfolio process is minimized under some expected terminal wealth.

For both approaches, the risk aversion of the investor needs to be characterized by some abstract, mathematical parameter. This is either some parameter in the utility function, or a combination of the expectation and the variance of the terminal wealth distribution. In this thesis we combine the two approaches by choosing a utility function of quadratic form. This leads to a target-based method, which has a one-to-one relation to the mean-variance approach and allows to identify the risk profile of the investor via the selection of a terminal wealth target, instead of the selection of some abstract risk aversion coefficient. This one-to-one correspondence was first introduced by [Zhou and Li, 2000] and extended thenceforth in both [Vigna, 2014] and [Menoncin and Vigna, 2017].

In most literature analyzing stochastic optimization problems, the wealth process is allowed to reach negative values as long as the terminal wealth outweighs this downside risk. This is not desirable, as in practice negative wealth means bankruptcy and only very few investors can continue borrowing and investing once bankruptcy is reached. Therefore we add the additional constraint of a non-negative wealth process, which in turn allows us to study the target-based optimization problem for a general lower bound on the wealth process.

The remainder of the thesis is organized as follows. Part I focuses on finding the optimal portfolio strategy in a self-financing manner, for a simple market model containing any number of risky stocks and one risk-free bank account. This problem has been well addressed in the literature, see e.g. [Korn, 1997], [Heunis, 2014] and [Bielecki et al., 2005]. In Chapter 2 a market structure is introduced and the portfolio process with its corresponding wealth process is formally defined. In Chapter 3, the optimization problem under the constraint of a non-negative wealth process is stated and the optimal investment strategy is derived. The non-negativity constraint is transformed into a constraint of a general lower bound in Chapter 4, where we show that the corresponding optimal strategy consists of a forward contract on the lower bound and the optimal portfolio process with a lower initial wealth.

Part II introduces stochastic inflation in terms of an indexed-linked bond to the market. Although inflation has been a strong focus of recent literature on stochastic optimal control, see e.g. [Zhang et al., 2007] and [Xue and Basimanebotlhe, 2015], the optimal investment strategy for the quadratic optimization problem under a non-negativity constraint has to our knowledge not yet been given. In Chapter 5, we lay the groundwork for the inflation index and define the price process of an inflation-linked bond. The corresponding optimization problem under the constraint of a non-negative wealth process is solved in Chapter 6.

Part III drops the self-financing assumption of Parts I and II and introduces stochastic contributions to the investment portfolio. Contrary to the optimal investment strategies derived so far, which are of mainly theoretical interest, the optimal portfolio process of Part III can directly be used as the strategy for a DC pension plan. The constraint of a non-negative wealth process is loosened and only non-negative terminal wealth is required. This allows to borrow against future contributions and in turn, much more risk is taken during the first years of the pension plan. After solving the optimization problem with and without this constraint in Chapter 9, we introduce the idea of "cut-shares", where a no-shorting constraint is imposed on the portfolio process ex-post. All the portfolio processes of the first three Parts are compared in Chapter 11.

The numerical analysis of the different portfolio processes is conducted in Part IV. Unlike in most existing literature, where parameters are chosen as constants, all optimal portfolio processes of this thesis allow for deterministic parameter processes. After outlining some possible models to predict the parameters in Chapter 13, the performance of constant and deterministic parameters are compared in Chapter 14. In Chapter 15 a performance methodology is developed in order to compare the optimal portfolio processes. This methodology is then used in Chapter 16 to show the importance of including the inflation-linked bond to the investment strategy. Finally, in Chapter 17, the performance of the target-based optimal portfolio processes is compared to the performance of lifecycle strategies and popular utility maximizing strategies.

Part I

Quadratic Minimization with Wealth Constraints

Chapter 2

The Financial Market Model

Remark (Notation)*.* We use the convention that random variables X are denoted by capital letters, whereas their realizations x are denoted by small letters. Additionally we use the following notation,

$$\begin{aligned}
M' &: \text{the transpose of any vector or matrix } M; \\
\|M\| &= \sqrt{\sum_{i,j} m_{i,j}^2} \text{ for any vector or matrix } M = (m_{i,j}); \\
\mathbb{1}\{A\} &: \text{the indicator function of some subset } A; \\
\mathbb{I} &= (1, \dots, 1)' \text{ as the unit vector}; \\
x^- &= \max\{-x, 0\} \text{ for any real number } x; \\
\mathbb{E}^{\mathbb{P}} &: \text{the expectation under the measure } \mathbb{P}; \\
f_t(t,x) &= \frac{\partial}{\partial t} f(t,x) \text{ as the derivative with respect to } t \text{ if it exists ;} \\
f(t,x) \in \mathcal{C}^{i,j} &: \text{the derivatives } f_t(t,x), \cdots f_{t^i}(t,x) \text{ and} \\
&\quad f_x(t,x), \cdots f_{x^j}(t,x) \text{ exist and are continuous.}
\end{aligned}$$

2.1 The Market Model

Choose a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we have an n-dimensional Brownian motion $W(t) = \big(W_1(t), \dots, W_n(t)\big), t \in [0,T]$, for a given, finite time horizon $T > 0$. The filtration $\mathcal{F}$ is the $\mathbb{P}$-augmentation of the filtration generated by the Brownian motion. We first assume the Black-Scholes model consisting of n stocks with price processes $(S_i(t))_{t\in[0,T]}$

and one risk-free bond with price process $(B(t))_{t\in[0,T]}$, where the dynamics are given by

$$dB(t) = r(t)B(t)dt, \tag{2.1}$$

$$dS_i(t) = S_i(t)\Big[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)\Big], \tag{2.2}$$

with $B(0) = 1$ and $S_i(0) = s_i > 0$, $\mathbb{P}$-a.s. In order to introduce an auxiliary measure $\mathbb{Q}$ which will be of high importance throughout this thesis, we need to make some assumptions on the coefficients of the market elements.

Assumption 2.1.1. The *interest rate* process $r(t)$, the vector of *mean rates of return* $\mu(t)$ and the *dispersion matrix* $\sigma(t)$ are uniformly bounded and $\mathcal{F}_t$-progressively measurable processes on $[0,T]\times\Omega$, with $r(t)\in\mathbb{R}, \mu(t)\in\mathbb{R}^n, \sigma(t)\in\mathbb{R}^{n\times n}$. Furthermore, $\sigma(t)\sigma(t)'$ shall be positive definite for all $t\in[0,T]$.

Consider an investor who starts with a fixed, strictly positive wealth x at time 0, who invests in the various securities and whose actions do not affect the market prices. At time $t\in[0,T]$ we denote the total wealth of this investor by $X(t)$ and the amount that is invested in the ith stock by $\pi_i(t)$, for $i = 1,\dots,n$.

Definition 2.1.2. A *portfolio process* $\pi(t) = \big(\pi_1(t),\dots,\pi_n(t)\big)_{t\in[0,T]}$ is a progressively measurable process with respect to $\{\mathcal{F}_t\}_{0\le t\le T}$. If π satisfies

$$\int_0^T \|\pi(t)\|^2 dt < \infty, \quad \mathbb{P}\text{-a.s.}$$

for all $t\in[0,T]$, then π is *admissible*. We denote the family of admissible portfolio processes by Π.

We assume for all of Part I and Part II, that the investor follows a *self-financing* strategy, i.e. any gains and losses arise solely from changes in the value of the stocks and the bond. Hence, the amount of wealth invested in the bond is given by $(X(t) - \sum_{i=1}^n \pi_i(t))_{t\in[0,T]}$.

Definition 2.1.3. Given a portfolio process π, the solution $X = X^\pi$ to

$$\begin{aligned} dX^\pi(t) &= \pi(t)'\frac{dS(t)}{S(t)} + \big(X^\pi(t) - \sum_{i=1}^n \pi_i(t)\big)\frac{dB(t)}{B(t)} \\ &= \Big(r(t)X^\pi(t) + \pi(t)'\big(\mu(t) - r(t)\mathbb{I}\big)\Big)dt + \pi(t)'\sigma(t)dW(t), \\ X^\pi(0) &= x, \end{aligned} \tag{2.3}$$

is called the *wealth process* corresponding to the portfolio process π and the initial capital $x > 0$.

We will see in (2.8) that a strong solution exists for (2.3) for any $\pi \in \Pi$. By the following proposition, the strong solution will in fact be unique as well. Note however, that the wealth process is not necessarily non-negative, which is of high importance for many practical applications, as normal investors cannot continue investing once bankrupt.

Proposition 2.1.4. *If there exists a strong solution to the stochastic differential equation* (2.3)*, then it is unique.*

Proof. Suppose X_1 and X_2 are solutions of (2.3) with $X_1(0) = X_2(0) = x$. Define $Y(t) = X_1(t) - X_2(t)$ for all $t \in [0, T]$. Then

$$dY(t) = r(t)Y(t)dt, \quad Y(0) = 0, \quad \mathbb{P}\text{-a.s.}$$

The unique solution of this stochastic differential equation is $Y(t) \equiv 0, \mathbb{P}$-a.s., for $t \in [0, T]$. □

At this point we emphasize the important difference in our definition of an admissible portfolio process π compared to [Cvitanic and Karatzas, 1992]. We have defined the portfolio process in terms of the *amounts* invested in the assets and with our definition of admissibility, the wealth process can reach negative values. Therefore, the non-negativity required needs to be posed as an additional constraint.

This is different to the definition of [Cvitanic and Karatzas, 1992], who instead define it in terms of the *proportions* invested. With this definition, it can be shown that the wealth at any time $t \geq 0$ is propotional to the wealth at time $t = 0$, in the sense that $X^{\pi}(t) = x\tilde{X}(t)$, where $\tilde{X}(t)$ is $\mathbb{P}$-a.s. positive for all $t \in [0, T]$. Consequently, as long as the initial wealth is positive, the whole wealth process X inherits that property. Note that the set of strategies which are defined as proportions is in fact a proper subclass of the set of strategies used here, see e.g. [Bielecki et al., 2005]. We will see later on, that the resulting optimal strategy cannot be written as a proportional strategy for out definition of admissibility.

2.2 Change of Measure

From an economic point of view, it is apparent that the non-negativity of the whole process will be a consequence of the non-negativity of the terminal value, once we have shown the existence of an equivalent martingale measure $\mathbb{Q}$. If this was not the case, an arbitrage opportunity would be necessary in order to reach a positive value at the end, with an almost sure probability. Henceforth, we denote by $\mathbb{E}^{\mathbb{P}}$ the expectation under the measure $\mathbb{P}$ and by $\mathbb{E}^{\mathbb{Q}}$ the expectation under the measure $\mathbb{Q}$.

To the end of deriving an equivalent martingale measure $\mathbb{Q}$, we introduce the risk premium process

$$\theta(t) = \sigma(t)^{-1}(\mu(t) - r(t)\mathbb{I}),$$

which exists and is bounded, measurable and adapted to $\mathcal{F}_t$ due to Assumption 2.1.1. Hence, the Novikov condition is fulfilled and we can apply Girsanov's theorem, utilizing the Doléan-Dade exponential

$$Z(t) = \exp\Big(- \int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\Big). \tag{2.4}$$

As the Novikov condition is fulfilled and $Z(0) = 1$, Z is a martingale.

Remark. The function $\mathbb{Q} : \Omega \to [0, 1]$ is defined by

$$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[Z(T)\mathbb{1}\{A\}], \quad \text{for all } A \in \mathcal{F}.$$

To prove that the two measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, we need to show

$$\mathbb{P}[A] = 0 \iff \mathbb{Q}[A] = 0, \quad \text{for any } A \in \mathcal{F}. \tag{2.5}$$

Let $A \in \mathcal{F}$ be such that $\mathbb{P}[A] = 0$. Then $\mathbb{P}[Z(T)\mathbb{1}\{A\}] = 0$, as it takes the value zero on the complement of A, which is a set of full measure. But then

$$0 = \mathbb{E}^{\mathbb{P}}[Z(T)\mathbb{1}\{A\}] = \mathbb{Q}[A].$$

The opposite direction follows by reversing the roles of $\mathbb{P}$ and $\mathbb{Q}$.. Therefore,

$$\mathbb{Q}[\Omega] = \mathbb{E}^{\mathbb{P}}[Z(T)\mathbb{1}\{\Omega\}] = \mathbb{E}^{\mathbb{P}}[Z(T)] = 1.$$

Lemma 2.2.1. *The process $\hat{W}(t) = W(t) + \int_0^t \theta(s)ds$ is an n-dimensional Brownian motion under $\mathbb{Q}$.*

Proof. We use that if the process $\big(Z(t)\hat{W}(t)\big)_{t\in[0,T]}$ is a martingale under $\mathbb{P}$, then $\big(\hat{W}(t)\big)_{t\in[0,T]}$ is a martingale under $\mathbb{Q}$. This follows from

$$\mathbb{E}^{\mathbb{Q}}[\hat{W}(t) \mid \mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}^{\mathbb{P}}[Z(t)\hat{W}(t) \mid \mathcal{F}_s] = \frac{1}{Z(s)}Z(s)\hat{W}(s) = \hat{W}(s).$$

By Itô's formula applied to the function $f(t,x) = \exp(-x - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds)$, we have

$$dZ(t) = -Z(t)\theta(t)'dW(t). \tag{2.6}$$

Now to show that $\big(Z(t)\hat{W}(t)\big)_{t\in[0,T]}$ is a martingale under $\mathbb{P}$ we use the product rule and compute

$$\begin{aligned} d\big(Z(t)\hat{W}(t)\big) &= Z(t)d\hat{W}(t) + \hat{W}(t)dZ(t) + d[Z,\hat{W}](t) \\ &= Z(t)dW(t) + Z(t)\theta(t)dt - \hat{W}(t)Z(t)\theta(t)'dW(t) - Z(t)\theta(t)dt \\ &= Z(t)dW(t) - \hat{W}(t)Z(t)\theta(t)'dW(t). \end{aligned}$$

Therefore, $\big(\hat{W}(t)\big)_{t\in[0,T]}$ is continuous and a martingale under $\mathbb{Q}$. Since we also have

$$d[\hat{W}_i, \hat{W}_j](t) = d[W_i, W_j](t) = \mathbb{1}\{i = j\}dt,$$

it follows from Lévy's characterization theorem of Brownian motion, see e.g. [Karatzas and Shreve, 1998, Theorem 3.16], that $(\hat{W}(t))_{t\in[0,T]}$ is a $\mathbb{Q}$-Brownian motion. □

This measure change allows us to solve the stochastic differential equation (2.3).

$$\begin{aligned} dX^\pi(t) &= r(t)X^\pi(t)dt + \pi(t)'\big(\mu(t) - r(t)\mathbb{I}\big)dt + \pi(t)'\sigma(t)dW(t) \\ &= r(t)X^\pi(t)dt + \pi(t)'\sigma(t)d\hat{W}(t). \end{aligned}$$

Introducing the bank account numéraire $\big(\beta(t)\big)_{t\in[0,T]}$ by

$$\beta(t) = \frac{1}{B(t)} = \exp(-\int_0^t r(s)ds), \tag{2.7}$$

for $t \in [0,T]$, we apply Itô's formula to the product of X^π and β and obtain

$$\begin{aligned} d(X^\pi(t)\beta(t)) &= \pi(t)'\sigma(t)\beta(t)d\hat{W}(t), \\ X^\pi(t)\beta(t) &= x + \int_0^t \beta(s)\pi(s)'\sigma(s)d\hat{W}(s). \end{aligned} \tag{2.8}$$

From this it is apparent that the process $(M(t))_{t\in[0,T]}$ defined by $M(t) = \beta(t)X(t)$ is a continuous local martingale with respect to the measure $\mathbb{Q}$. Together with (2.5), this proves that $\mathbb{Q}$ is an equivalent martingale measure.

Define the state price deflator

$$\xi(t) = \beta(t)Z(t) = \exp\Big(-\int_0^t \theta(s)'dW(s) - \int_0^t (r(s) + \frac{1}{2}\|\theta(s)\|^2)ds\Big), \tag{2.9}$$

for $t \in [0, T]$, we see by Bayes' rule, see e.g. [Karatzas and Shreve, 1998, Lemma 5.3], that the process $(N(t))_{t\in[0,T]}$, defined by $N(t) = \xi(t)X(t)$ is a continuous local martingale under $\mathbb{P}$.

Chapter 3

The Constrained Optimal Strategy: Non-Negativity

We start by analyzing the trading strategy which minimizes the mean-square difference of the terminal wealth to some predetermined target, under all portfolio processes with affordable, non-negative paths. The resulting strategy and terminal wealth distribution will then be investigated in more detail and compared to the optimal strategy when the non-negativity constraint is dropped.

3.1 Problem Formulation

3.1.1 The Constrained Portfolio Problem

In order to mathematically formulate the quadratic optimization problem, we still need to formally prove the claim of Section 2.2 that we can focus solely on the non-negativity of the terminal wealth $X(T)$, rather than the non-negativity of the process $X(t)$ over the entire time interval $t \in [0,T]$. In order to do this, we apply results on backward stochastic differential equations from [El Karoui et al., 1997].

Proposition 3.1.1. *Let X^π be a wealth process under an admissible portfolio π. If $X^\pi(T) \geq 0$ a.s., then $X^\pi(t) \geq 0$ a.s., for all $t \in [0,T]$.*

Proof. Fix some $\pi \in \Pi$ and assume that the wealth process X^π corresponding to (2.3) exists and satisfies $X^\pi(T) = \Psi \geq 0$, a.s. By defining $P(t) = \sigma(t)'\pi(t)$ for all $t \in [0,T]$ in (2.3), we obtain the backward stochastic differential equation

$$\begin{aligned} dX^\pi(t) &= \big(r(t)X^\pi(t) + \theta(t)P(t)\big)dt + P(t)'dW(t), \\ X^\pi(T) &= \Psi. \end{aligned}$$

By [El Karoui et al., 1997, Theorem 2.1], there exists a unique, square integrable solution (X, P). Furthermore, by Assumption 2.1.1,

$$\mathbb{E}^{\mathbb{P}}\big[(\sup_{t\in[0,T]} |X^\pi(t)|)^2\big] < \infty, \quad \mathbb{E}^{\mathbb{P}}\big[(\sup_{t\in[0,T]} |\xi(t)|)^2\big] < \infty.$$

Hence, the local martingale $\xi(t)X^\pi(t)$ is uniformly integrable and therefore equal to the conditional expectation of its terminal value, i.e.

$$X^\pi(t) = \xi(t)^{-1}\mathbb{E}^{\mathbb{P}}\big[\xi(T)X^\pi(T)\big|\mathcal{F}_t\big], \qquad \text{for all } t \in [0,T]. \tag{3.1}$$

It follows from the definition of the state price deflator ξ in (2.9) that $X^\pi(t) \geq 0$ a.s., for all $t \in [0,T]$. □

Proposition 3.1.1 states that a.s. non-negative terminal wealth leads to an a.s. non-negative wealth process. Therefore, we only need to show non-negativity, $\mathbb{P}$-a.s., of the terminal wealth of some optimal portfolio process in order to solve the optimization problem.

Problem 3.1.2. Choose $x > 0$ and let the family of all admissible portfolio processes that lead to non-negative terminal wealth be denoted by

$$\mathcal{A}(x) = \Big\{\pi \in \Pi \mid X^\pi(0) \leq x \text{ and } X^\pi(T) \geq 0, \quad \mathbb{P}\text{-a.s.}\Big\}.$$

Given a constant $C > 0$, the problem is to determine a portfolio process $\hat\pi \in \mathcal{A}(x)$ such that

$$\mathbb{E}^{\mathbb{P}}\big[(C - X^{\hat\pi}(T))^2\big] = \inf_{\pi\in\mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[(C - X^\pi(T))^2\big], \tag{3.2}$$

and the pair $\big(X^{\hat\pi}(t), \hat\pi(t)\big)$ satisfies the stochastic differential equation (2.3).

One application of this problem, and the main focus of this thesis, is an investor who does not have enough money to reach the target wealth C at maturity by solely investing in the bank account. The optimal portfolio process of Problem 3.1.2 will then be the investment strategy which comes as close to C as possible, while assuring the investor to never become bankrupt.

3.1.2 Conditions on Admissibility

In order to solve Problem 3.1.2, we express the condition $\pi \in \mathcal{A}(x)$ by an inequality, which then allows us to use Lagrangian techniques to find the optimal terminal wealth. We use the fact that the continuous local martingales M and N (under $\mathbb{Q}$ and $\mathbb{P}$, respectively), introduced in Section 2.2, are bounded from below for $\pi \in \mathcal{A}(x)$ to show that they are in fact supermartingales.

Lemma 3.1.3. *Let $M(t)$ be a continuous local martingale, i.e. there is a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity a.s. such that $M(t\wedge \tau_n)$ is a martingale for all n. Suppose that $M(t)$ is bounded from below by some fixed constant c. Then $M(t)$ is a supermartingale.*

Proof. Since this proof works for general probability measures $\mathbb{P}$, we forgo the specification of the underlying probability measure for the expecation. Let $\tilde{M}(t) = M(t) - c$, which is also a continuous local martingale for some sequence of stopping times $(\tilde{\tau}_n)_{n\in\mathbb{N}}$, i.e. $\tilde{M}_n(t) := \tilde{M}(t \wedge \tau_n)$ is a martingale for all $n \in \mathbb{N}$. That is, for all $0 \le s \le t$ we have

$$\mathbb{E}[\tilde{M}_n(t) \mid \mathcal{F}_t] = \tilde{M}_n(s).$$

By Fatou's lemma and the positivity of $\tilde{M}$, we have

$$\mathbb{E}[|\tilde{M}(t)|] = \mathbb{E}[\tilde{M}(t)] = \mathbb{E}[\liminf_n \tilde{M}_n(t)] \le \liminf_n \mathbb{E}[\tilde{M}_n(t)] = \mathbb{E}[\tilde{M}(0)].$$

Moreover,

$$\mathbb{E}\big[\tilde{M}(t) \mid \mathcal{F}_s\big] = \mathbb{E}\big[\liminf_n \tilde{M}_n(t) \mid \mathcal{F}_s\big] \le \liminf_n \mathbb{E}\big[\tilde{M}_n(t) \mid \mathcal{F}_s\big] = \tilde{M}(s).$$

Hence, $\tilde{M}$ is a supermartingale and so is M. □

We use the previous lemma in order to obtain an initial condition for the optimal wealth process. Applying Itô's formula to the product of $Z(t)$ and $\beta(t)X^\pi(t)$, we write

$$\begin{aligned} d(Z(t)\beta(t)X^\pi(t)) =& Z(t)d(\beta(t)X^\pi(t)) + \beta(t)X^\pi(t)dZ(t) + d[Z,\beta X^\pi](t) \\ =& Z(t)\pi(t)'\sigma(t)\beta(t)dW(t) + Z(t)\pi(t)'\sigma(t)\beta(t)\theta(t)'dt \\ & - \beta(t)X^\pi(t)Z(t)\theta(t)'dW(t) - Z(t)\pi(t)'\sigma(t)\beta(t)\theta(t)'dt, \end{aligned}$$

where the second step follows from (2.8) and (2.6). Hence, the process $(N(t))_{t\in[0,t]}$ satisfies

$$N(t) = \xi(t)X^\pi(t) = x + \int_0^t \xi(s)\big(\pi(s)'\sigma(s) - X^\pi(s)\theta(s)'\big)dW(s).$$

Since N(t) is a supermartingale for all $\pi \in \mathcal{A}(x)$, we have shown that

$$\mathbb{E}^{\mathbb{P}}\Big[\xi(T)X^{\pi}(T)\Big] \leq x, \qquad \text{for all } \pi \in \mathcal{A}(x). \tag{3.3}$$

A similar inequality can be shown under the measure $\mathbb{Q}$ for the discount factor $\beta(t)$. As ξ is the state price deflator, the inequality (3.3) takes the role of a budget constraint, i.e. the expected value of the current wealth up to date, deflated to $t = 0$, does not exceed the initial capital. We now prove that this condition is in fact also sufficient for admissibility in the sense that if it is fulfilled by a wealth process, there will exist a corresponding admissible portfolio process π.

Theorem 3.1.4. *For every non-negative, $\mathcal{F}_T$-measurable Ψ which satisfies $\mathbb{E}[\xi(T)\Psi] = x$, there exists a unique $\pi \in \mathcal{A}(x)$ such that the corresponding wealth process satisfies $X^{\pi}(T) = \Psi$, a.s.*

Proof. By [El Karoui et al., 1997, Theorem 2.1], the linear backward stochastic differential equation

$$\begin{aligned} dX(t) &= \big(r(t)X(t) + \theta(t)P(t)\big)dt + P(t)'dW(t), \\ X(T) &= \Psi, \end{aligned}$$

admits a unique, square integrable, $\mathcal{F}_t$-adapted solution (X, P), since the coefficients are uniformly bounded due to Assumption 2.1.1 (see proof of Proposition 3.1.1). Furthermore, $\big(X(t)\big)_{t\in[0,T]}$ is a continuous, adapted process and $\big(P(t)\big)_{t\in[0,T]}$ is a progressively measurable process satisfying $\int_0^T \|P(t)\|^2 dt < \infty$. Define

$$\pi(t) = (\sigma(t)')^{-1}P(t),$$

which is square integrable due to the uniform boundedness of $(\sigma(t)')^{-1}$ and the square-integrability $P(t)$. Moreover, it follows from (3.1) that

$$X(0) = \mathbb{E}^{\mathbb{P}}[\xi(T)\Psi] = x.$$

Hence, $\pi \in \mathcal{A}(x)$ and $\big(X(t), \pi(t)\big)$ satisfies the dynamics of (2.3). □

3.1.3 Feasibility

We follow [Bielecki et al., 2005, Section 3] to determine under what conditions a solution to Problem 3.1.2 exists and if such a solution is unique.

By Theorem 3.1.4, we may study the feasibility of the following problem instead.

$$\begin{aligned} \text{Minimize} \quad & \mathbb{E}^{\mathbb{P}}\big[(C-\Psi)^2\big], \\ \text{subject to} \quad & \mathbb{E}^{\mathbb{P}}\big[\xi(T)\Psi\big] = x \text{ and } \Psi \geq 0,\ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{3.4}$$

over all $\mathcal{F}_T$-measurable processes Ψ.

Proposition 3.1.5. *Problem* (3.4) *either has no feasible solution, or it admits a unique optimal solution.*

Proof. Let

$$D = \Big\{\Psi \in \mathcal{F}_T \mid \mathbb{E}^{\mathbb{P}}\big[\xi(T)\Psi\big] = x \text{ and } \Psi \geq 0,\ \mathbb{P}\text{-a.s.}\Big\},$$

be the constraint set of Problem (3.4), where we write $\Psi \in \mathcal{F}_T$ to denote that Ψ is $\mathcal{F}_T$-measurable. If there exists some $\Psi_1 \in D$, then an optimal solution must be in the set

$$D' = D \cap \Big\{\Psi \in \mathcal{F}_T \mid \mathbb{E}^{\mathbb{P}}\big[(C-\Psi)^2\big] \leq \mathbb{E}^{\mathbb{P}}\big[(C-\Psi_1)^2\big]\Big\}.$$

If there exists some $\Psi_0 \in D'$, then D' is non-empty, closed and convex. As the quadratic function $\big(C-\Psi\big)^2$ is strictly convex with a lower bound of zero, the optimal solution to (3.4) must be unique. $\square$

Define

$$\begin{aligned} a &= \inf_{\Psi \geq 0, \mathbb{P}\text{-a.s.}} \mathbb{E}^{\mathbb{P}}\big[\xi(T)\Psi\big], \\ b &= \sup_{\Psi \geq 0, \mathbb{P}\text{-a.s.}} \mathbb{E}^{\mathbb{P}}\big[\xi(T)\Psi\big]. \end{aligned} \tag{3.5}$$

Proposition 3.1.6. *If $a < x < b$, then there must be a feasible solution to Problem* (3.4) *and hence to Problem 3.1.2.*

Proof. By definition of a and b, there exist non-negative, $\mathcal{F}_T$-measurable processes Ψ_1 and Ψ_2 such that for any $x > 0$,

$$\mathbb{E}^{\mathbb{P}}\big[\xi(T)\Psi_1\big] < x < \mathbb{E}^{\mathbb{P}}\big[\xi(T)\Psi_2\big].$$

Define the function $f : [0,1] \to [0,\infty]$ by

$$f(\lambda) = \mathbb{E}^{\mathbb{P}}\Big[\xi(T)\big(\lambda\Psi_1 + (1-\lambda)\Psi_2\big)\Big],$$

which is continuous and $f(1) < x < f(0)$. Hence, there exists some $\lambda_0 \in (0, 1)$ such that $x = f(\lambda_0)$. Set

$$\Psi_0 = \lambda_0 \Psi_1 + (1 - \lambda_0)\Psi_2.$$

Clearly, Ψ_0 is $\mathcal{F}_T$ measurable, non-negative and satisfies $\mathbb{E}^{\mathbb{P}}[\xi(T)\Psi_0] = x$. □

In order to guarantee the feasibility of the problem, we restrict ourselves to a subset of parameter values, such that $a < x < b$ for all $x > 0$.

Assumption 3.1.7. The risk premium process $\theta(t)$ is deterministic and satisfies

$$\int_0^T \|\theta(s)\|^2 ds \neq 0.$$

Under Assumption 3.1.7, the state price deflator (2.9) at time T is the sum of a bounded random variable and a normal random variable with strictly positive variance. In that case we have $a = 0$ and $b = \infty$, and hence Problem 3.1.2 has a unique optimal solution for any initial wealth $x > 0$.

Remark. Note that the assumption of a deterministic risk premium process is very strong. As applications usually have very long time horizons, it is unreasonable to expect to know the risk profile for the whole duration at the beginning. We evaluate the effect of a deterministic risk premium process during the numerical analysis in Chapter 14 and give some insight about when we can reduce on the assumption of a deterministic risk premium process.

3.2 Optimization of Terminal Wealth

We give an overview of the martingale method in finding a solution to the constrained optimization problem of terminal wealth, as it was done in [Karatzas, 1989] and [Korn and Trautmann, 1995]. Solving the problem at hand will then be a special case where a specific utility function is chosen. The main idea of this section is to reduce the problem at hand as close as possible to a Lagrange problem. If a Lagrange multiplier for this problem exists, the Kuhn-Tucker optimality conditions immediately give necessary and sufficient conditions for optimality. These conditions are then used to construct an optimal portfolio in terms of the Lagrange multiplier.

Definition 3.2.1. Let $U : (0, \infty) \to \mathbb{R}$ be strictly concave and $\mathcal{C}^1$ with

- the derivative $U'(c)$ satisfies $U'(0) = \lim_{c \to 0} U'(c) > 0$;
- there exists $z \in \mathbb{R} \cup \{+\infty\}$ with $U'(z) = 0$.

Then U is called a *generalized utility function*.

Assumption 3.2.2. We impose the additional assumptions

$$U \in \mathcal{C}^2 \quad \text{and} \quad U'' \text{ is non-decreasing on } (0, \infty).$$

Remark. $U' : [0, z] \to [0, U'(0)]$ is strictly decreasing on $[0, z]$ and hence has a strictly decreasing inverse function $\hat{I} : [0, U'(0)] \to [0, z]$. By Assumption 3.2.2, $\hat{I}$ is convex and of class $\mathcal{C}^1$. Denote by

$$I(y) = \begin{cases} \hat{I}(y) & \text{if } y \in [0, U'(0)], \\ 0 & \text{otherwise}, \end{cases} \tag{3.6}$$

the *truncated inverse function* of U'.

Problem 3.2.3. Consider the utility function U and maximize

$$J(x, \pi) = \mathbb{E}^{\mathbb{P}}\big[U(X^\pi(T))\big], \tag{3.7}$$

over the class

$$\mathcal{A}_2(x) = \Big\{\pi \in \mathcal{A}(x) \mid \mathbb{E}^{\mathbb{P}}[U^-(X^\pi(T))] < \infty\Big\}.$$

We denote by

$$V(x) = \sup_{\pi \in \mathcal{A}_2(x)} J(x, \pi), \tag{3.8}$$

the optimal value function of this problem.

Remark. By choosing a utility function of the form

$$U(x) = -\frac{1}{2}(C - x)^2,$$

optimization Problem 3.1.2 is equivalent to the constrained portfolio Problem 3.2.3.

Recall the notation of the state price deflator ξ from (2.9). In light of the admissibility constraint (3.3) we obtain the Lagrangian expression

$$\mathcal{L} = \mathbb{E}^{\mathbb{P}}\big[U(X^\pi(T))\big] + y\big(x - \mathbb{E}^{\mathbb{P}}\big[X^\pi(T)\xi(T)\big]\big), \tag{3.9}$$

for Problem 3.2.3. The maximization of $\mathcal{L}$ is achieved by choosing $X^\pi(T)$ to maximize $U(X^\pi(T)) - yX^\pi(T)\xi(T)$, which leads to the first order constraint $U'(X^\pi(T)) = y\xi(T)$. We determine how to specify y to make sure that $\mathbb{E}^{\mathbb{P}}[X^\pi(T)\xi(T)] = x$. In that regard, define $\mathcal{H}(y) = \mathbb{E}^{\mathbb{P}}[\xi(T)I(y\xi(T))]$ for all $y \in (0, \infty)$.

Lemma 3.2.4. *Assume $\mathcal{H}(y) < \infty$ for all $y \in (0,\infty)$. The function $\mathcal{H}$ is continuous and strictly decreasing. Furthermore,*

$$\mathcal{H}(\infty) = \lim_{y\to\infty} \mathcal{H}(y) = 0, \tag{3.10}$$

$$\mathcal{H}(0) = \lim_{y\to 0} \mathcal{H}(y) = \begin{cases} \infty & \text{if } \lim_{z\to\infty} U'(z) = 0, \\ \tilde{z}\mathbb{E}^{\mathbb{P}}[\xi(T)] & \text{else}, \end{cases} \tag{3.11}$$

where $\tilde{z}$ is defined by $U'(\tilde{z}) = 0$.

Proof.

- The continuity of $\mathcal{H}$ follows from the continuity of I by the dominated convergence theorem.

- Because I is non-increasing and strictly decreasing on $(0, U'(0))$, once we have shown that

$$\mathbb{P}\left[\xi(T) < \frac{U'(0)}{y}\right] > 0, \tag{3.12}$$

for every fixed $y \in (0,\infty)$, it follows that $\mathcal{H}$ is strictly decreasing, as $\xi(T)I(y\xi(T))$ is then strictly decreasing on the set in (3.12) and identically zero everywhere else. But

$$\begin{aligned} \log\big(\xi(T)\big) &= -\int_0^T \theta'(s)dW(s) - \frac{1}{2}\int_0^T \|\theta(s)\|^2 ds - \int_0^T r(s)ds \\ &= \tilde{W}_{D_T} - \int_0^T (r(s) + \frac{1}{2}\|\theta(s)\|^2)ds, \end{aligned}$$

where $\tilde{W}$ is a standard Brownian motion and $D(t) = \int_0^t \|\theta(s)\|^2 ds$. As by Assumption 2.1.1 $r(t)$ and $\sigma(t)$ are uniformly bounded and $D(t)$ is positive and by Assumption 3.4.1 deterministic, we have

$$\mathbb{P}\big[\log(\xi(T)) < u\big] > 0,$$

for $u > 0$ arbitrary and especially for u of the form $u = \exp(\frac{U'(0)}{y})$.

- As $I(\infty) = 0$, (3.10) follows from the monotone convergence theorem.

- To prove (3.11), we distinguish two cases. First, assume $\lim_{z\to\infty} U'(z) = 0$. As I is non-negative, we can apply Fatou's lemma to obtain

$$\liminf_{y\to 0} \mathcal{H}(y) \geq \mathbb{E}^{\mathbb{P}}\big[\xi(T) \liminf_{y\to 0} I(y\xi(T))\big] = \infty.$$

  In the other case we have $\limsup_{y\to 0} \mathcal{H}(y) \leq \tilde{z}\mathbb{E}^{\mathbb{P}}[\xi(T)]$, because for every $x \in [0, U'(0)]$ we have $I(x) \leq \tilde{z}$. Applying Fatou's lemma again yields

$$\liminf_{y\to 0} \mathcal{H}(y) \geq \mathbb{E}^{\mathbb{P}}\big[\xi(T) \liminf_{y\to 0} I(y\xi(T))\big] = \tilde{z}\mathbb{E}^{\mathbb{P}}[\xi(T)].$$

□

Note that ξ is fully determined by the market and not by the utility function of the investor. In turn, ξ determines the number $y > 0$ that serves as initial value of the process

$$Y(t) = y\xi(t).$$

Once the constant $y > 0$ is determined, the further evolution of the process Y and in turn the wealth process X depends only on the market. The following theorem shows how y is chosen for the maximization of the utility from wealth.

Theorem 3.2.5. *For any $x > 0$ define*

$$\Psi = \begin{cases} \tilde{z} & \textit{if } x \geq \mathcal{H}(0), \\ I\big(\mathcal{Y}(x)\xi(T)\big) & \textit{else }, \end{cases} \tag{3.13}$$

where $\mathcal{Y} : (0, \mathcal{H}(0)) \to (0, \infty)$ denotes the inverse of $\mathcal{H}$. Then, there exists a portfolio process $\pi \in \mathcal{A}_2(x)$ with corresponding wealth process $\{X^\pi, 0 \leq t \leq T\}$ such that

$$X^\pi(T) = \Psi, \quad \textit{a.s.},$$

and X^π solves Problem 3.2.3, i.e. $V(x) = \mathbb{E}^{\mathbb{P}}[U(X^\pi(T))]$.

Proof. Case 1: $x \geq \mathcal{H}(0)$

We know that U achieves its maximum at $\tilde{z}$, as it is the only extremum of a strictly concave function. Hence

$$U(\tilde{z}) \geq U(X^\pi(T)), \quad \text{a.s.},$$

for every $\pi \in \mathcal{A}_2(x)$. Therefore, $X^\pi(T) = \tilde{z}$ is optimal for the constrained portfolio, for $x \geq \mathcal{H}(0)$. Notice further that $X^\pi(T)$ is in this case deterministic, which for $x = \mathcal{H}(0)$ leads to

$$\mathbb{E}^{\mathbb{P}}\big[U(X^\pi(T))^-\big] = U(\tilde{z})^- < \infty.$$

The existence of a portfolio process $\pi \in \mathcal{A}_2(x)$ and a corresponding wealth process with $X^\pi(T) = \Psi$ follows from Theorem 3.1.4.
Case 2: $x < \mathcal{H}(0)$

By construction, there is exactly one number $y = \mathcal{Y}(x)$ such that $\mathcal{H}(y) = x$. Hence, $\mathbb{E}^{\mathbb{P}}[\xi(T)\Psi] = x$ and by Theorem 3.1.4, there exists a unique admissible portfolio process $\pi \in \mathcal{A}(x)$ such that $X^\pi(T) = \Psi$. Furthermore, for any utility function, we have for all $y \in (0, \infty)$

$$U(I(y)) \geq U(c) + y(I(y) - c), \qquad \text{for all } c \geq 0.$$

Hence, for any other wealth process $\tilde{X}$, we have

$$\mathbb{E}^{\mathbb{P}}\big[U(X^\pi(T))\big] \geq \mathbb{E}^{\mathbb{P}}\big[U(\tilde{X}(T))\big] + \mathcal{Y}(x)\Big(x - \mathbb{E}^{\mathbb{P}}[\xi(T)\tilde{X}(T)]\Big)$$

as the term inside the brackets is positive, we are finished. □

3.3 Solution to the Constrained Problem

Returning to Problem 3.1.2 and especially equation (3.2) we now have the tools to show that a solution exists and to characterize the corresponding optimal terminal wealth.

Proposition 3.3.1. *Let the initial wealth $x > 0$ and assume the deterministic mean-variance trade-off of Assumption 3.1.7. Then, there exists a portfolio process $\pi \in \mathcal{A}(x)$ that solves Problem 3.1.2. The corresponding optimal terminal wealth $X^\pi(T)$ is given by*

$$X^\pi(T) = \begin{cases} C & \text{if } x \geq \mathbb{E}^{\mathbb{P}}[\xi(T)C], \\ \big(C - \mathcal{Y}(x)\xi(T)\big)^+ & \text{else.} \end{cases} \tag{3.14}$$

Proof. As the preference ordering derived from utility functions is invariant under affine transformations, we choose as utility function

$$U(x) = -\frac{1}{2}(C - x)^2.$$

Then $U'(x) = C - x$ and $\hat{I}(y) = C - y$ for $y \in [0, U'(0)]$. As in (3.6) we define the truncated inverse function of U' by

$$I(y) = \begin{cases} C - y & \text{if } y \leq C, \\ 0 & \text{else.} \end{cases}$$

Optimization Problem 3.1.2 reads as

$$\inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[(C - X^{\pi}(T))^2\big].$$

Hence, it is equivalent to the constrained portfolio problem

$$\sup_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[U(X^{\pi}(T))\big],$$

and we may proceed with the help of our findings in Section 3.2. Define $\mathcal{H}(y) = \mathbb{E}^{\mathbb{P}}[\xi(T)I(y\xi(T))] = \mathbb{E}^{\mathbb{P}}[\xi(T)\big(C - y\xi(T)\big)^+]$ which has a strictly decreasing inverse function $\mathcal{Y}(x)$ on $(0, \mathcal{H}(0))$. We have $\mathcal{H}(0) = C\mathbb{E}^{\mathbb{P}}[\xi(T)]$. In the case of $x \geq \mathcal{H}(0)$, Theorem 3.2.5 implies

$$X^{\pi}(T) = C.$$

In the case of $x < \mathcal{H}(0)$, Theorem 3.2.5 implies

$$X^{\pi}(T) = I(\mathcal{Y}(x)\xi(T)) = \big(C - \mathcal{Y}(x)\xi(T)\big)^+.$$

□

Remark. Note that using the truncated inverse function from (3.6) assures non-negativity of the wealth process X^{π}. If we use the inverse function $\hat{I}(y) = C - y$ instead, we would obtain as optimal solution

$$\hat{X}(T) = \begin{cases} C & \text{if } x \geq \mathbb{E}^{\mathbb{P}}[\xi(T)C], \\ C - \hat{\mathcal{Y}}(x)\xi(T) & \text{else.} \end{cases} \tag{3.15}$$

It is clearly apparent that in this case, non-negativity of the wealth process is not guaranteed. For the unconstrained terminal wealth (3.15), we can express the function $\hat{\mathcal{Y}}(x)$ explicitly. Note that $\hat{\mathcal{H}}(y) = \mathbb{E}[\xi(T)(C - y\xi(T))]$, and hence

$$\hat{\mathcal{Y}}(x) = \frac{C\mathbb{E}^{\mathbb{P}}[\xi(T)] - x}{\mathbb{E}^{\mathbb{P}}[\xi(T)^2]}. \tag{3.16}$$

Unlike in the unconstrained portfolio problem, where the function $\hat{\mathcal{Y}}(x)$ can be calculated explicitly by (3.16), the function $\mathcal{Y}(x)$ must be calculated numerically. It will be shown in Proposition 3.4.11 how this can be done under Assumption 3.1.7.

In order to compare the optimal portfolio problem leading to non-negative terminal wealth to the unconstrained portfolio problem we need to formally define the probability of ruin, as well as the probability of ending with a higher terminal wealth than the strategy of only investing in the bank account.

Definition 3.3.2. The *ruin probability* is defined by

$$\mathbb{P}[X^\pi(T) < 0], \tag{3.17}$$

whereas the *probability of success* is defined by

$$\mathbb{P}[X^\pi(T) > xe^{\int_0^T r(t)dt}]. \tag{3.18}$$

Remark. It is clear, that the probability of ruin is zero for the constrained portfolio process. However, it is useful in order to quantify the advantage over the unconstrained process.

Example 3.3.3. In order to illustrate the effect of the restriction on non-negative terminal wealth, we plot the empirical distribution of the optimal terminal wealth for both the constrained process (3.14) as well as the unconstrained process (3.15).

For this example we suppose that all parameters are constant over time and that there is only one stock in the market. We set the market parameters as $r = 0.05$, $\mu = 0.08$ and $\sigma = 0.15$. The investor starts with an initial wealth $x = 1000$ and tries to reach $C = 5000$ over a time horizon of $T = 10$ years. In Figure 3.1 we plot the empirical terminal wealth distributions for 10'000 realizations. For both strategies, the target C acts as an upper bound of the terminal wealth. This follows directly from (3.14) and (3.15), as both $\xi(T)$ and $\mathcal{Y}(x)$, resp. $\hat{\mathcal{Y}}(x)$, are strictly positive functions. Therefore, the probability of reaching or overshooting the target C is zero. We also note, that even though the ruin probability of the constrained strategy is zero, the probability of ending up with very little money is fairly high.

In order to study the advantages and disadvantages more thoroughly, we report some statistics of the final wealth out of 10'000 simulations in Table 3.1. On average, the constrained strategy leads to a lower terminal wealth, as expected. From this we see that the no-ruin option has its price,

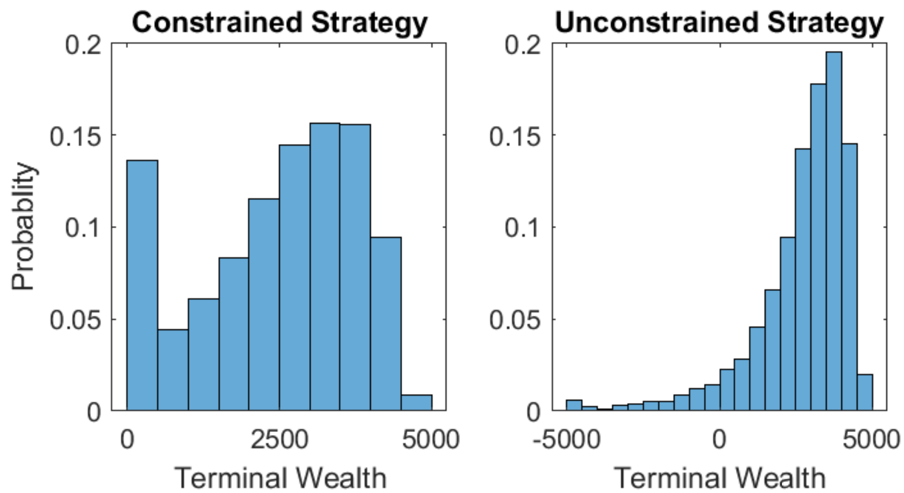


Figure 3.1: Histograms of the terminal wealth distribution for the constrained and unconstrained portfolio problem.

and in particular results in lower expected terminal wealth. On the other hand we see that the terminal distribution of the unconstrained strategy is much broader and the underlying risk is larger than for the constrained strategy. In Chapter 15 we will introduce a performance methodology which will enable us to compare the risks and the rate of return simultaneously, to decide when one of the strategies outperforms the other.

	Unconstrained	**Constrained**
2.5% Quantile	-1'560	0
Mean	2'710	2'450
97.5% Quantile	4'470	4'360
$\sqrt{\textbf{L2}}$**-Distance**	2'810	2'870
Median Rate of Return	11.4%	10.0%
Ruin Probability	5.7%	0.0%
Success Probability	82.9%	74.0%

Table 3.1: Properties of the empirical terminal wealth distribution for the constrained and unconstrained portfolio problem.

3.4 The Optimal Portfolio Process

In order to understand the effects of requiring the wealth process to be non-negative, we explicitly derive the optimal portfolio process and the corresponding wealth process with and without the non-negativity requirement. Without the additional requirement, we can use (3.15) and solve the stochastic differential equation (2.3) directly with the help of the Hamilton-Jacobi-Bellman equation. On the other hand, the optimal portfolio process leading to the non-negative payoff (3.14) can be seen as a put-option on a certain type of underlying, which we will use in order to find an explicit solution.

In order to explicitly derive the optimal portfolio processes we add an additional assumption to the market model.

Assumption 3.4.1. The interest rate process $r(t)$ and the risk premium process $\theta(t)$ are deterministic functions.

3.4.1 Without Bankruptcy Prohibition

To find an explicit form for the portfolio process $\hat{\pi}$ leading to the optimal terminal wealth $\hat{X}^{\hat{\pi}}(T)$ of (3.14) we use the Hamilton-Jacobi-Bellman (HJB) equation. The main idea is to expand the problem by not only considering a wealth process starting from $t = 0$, but from any $t \in [0, T]$. Hence for this section only, we denote the value of the wealth process at time t by $X^{\pi}(t) = y$. With this, the wealth process satisfies

$$\begin{aligned} dX^{\pi}(s) &= \Big(X^{\pi}(s)r(s) + \pi(s)'\big(\mu(s) - r(s)\mathbb{I}\big)\Big)ds + \pi(s)'\sigma(s)dW(s), \\ X^{\pi}(t) &= y, \end{aligned} \tag{3.19}$$

for $s \in [t, T]$. We give a summary of Theorem 14.5 from [Björk, 2007] in showing that the task of finding the optimal portfolio process is equivalent to finding a solution to the HJB equation. This approach is henceforth called the *dynamic programming approach.*

Theorem 3.4.2. *Extending the definition of the optimal value function from* (3.8)*, we write*

$$V(t, y) = \sup_{\pi \in \Pi} \mathbb{E}^{\mathbb{P}}\big[U(X^{\pi}(T))|X^{\pi}(t) = y\big].$$

Under Assumption 2.1.1 and assuming $V \in \mathcal{C}^{1,2}$ and $V < \infty$ the optimal *value function V satisfies the HJB equations given by*

$$\begin{aligned} U(y) =& V(T,y), \\ 0 =& V_t(t,y) + r(t)yV_y(t,y) \\ &+ \max_{\pi\in\Pi}\Big\{\pi(t)'\big(\mu(t)-r(t)\mathbb{I}\big)V_y(t,y) + \frac{1}{2}\pi(t)'\sigma(t)\sigma(t)'\pi(t)V_{yy}(t,y)\Big\}, \end{aligned} \tag{3.20}$$

where we introduced the notation $V_y(t,y) = \frac{\partial}{\partial y}V(t,y)$, $V_t(t,y) = \frac{\partial}{\partial t}V(t,y)$ and $V_{yy}(t,y) = \frac{\partial^2}{\partial y^2}V(t,y)$.

Proof. Let $\hat{\pi}$ be the optimal portfolio process and fix any admissible strategy π. For $h \in \mathbb{R}$ such that $t+h < T$, define

$$\pi^*(s) = \begin{cases} \hat{\pi}(s), & \text{if } s \in [t+h, T], \\ \pi(s), & \text{if } s \in [t, t+h). \end{cases}$$

It is clear by definition, that the value function of the strategy π^* is bounded from below by the optimal value function. If π^* takes the value $X^\pi(t)$ at time t to $X^\pi(t+h)$, the expected utility at maturity T is then

$$\mathbb{E}^{\mathbb{P}}\Big[U\big(X^{\pi^*}(T)\big)\Big|X^{\hat{\pi}}(t+h) = X^\pi(t+h)\Big] = V\big(t+h, X^\pi(t+h)\big),$$

and therefore the value function of π^* satisfies

$$\mathbb{E}^{\mathbb{P}}\Big[V(t+h, X^\pi(t+h))\Big|X^\pi(t) = y\Big] \le V(t,y). \tag{3.21}$$

Using Itô's formula and taking the conditional expectation we obtain

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\big[V(t+h, X^\pi(t+h))|X^\pi(t) = y\big] \\ &= V(t,y) + \mathbb{E}^{\mathbb{P}}\Big[\int_t^{t+h}\Big(V_t(s, X^\pi(s)) + r(s)X^\pi(s)V_y(s, X^\pi(s)) \\ &\qquad + \pi(s)'(\mu(s) - r(s)\mathbb{I}))V_y(s, X^\pi(s)) \\ &\qquad + \frac{1}{2}\pi(s)'\sigma(s)\sigma(s)'\pi(s)V_{yy}(s, X^\pi(s))\Big)ds\Big|X^\pi(t) = y\Big]. \end{aligned}$$

Now we divide by h and take the limit as $h \to 0$. By our assumptions on the coefficients and the regularity of V we can interchange limit and expectation. Hence by (3.21) we get

$$0 \ge V_t(t,y) + \big(r(t)y + \pi(t)'\big(\mu(t) - r(t)\mathbb{I}\big)\big)V_y(t,y) + \frac{1}{2}\pi(t)'\sigma(t)\sigma(t)'\pi(t)V_{yy}(t,y).$$

This inequality holds for all admissible strategies and equality holds if and only if $\pi = \hat{\pi}$. □

Remark. For a more general, $\mathbb{R}^m$-valued stochastic process, given by

$$\begin{aligned} dX^\pi(t) &= \mu(t, X^\pi(t), \pi(t))dt + \sigma(t, X^\pi(t), \pi(t))dW(t), \\ X(t) &= x, \end{aligned}$$

one can generalize Theorem 3.4.2, so that under the same assumptions V satisfies the HJB equation

$$\begin{aligned} U(y) &= V(T, y), \\ 0 &= V_t(t, x) + \max_{\pi \in \Pi} H^\pi V(t, x), \qquad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^m. \end{aligned} \tag{3.22}$$

Here, for any $\pi \in \Pi$, H^π is the partial differential operator defined by

$$H^\pi = \sum_{i=1}^m \mu(t, x, \pi) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m \sigma(t, X^\pi(t), \pi(t))\sigma(t, X^\pi(t), \pi(t))' \frac{\partial^2}{\partial x_i \partial x_j}. \tag{3.23}$$

Note that we will use the partial differential operator (3.23) in Section 9.2, as the presence of an additional stochastic factor in the wealth equation makes the HJB equation given by (3.20) unusable.

For more information on the Hamilton-Jacobi-Bellman equation and the derivation of the more general equation (3.22), we refer to [Björk, 2007], [Fleming and Soner, 2006] and [Yong and Zhou, 1999].

By Theorem 3.4.2, the optimal portfolio process $\hat{\pi}$ needs to satisfy the first order conditions

$$\begin{aligned} 0 &= (\mu(t) - r(t)\mathbb{I})V_y(t, y) + \sigma(t)\sigma(t)'\hat{\pi}(t)V_{yy}(t, y), \\ 0 &> \sigma(t)\sigma(t)'V_{yy}(t, y), \end{aligned}$$

and is of the form

$$\hat{\pi}(t) = -\big(\sigma(t)\sigma(t)'\big)^{-1}\big(\mu(t) - r(t)\mathbb{I}\big)\frac{V_y(t, y)}{V_{yy}(t, y)} \tag{3.24}$$

Recall the notation for the risk premium process $\theta(t) = \sigma(t)^{-1}\big(\mu(t) - r(t)\mathbb{I}\big)$. Inserting (3.24) into the HJB equation (3.20), we have

$$0 = V_t(t, y) + r(t)yV_y(t, y) - \frac{1}{2}\|\theta(t)\|^2 \frac{V_y(t, y)^2}{V_{yy}(t, y)}. \tag{3.25}$$

Theorem 3.4.3. *In the financial market* (2.2) *and under Assumptions 2.1.1 and 3.4.1, the optimal portfolio process for Problem 3.1.2 is given by*

$$\hat{\pi}(t) = -\big(\sigma(t)\sigma(t)'\big)^{-1}\big(\mu(t) - r(t)\mathbb{I}\big)\big(\hat{X}^{\hat{\pi}}(t) + h(t)\big), \tag{3.26}$$

where $h(t) = -C\frac{\beta(T)}{\beta(t)}$.

To prove this theorem we need to first find a solution $V(t,y)$ to the HJB equation (3.25). This will be given in the following lemma.

Lemma 3.4.4. *The optimal value function*

$$V(t,y) = e^{-\int_t^T \left(\|\theta(s)\|^2\right)ds}\Big(-\frac{1}{2}\big(y\frac{\beta(t)}{\beta(T)}\big)^2 + Cy\frac{\beta(t)}{\beta(T)} - \frac{1}{2}C^2\Big), \tag{3.27}$$

is a solution to the HJB equation (3.25).

Proof. Assume the optimal value function to be of quadratic form, i.e. $V(t,y) = a(t)y^2 + b(t)y + c(t)$ for some functions $a(t)$, $b(t)$ and $c(t)$ with initial values $a(T) = -\frac{1}{2}$, $b(T) = C$ and $c(T) = -\frac{1}{2}C^2$. Then

$$\begin{aligned} V_y(t,y) &= 2a(t)y + b(t), \\ V_{yy}(t,y) &= 2a(t), \\ V_t(t,y) &= a_t(t)y^2 + b_t(t)y + c_t(t). \end{aligned}$$

The HJB equation then becomes

$$\begin{aligned} 0 =& a_t(t)y^2 + b_t(t)y + c_t(t) \\ &+ 2r(t)a(t)y^2 + r(t)b(t)y - \frac{1}{2}\|\theta(t)\|^2\Big(2a(t)y^2 + 2b(t)y + \frac{b(t)^2}{2a(t)}\Big). \end{aligned}$$

For this to be equal zero, we need to eliminate the dependencies on y. Hence we obtain three differential equations

$$\begin{aligned} &a_t(t) + 2r(t)a(t) - \|\theta(t)\|^2 a(t) = 0, \\ &a(T) = -\frac{1}{2}, \end{aligned}$$

$$\begin{aligned} &b_t(t) + r(t)b(t) - \|\theta(t)\|^2 b(t) = 0, \\ &b(T) = C, \end{aligned}$$

$$\begin{aligned} &c_t(t) - \frac{1}{2}\|\theta(t)\|^2\frac{b(t)^2}{2a(t)} = 0, \\ &c(T) = -\frac{1}{2}C^2. \end{aligned}$$

Either solving these differential equations or inserting

$$\begin{aligned} a(t) &= -\frac{1}{2} e^{\int_t^T \left(2r(s) - \|\theta(s)\|^2\right) ds}, \\ b(t) &= C e^{\int_t^T \left(r(s) - \|\theta(s)\|^2\right) ds}, \\ c(t) &= -\frac{1}{2} C^2 e^{-\int_t^T \left(\|\theta(s)\|^2\right) ds}, \end{aligned}$$

directly into the HJB equation solves the lemma. □

In order to prove Theorem 3.4.3, we return to the notation of (2.3) by replacing the starting point y of the optimal wealth process (3.19) by its value $\hat{X}^{\hat{\pi}}(t)$.

Proof of Theorem 3.4.3. Recall that

$$\hat{\pi}(t) = -\big(\sigma(t)\sigma(t)'\big)^{-1}\big(\mu(t) - r(t)\mathbb{I}\big)\frac{V_y(t, \hat{X}^{\hat{\pi}}(t))}{V_{yy}(t, \hat{X}^{\hat{\pi}}(t))}$$

and with the help of Lemma 3.4.4, this is

$$\hat{\pi}(t) = -\big(\sigma(t)\sigma(t)'\big)^{-1}\big(\mu(t) - r(t)\mathbb{I}\big)\big(\hat{X}^{\hat{\pi}}(t) + \frac{b(t)}{2a(t)}\big).$$

Define $h(t) = \frac{b(t)}{2a(t)}$, with initial value $h(T) = -C$. Then $h_t(t) - r(t)h(t) = 0$ and

$$h(t) = \frac{b(t)}{2a(t)} = -C\frac{\beta(T)}{\beta(t)}.$$

□

Theorem 3.4.5. *Under the same assumptions as Theorem 3.4.3 and for $x < C\beta(T)$, the optimal wealth process to the unconstrained optimization problem is given by*

$$\hat{X}^{\hat{\pi}}(t) = \big(x - C\beta(T)\big)e^{\int_0^t \left(r(s) - \frac{3}{2}\|\theta(s)\|^2\right) ds - \int_0^t \theta(s)' dW(s)} + C\frac{\beta(T)}{\beta(t)}. \qquad (3.28)$$

for all $t \in [0, T]$.

Proof. To derive an explicit form for the wealth process $\hat{X}^{\hat{\pi}}$, we insert the optimal portfolio process from (3.26), into (2.3), which becomes

$$\begin{aligned} d\hat{X}^{\hat{\pi}}(t) &= \Big(r(t)\hat{X}^{\hat{\pi}}(t) - \|\theta(t)\|^2\frac{V_y(t, \hat{X}^{\hat{\pi}}(t))}{V_{yy}(t, \hat{X}^{\hat{\pi}}(t))}\Big)dt - \theta(t)'\frac{V_y(t, \hat{X}^{\hat{\pi}}(t))}{V_{yy}(t, \hat{X}^{\hat{\pi}}(t))}dW(t) \\ &= \Big(r(t)\hat{X}^{\hat{\pi}}(t) - \|\theta(t)\|^2 Z(t)\Big)dt - \theta(t)'Z(t)dW(t), \end{aligned}$$

where we define the auxiliary process $Z(t) = \hat{X}^{\hat{\pi}}(t) + h(t)$ with initial value $Z(0) = x - Ce^{-\int_0^T r(s)ds}$. By Itô's lemma we obtain

$$dZ(t) = \Big(Z(t)\big(r(t) - \|\theta(t)\|^2\big)\Big)dt - \theta(t)'Z(t)dW(t),$$

which is the expression for a geometric Brownian motion, with solution

$$Z(t) = Z(0)e^{\int_0^t \big(r(s) - \frac{3}{2}\|\theta(s)\|^2\big)ds - \int_0^t \theta(s)'dW(s)}.$$

Hence, we can write the optimal wealth process corresponding to the portfolio process $\hat{\pi}$ of (3.26) as

$$\hat{X}^{\hat{\pi}}(t) = \big(x - C\beta(T)\big)e^{\int_0^t \big(r(s) - \frac{3}{2}\|\theta(s)\|^2\big)ds - \int_0^t \theta(s)'dW(s)} + C\frac{\beta(T)}{\beta(t)}.$$

□

Recall the expression for the state price deflator

$$\xi(t) = \exp\Big(- \int_0^t \theta'(s)dW(s) - \int_0^t (r(s) + \frac{1}{2}\|\theta(s)\|^2)ds\Big).$$

Now, for $t = T$ we obtain

$$\begin{aligned} \hat{X}^{\hat{\pi}}(T) &= \frac{\xi(T)e^{-\int_0^T \|\theta(t)\|^2 dt}}{\beta(T)^2}\big(x - C\beta(T)\big) + C \\ &= C - \hat{\mathcal{Y}}(x)\xi(T), \end{aligned} \tag{3.29}$$

for

$$\hat{\mathcal{Y}}(x) = \big(\frac{C}{\beta(T)} - \frac{x}{\beta(T)^2}\big)e^{-\int_0^T \|\theta(t)\|^2 dt} = \frac{C\mathbb{E}[\xi(T)] - x}{\mathbb{E}[\xi(T)^2]},$$

which is exactly the expression (3.16).

Remark. Note that this is the same inverse function $\hat{\mathcal{Y}}$ as in [Schweizer, 1997], which discusses much more general L^2-approximation of random variables. Even though Schweizer's approach cannot guarantee non-negativity, it applies to much broader settings than the ones treated in this thesis, as noted in [Korn, 1997].

3.4.2 With Bankruptcy Prohibition

To find the optimizing portfolio in (3.14) for the case of $x < C\beta(T)$, it suffices to find a portfolio process $\hat{\pi}$ with a corresponding wealth process $X^{\hat{\pi}}$ satisfying

$$\begin{aligned} dX^{\hat{\pi}}(t) &= \Big(X^{\hat{\pi}}(t)r(t) + \pi(t)'\big(\mu(t) - r(t)\mathbb{I}\big)\Big)dt + \pi(t)'\sigma(t)dW(t), \\ X^{\hat{\pi}}(T) &= (C - \mathcal{Y}(x)\xi(T))^{+}. \end{aligned}$$

In general, one is not able to express $X^{\hat{\pi}}$ and $\hat{\pi}$ in closed form. However, adapting the approach of [Bielecki et al., 2005] to the case of a quadratic utility function, we will show the existence of an explicit solution if the market coefficients are deterministic, albeit possibly time-varying. The approach taken is henceforth called the *martingale approach*.

Theorem 3.4.6 (Law of One Price)**.** *Two wealth processes with the same value, $\mathbb{P}$-a.s., at some point in the future must have the same value today.*

Proof. This immediately follows from (3.1), since if two wealth processes $\big(Y(t)\big)_{t\in[0,T]}$ and $\big(X(t)\big)_{t\in[0,T]}$ have the same terminal value $X(T) = Y(T) = \Psi$, $\mathbb{P}$-a.s., we also have

$$X(t) = Y(t) = \xi(t)^{-1}\mathbb{E}^{\mathbb{P}}\big[\xi(T)\Psi\big|\mathcal{F}_t\big].$$

□

Therefore we reduce the problem to finding a replicating portfolio process π, which yields the same terminal value as the optimal portfolio process $\hat{\pi}$. By the Law of One Price, the wealth processes of the two portfolio processes will then have the same value for each $t \in [0, T]$.

Proposition 3.4.7. *The unique optimizing portfolio $\hat{\pi}$ for Problem 3.1.2 corresponding to the case $x < C\beta(T)$ is a replicating portfolio for a European put option written on the asset $\mathcal{Y}(x)\xi(t)$ with strike price C and time to maturity T.*

Proof. This follows immediately from the First Fundamental Theorem of Asset Pricing, the Law of One Price and Proposition 3.3.1. □

Theorem 3.4.8. *Under Assumptions 2.1.1, 3.1.7 and 3.4.1, the optimal wealth process to Problem 3.1.2 is given by*

$$X^{\hat{\pi}}(t) = C\Phi(-d_{-}(t, y(t)))\frac{\beta(T)}{\beta(t)} - \Phi(-d_{+}(t, y(t)))y(t), \tag{3.30}$$

for all $t \in [0,T]$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{v^2}{2}} dv$ is the cumulative distribution function of the standard normal distribution and

$$d_+(t,y) = \frac{\log(\frac{y}{C}) + \int_t^T (r(s) + \frac{1}{2}\|\theta(s)\|^2) ds}{\sqrt{\int_t^T \|\theta(s)\|^2 ds}},$$
$$d_-(t,y) = d_+(t,y) - \sqrt{\int_t^T \|\theta(s)\|^2 ds}.$$

Furthermore, the process y is given by

$$y(t) = \mathcal{Y}(x) \exp\Big(- \int_0^T (2r(s) - \|\theta(s)\|^2) ds\Big)$$
$$\exp\Big(\int_0^t (r(s) - \frac{3}{2}\|\theta(s)\|^2) ds - \int_0^t \theta(s)' dW(s)\Big). \tag{3.31}$$

Proof. Consider y as given above. By applying Itô's formula, y satisfies

$$dy(t) = y(t)\Big(\big(r(t) - \|\theta(t)\|^2\big) dt - \theta(t)' dW(t)\Big),$$
$$y(0) = \mathcal{Y}(x) \exp\Big(- \int_0^T \big(2r(s) - \|\theta(s)\|^2\big) ds\Big), \quad y(T) = \mathcal{Y}(x)\xi(T).$$

Then, by Proposition 3.4.7 and again by the Law of One Price, the optimizing portfolio corresponding to $x < C\beta(T)$ is a replicating portfolio for a European put option written on y with strike C and time to maturity T.

Now let $X^{\hat{\pi}}(t) = f(t, y(t))$ for some function $f \in \mathcal{C}^{1,2}$. Applying Itô's formula to $f(t, y(t))$ we obtain

$$df(t,y) = \Big(y f_y(t,y)\big(r(t) - \|\theta(t)\|^2\big) + f_t(t,y) + \frac{1}{2} y^2 f_{yy}(t,y)\|\theta(t)\|^2\Big) dt$$
$$+ y f_y(t,y)\theta(t)' dW(t),$$

and comparing the drift and diffusion terms to (2.3) we have

$$\hat{\pi}(t) = -\big(\sigma(t)\sigma(t)'\big)^{-1}(\mu(t) - r(t)\mathbb{I})\big(f_y(t, y(t)))y(t)\big), \tag{3.32}$$

and

$$f_t(t,y) + r(t) y f_y(t,y) + \frac{1}{2}\|\theta(t)\|^2 y^2 f_{yy}(t,y) - r(t) f(t,y) = 0,$$
$$f(T,y) = \big(C - y\big)^+. \tag{3.33}$$

But (3.33) is exactly the Black Scholes equation for a European put option and hence allows for the explicit solution

$$f(t,y) = C\Phi(-d_-(t,y))\frac{\beta(T)}{\beta(t)} - \Phi(-d_+(t,y))y. \tag{3.34}$$

□

Corollary 3.4.9. *Under the same notations and assumptions as in Theorem 3.4.8, the optimal portfolio process to Problem 3.1.2 is given by*

$$\hat{\pi}(t) = \Phi(-d_+(t,y(t)))(\sigma(t)\sigma(t)')^{-1}(\mu(t) - r(t)\mathbb{I})y(t) \tag{3.35}$$

$$= -(\sigma(t)\sigma(t)')^{-1}(\mu(t) - r(t)\mathbb{I})\Big(X^{\hat{\pi}}(t) - C\frac{\beta(T)}{\beta(t)}\Phi(-d_-(t,y(t)))\Big). \tag{3.36}$$

Proof. For notational purposes we define the process $a(t) = \sqrt{\int_t^T \|\theta(s)\|^2 ds}$. Note, that (3.36) follows immediately from (3.35) by using $X^{\hat{\pi}}(t) = f(t,y(t))$ in (3.34). To prove (3.35) we start from (3.32) and only need to show that $f_y(t,y) = -\Phi(-d_+(t,y))$. We have

$$\frac{\partial d_+(t,y)}{\partial y} = \frac{1}{ya(t)} = \frac{\partial d_-(t,y)}{\partial y},$$

and, defining $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ for all $x \in \mathbb{R}$,

$$f_y(t,y) = \frac{1}{ya(t)}\Big(y\phi(-d_+(t,y)) - C\frac{\beta(T)}{\beta(t)}\phi(-d_-(t,y)\Big) - \Phi(-d_+(t,y)).$$

Now

$$\begin{aligned}
C\frac{\beta(T)}{\beta(t)}\phi(-d_-(t,y)) &= C\frac{\beta(T)}{\beta(t)}\phi(-d_+(t,y) + a(t)) \\
&= C\frac{\beta(T)}{\beta(t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+(t,y)^2}{2}}e^{d_+(t,y)a(t) - \frac{1}{2}a(t)^2} \\
&= C\phi(d_+(t,y))e^{\log(\frac{y}{C})} \\
&= y\phi(d_+(t,y).
\end{aligned}$$

And hence by symmetry of ϕ we have shown that

$$f_y(t,y) = -\Phi(-d_+(t,y)).$$

□

Remark. Note that if there is only one stock available in the market, and if $x < C\beta(T)$, both the constrained optimal portfolio (3.36) and the unconstrained optimal portfolio (3.26) include a positive investment in the stock, as long as $\mu(t) > r(t)$. Therefore, no shorting of the stock occurs. On the other hand, it is possible that $\hat{\pi}(t) > 1$, i.e. money is borrowed from the bank account in order to finance the investment in the stock.

If a second stock is introduced to the market, both the constrained and the unconstrained optimal portfolio may require going short in one of the stocks. As this might not be possible in reality, due to regulatory restrictions, an additional no-shorting constraint could be added to the Problem 3.1.2. This new portfolio problem has been analyzed in [Heunis, 2014] and [Liang and Sheng, 2015].

Example 3.4.10. To gain a general impression of the behavior of the portfolio processes and the correlation between the underlying stock price and the amount of wealth invested, we calibrate the optimal portfolio processes for a specific market scenario. We assume the only stock to be traded in the market to be the FTSE-Actuaries All Share Index and calculate the optimal portfolio strategies for a investment horizon of ten years.

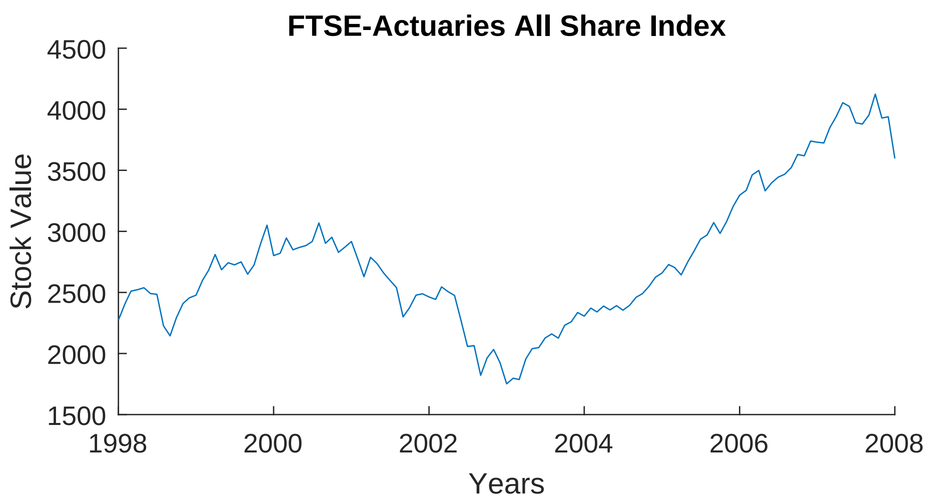


Figure 3.2: Monthly historical time series of the FTSE-Actuaries All Share Index between January 1998 and December 2007.

The period between 1998 and 2008 was chosen to capture the dot-com bubble and the subsequent effects of the September 11 attacks in 2001, which led to a drop of the FTSE-Actuaries All Share Index of over 40%. We

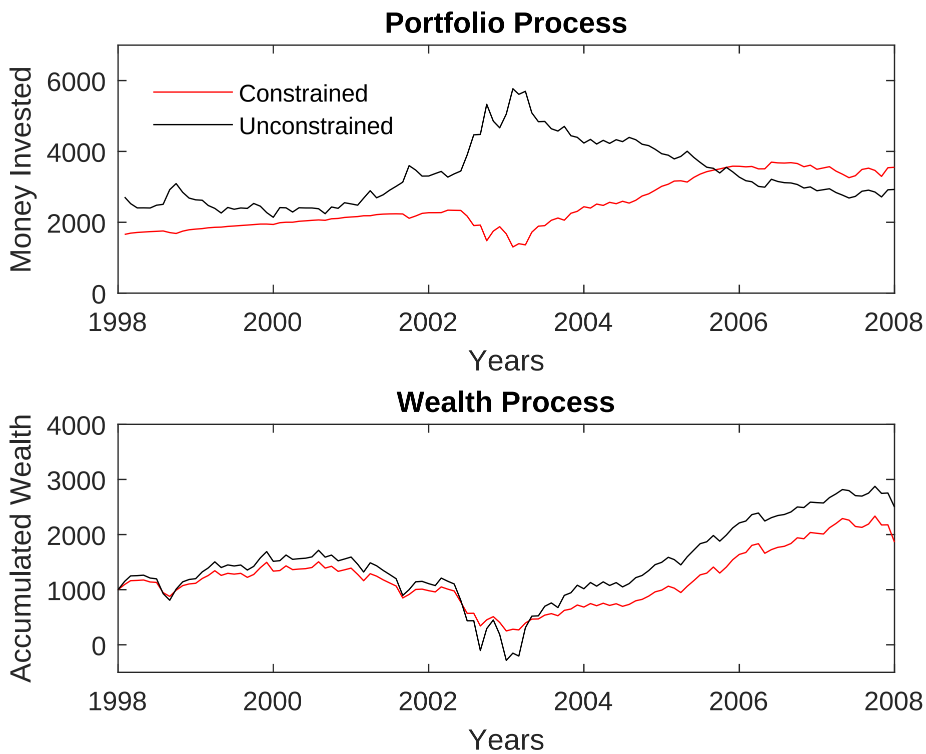


Figure 3.3: In the upper graph, the amount invested in the stock can be seen for both the constrained and the unconstrained optimal portfolio. The resulting wealth process is plotted below.

assume the underlying parameters of the stock process to be constants and calculate them empirically. We obtain $r = 5\%$, $\mu = 8\%$ and $\sigma = 0.15$. The investor starts with $x = 1000$ and the target wealth is set to $C = 5000$.

It can be observed in Figure 3.3, that there is some negative correlation between the stock process and the unconstrained portfolio process. During the bear market of 2000 to 2002, the amount invested in the stock rises quickly. By looking at the equation (3.26) for the portfolio process, we see that this correlation is in fact between the current and the target wealth. During periods where the stock price falls, the wealth falls as well and in turn, the distance to the target wealth increases. This leads to more willingness to take on risk and higher investment in the stock.

This negative correlation also exists for the constrained portfolio problem. However, the amount invested depends not directly on the discounted target wealth, but rather on $C\frac{\beta(T)}{\beta(t)}\Phi(-d_{-}(t, y(t)))$, as seen in (3.36). The additional factor $\Phi(-d_-(t, y(t)))$ acts as a "cap" to the amount invested and guarantees the non-negativity of the wealth process.

We also observe that while the constrained wealth process comes close to zero in 2002, the unconstrained wealth process actually becomes negative. Due to the recovery of the market and heavy borrowing, the unrestricted portfolio process still manages to outperform the restricted process in the end. In practice, the investor might not have been able to continue investing after bankruptcy.

We have represented both the portfolio process and the corresponding wealth process explicitly in terms of the function $\mathcal{Y}(x)$. Unlike the case where we do not exclude bankruptcy, this function can only be found numerically.

Proposition 3.4.11. *If $x < C\beta(T)$ and if the Assumptions 2.1.1, 3.1.7 and 3.4.1 are satisfied, $\mathcal{Y}(x)$ is the unique solution to the equation*

$$\begin{aligned} xe^{\int_0^T r(t)dt} &= C\Phi\Big(\frac{\log(\frac{C}{\mathcal{Y}(x)}) + \int_0^T \big(r(t) - \frac{1}{2}\|\theta(t)\|^2\big)dt}{\sqrt{\int_0^T \|\theta(t)\|^2 dt}}\Big) \\ &\quad - \mathcal{Y}(x)e^{-\int_0^T \big(r(t) - \|\theta(t)\|^2\big)dt}\Phi\Big(\frac{\log(\frac{C}{\mathcal{Y}(x)}) + \int_0^T \big(r(t) - \frac{3}{2}\|\theta(t)\|^2\big)dt}{\sqrt{\int_0^T \|\theta(t)\|^2 dt}}\Big). \end{aligned} \tag{3.37}$$

Proof. In the proof of Theorem 3.4.8 we saw, that the optimal wealth process satisfies, $x = X^{\hat{\pi}}(0) = f(0, y(0))$, for y and f given in (3.31) and (3.34) respectively. Hence

$$x = C\Phi\big(-d_-(0, y(0))\big)\beta(T) - \Phi\big(-d_+(0, y(0))\big)y(0).$$

As $y(0)$ is given by $y(0) = \mathcal{Y}(x)\exp\Big(-\int_0^T \big(2r(s) - \|\theta(s)\|^2\big)ds\Big)$, calculating $d_-(0, y(0))$ and $d_+(0, y(0))$ explicitly, yields the claim. □

Chapter 4

The Constrained Optimal Strategy: Lower Bound

As a next step we introduce a lower constraint to limit the terminal wealth from below. We will see that this problem is again closely related to the optimization problem of maximizing expected utility from terminal wealth when terminal wealth is bounded from below. We retain all notation from Chapter 3.

4.1 Problem Formulation

Problem 4.1.1. Given a constant C and a positive real number K, we consider the problem of finding a portfolio process $\hat{\pi}^l \in \mathcal{A}(x)$ such that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\big[(C - X^{\hat{\pi}^l}(T))^2\big] &= \inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[(C - X^{\pi}(T))^2\big], \\ \text{subject to} \qquad & X^{\hat{\pi}^l}(T) \geq K, \quad \text{a.s.,} \end{aligned} \tag{4.1}$$

and the pair $\big(X^{\hat{\pi}^l}(t), \hat{\pi}^l(t)\big)$ satisfies the stochastic differential equation (2.3).

Remark.

- Note that under the restriction that $X^{\hat{\pi}^l}(T) \geq K$ a.s., the non-negativity constraint is non-binding, due to Proposition 3.1.1.
- By Lemma 3.1.3 we know that the process $\beta(t)X^{\pi}(t)$ is a supermartingale under $\mathbb{Q}$ and needs to satisfy

$$\mathbb{E}^{\mathbb{Q}}\big[\beta(T)X^{\pi}(T)\big] \leq x.$$

This budget constraint implies, that the class of optimal portfolio processes for Problem 4.1.1 is empty unless we have $\beta(T)K \le x$. The case of $\beta(T)K = x$ is also trivial, as then the only optimal portfolio process is $\hat{\pi}^l \equiv 0$, i.e. everything is invested into the bank account. For the rest of this chapter, we will hence assume that

$$\beta(T)K < x. \tag{4.2}$$

- Define the stopping time

$$\tau(\pi) = \inf_{0\le t\le T}\{\beta(t)X^\pi(t) = \beta(T)K\}.$$

As soon as the wealth process $X^\pi(t)$ hits the boundary $Ke^{-\int_t^T r(s)ds}$, the only viable process is to invest everything in the bank account, i.e.

$$\pi(s) = 0, \quad s \in [\tau, T], \text{ a.s.},$$
$$X^\pi(s) = K\frac{\beta(T)}{\beta(s)}, \quad s \in [\tau, T], \text{ a.s.}$$

4.2 Solution of the Constrained Problem

To find the optimal wealth process for the constraint problem, we adapt [Grossman and Zhou, 1996, Lemma 2] for a quadratic utility function. We will see that the optimal terminal wealth consists of an unconstrained wealth process (which is allowed to take on negative values) plus a put option which is in-the-money if the constrained wealth process dips below K. Finally, the purchase of this put option is financed by using a lower initial wealth for the constrained wealth process.

We first proof the general case of finding a solution to Problem 3.2.3 under an additional lower constraint for the terminal wealth. Note that the admissibility constraint (3.3) still needs to hold for the optimal wealth process $X^{\hat{\pi}}(t)$.

Proposition 4.2.1. *The solution to*

$$\sup_{\pi\in\mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[U(X^\pi(t))\big] \quad \textit{subject to,} \quad X^\pi(T) \ge K, \ \textit{a.s.} \tag{4.3}$$

is given by the following expression for the optimal terminal wealth

$$X^{\hat{\pi}}(T) = \max\big(K, I(y\xi(T))\big), \tag{4.4}$$

where y is chosen in such a way that the terminal wealth given by (4.4) satisfies the admissibility constraint $\mathbb{E}^{\mathbb{P}}[\xi(T)X^{\hat{\pi}}(T)] = x$.

Proof. Thanks to the assumption in (4.2) there is at least one attainable terminal wealth $X^\pi(T)$ with an admissible portfolio process π such that $X^\pi(T) \geq K$ and $\mathbb{E}^{\mathbb{P}}[\xi(T)X^\pi(T)] = x$. Using that for the concave function U we have for any a and b, $U(a) - U(b) \leq U'(b)(b-a)$, we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[U(X^\pi(T))] - \mathbb{E}^{\mathbb{P}}[U(X^{\hat\pi}(T)] \\
&\qquad \leq \mathbb{E}^{\mathbb{P}}[U'(X^{\hat\pi}(T))\big(X^\pi(T) - X^{\hat\pi}(T)\big)] \\
&\qquad = \mathbb{E}^{\mathbb{P}}[U'(X^{\hat\pi}(T))\big(X^\pi(T) - X^{\hat\pi}(T)\big)|X^{\hat\pi}(T) > K]\mathbb{P}[X^{\hat\pi}(T) > K] \\
&\qquad + \mathbb{E}^{\mathbb{P}}[U'(X^{\hat\pi}(T))\big(X^\pi(T) - X^{\hat\pi}(T)\big)|X^{\hat\pi}(T) \leq K]\mathbb{P}[X^{\hat\pi}(T) \leq K].
\end{aligned}$$

Now, in the event $X^{\hat\pi}(T) > K$, we have that $U'(X^{\hat\pi}(T)) = y\xi(T)$, whereas in the event $X^{\hat\pi}(T) \leq K$, we have that $X^\pi(T) - X^{\hat\pi}(T) \geq 0$ and that $U'(X^{\hat\pi}(T)) \leq y\xi(T)$ as U' is a decreasing function. Therefore

$$\mathbb{E}^{\mathbb{P}}[U(X^\pi(T))] - \mathbb{E}^{\mathbb{P}}[U(X^{\hat\pi}(T)] \leq y\mathbb{E}^{\mathbb{P}}[\xi(T)\big(X^\pi(T) - X^{\hat\pi}(T)\big)] = 0.$$

So, $\mathbb{E}^{\mathbb{P}}[U(X^\pi(T))] \leq \mathbb{E}^{\mathbb{P}}[U(X^{\hat\pi}(T))]$ for all admissible strategies π. □

Corollary 4.2.2. *For Problem 4.1.1, the optimal terminal wealth is of the form*

$$X^{\hat\pi^l}(T) = \hat X^{\hat\pi}(T) + \big(K - \hat X^{\hat\pi}(T)\big)^+, \tag{4.5}$$

where $\hat X^{\hat\pi}(t)$ is the optimal wealth process from (3.28) *with*

$$\hat x_0 = C\mathbb{E}^{\mathbb{P}}[\xi(T)] - y\mathbb{E}^{\mathbb{P}}[\xi(T)^2],$$

where y is chosen in such a way that the terminal wealth given by (4.5) *satisfies the admissibility constraint $\mathbb{E}^{\mathbb{P}}[X^{\hat\pi^l}(T)\xi(T)] = x$.*

Proof. Let $\hat I(y) = C - y$ be the inverse of $U'(x)$. Then by Proposition 4.2.1,

$$X^{\hat\pi^l}(T) = \hat I(y\xi(T)) + \max\{K - \hat I(y\xi(T)), 0\}.$$

Hence, determine $\hat x_0$ such that $\hat I(y\xi(T)) = \hat X^{\hat\pi}(T)$, which is the case if

$$\begin{aligned}
&\hat I(y\xi(T)) = C - y\xi(T) = C - \frac{C\mathbb{E}^{\mathbb{P}}[\xi(T)] - \hat x_0}{\mathbb{E}^{\mathbb{P}}[\xi(T)^2]}\xi(T) \\
&\iff y\mathbb{E}^{\mathbb{P}}[\xi(T)^2] = C\mathbb{E}^{\mathbb{P}}[\xi(T)] - \hat x_0
\end{aligned}$$

and finally y is chosen such that the budget constraint $\mathbb{E}^{\mathbb{P}}[X^{\hat\pi^l}(T)\xi(T)] = x$ is fulfilled. □

4.3 The Optimal Portfolio Process

Using (4.5) we see that the optimal portfolio process corresponding to Problem 4.1 can be solved quickly with the help of the optimal portfolio process (3.36). Namely, rewriting (4.5) with the help of the findings in Section 3.3, we have

$$X^{\hat{\pi}^l}(T) = \begin{cases} C & \text{if } x \geq C\beta(T), \\ \max\big(C - y\xi(T), K\big) & \text{else .} \end{cases} \tag{4.6}$$

where y is chosen such that

$$\mathbb{E}^{\mathbb{P}}[\xi(T)\big((C-K) - y\xi(T)\big)^+] = x - K\mathbb{E}^{\mathbb{P}}[\xi(T)].$$

Hence, we state the optimal wealth process as well as the optimal portfolio process in terms of the optimal processes under the non-negativity constraint. Define

$$\begin{aligned} \hat{x} &= x - K\beta(T), \\ \hat{C} &= C - K. \end{aligned}$$

Theorem 4.3.1. *Denote by $\hat{X}^{\hat{\pi}}(t;\hat{x},\hat{C})$ the optimal wealth process* (3.30) *at time t with initial wealth $\hat{x}$ and fixed claim $\hat{C}$. Then under the Assumptions 2.1.1, 3.1.7 and 3.4.1 the optimal wealth process to Problem 4.1 is given by*

$$X^{\hat{\pi}^l}(t) = \hat{X}^{\hat{\pi}}(t;\hat{x},\hat{C}) + K\frac{\beta(T)}{\beta(t)}. \tag{4.7}$$

Similarly, denoting by $\hat{\pi}(t;\hat{x},\hat{C})$ the optimal portfolio process (3.36) *at time t with initial wealth $\hat{x}$ and fixed claim $\hat{C}$, the optimal portfolio process to Problem 4.1 is given by*

$$\hat{\pi}^l(t) = \hat{\pi}(t;\hat{x},\hat{C}) + Ke^{-\int_t^T r(s)ds}. \tag{4.8}$$

Proof. Looking at (4.6) we see that the lower constraint is fulfilled by investing just enough money in the bank account to reach K a.s. The rest of the money is then invested in order to reach $(C-K)$ with maximum probability. As this is done in the same way as in (3.36), non-negativity is guaranteed, and the whole wealth process never falls below $Ke^{-\int_t^T r(s)ds}$. Finally the Law of one Price of Theorem 3.4.6 guarantees that the replicating portfolio and the optimal portfolio are identical. □

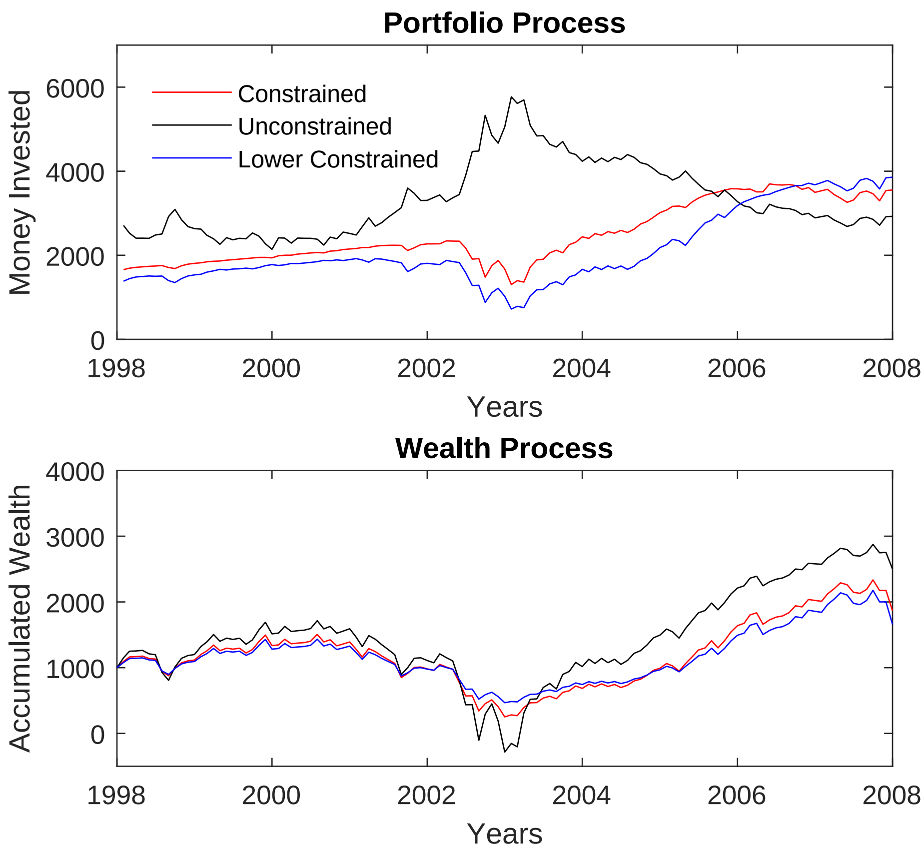


Figure 4.1: In the upper graph, the amount invested in the stock can be seen for the constrained portfolio, the unconstrained portfolio and the portfolio process with a lower constraint of $K = 500$. The resulting wealth process is plotted below.

Example 4.3.2. We repeat the analysis of Example 3.4.10, including the portfolio and wealth process of the optimal strategy with a positive constraint. In Figure 4.1 we plot the optimal portfolio strategy with a lower bound of $K = 500$ in addition to the constrained and the unconstrained optimal portfolio processes. We observe that during the bear market of 2000 to 2002, the amount invested in the stock for the portfolio including the lower bound is even lower than the amount invested for the constrained strategy. Although this ensures the wealth to stay above the lower bound, this guarantee comes at a price. The minimal wealth attained during the investment period is the highest for the portfolio process which includes a

	Unconstrained	**Constrained**	**Lower Bound**
Terminal Wealth	2’500	1’860	1’660
Rate of Return	9.2%	6.2%	5.1%
Minimal Wealth	-283	252	467

Table 4.1: Properties of the different portfolio processes calculated for the period 1998-2008 with the FTSE-Actuaries All Share Index as the sole underlying stock.

lower bound, but this is paid for by the rate of return and in turn, by lower terminal wealth, as observed in Table 4.1. Note that the minimal wealth attained by the strategy with a constraint of $K = 500$ is below that lower bound. This is due to the discretization used in the calculation, which does not allow to change the investment quick enough to prevent this.

Notably, after the market had recovered in 2003, the amount invested in the stock for the two constrained portfolio processes only gradually becomes larger than the amount invested for the unconstrained strategy, even though the distance to the target wealth is higher. This might be one weakness of the constrained portfolio processes, as much less risk can be taken during periods of low wealth to assure to stay above the lower bound. This in turn makes it harder to recover, compared to the unconstrained portfolio process.

Part II

Introducing Inflation

Chapter 5

The Financial Market Model

In Part I, we analyzed the problem of finding an optimal investment strategy in order to obtain the terminal wealth as close to a predetermined value as possible. Setting terminal wealth constraints, we were able to find explicit solutions for both the investment strategy as well as the corresponding wealth process.

A possible application of the quadratic optimization scheme is a pension plan, in which the plan member determines an optimal wealth at retirement, which will then be the objective to be reached by the investment strategy. However, such pension plans usually have a long duration and hence the plan members bear considerable risk due to inflation. In order to analyze how to best invest in the presence of inflation we will therefore introduce an additional market element, a so called inflation-linked bond, which allows us to hedge the risk of inflation. On the real financial market, Treasury Inflation Protected Securities (TIPS) or UK inflation-linked, gilt-edged securities are possible available derivatives.

The optimal investment strategy in the accumulation phase of such a pension plan under the presence of inflation has been studied in [Nkeki, 2012], [Liang and Sheng, 2015], [Pan and Xiao, 2017] and [Xu and Wu, 2014], which study the mean-variance framework. While most research concludes that the addition of an inflation-linked bond to the market is beneficial in practice, [Zhang, 2012] states that for the expectation maximization of some utility functions, the optimal terminal wealth is the same in real and in nominal terms. We show that for the target-based approach, including an inflation-linked bond actually transforms the optimal portfolio strategy and that the additional market element cannot be modeled as another risky stock. Most additional literature on the topic of optimal portfolio management under inflation additionally includes some constant, time-varying or random contribution process from the plan members during the accumulation phase. The optimization problem including contributions with or without inflation will be the subject of Part III.

We use the same approach as in Part I to solve the problem of quadratic optimization with non-negativity of the terminal wealth under the presence of inflation. Transforming the constraint of the wealth process to the constraint on terminal wealth, we restate the optimization problem. As the terminal inflation is now included in the expectation, we consider an equivalent optimization problem, applying the martingale approach of Part I in order to find a solution.

5.1 Introduction to Inflation and Inflation-Linked Bonds

5.1.1 Inflation

Inflation is defined as the increase in the general price level over a period of time and is usually measured by a certain price index $I(t)$. The percentage change of the index between times t and Δt is then the inflation over this time period, i.e.

$$i_s(t, t + \Delta t) = \frac{I(t + \Delta t) - I(t)}{I(t)},$$

where $\big(i_s(t, t')\big)_{t<t'\in[0,T]}$ denotes the *simple inflation rate*. We note the similarities between inflation rate and interest rate theory and define the *continuously compounded inflation rate* $\big(i_c(t, t')\big)_{t<t'\in[0,T]}$, by the solution to

$$e^{i_c(t+\Delta t,t)\Delta t} = \frac{I(t + \Delta t)}{I(t)}.$$

The *instantaneous inflation rate* $\big(i(t)\big)_{t\in[0,T]}$ is then defined similarly to the way instantaneous short rate is defined in interest rate theory, see e.g. [Brigo and Mercurio, 2001][Chapter 1], by

$$i(t) = \lim_{\Delta\to 0} i_c(t + \Delta t, t) = \frac{d \log(I(t))}{dt},$$

where we assume that the limit exists for all $t \geq 0$ and is well defined. In the following discussion, only the instantaneous rate of inflation will be used and is henceforth called the inflation rate.

A well-known result from microeconomics in the area of inflation is the Fisher effect, first introduced by [Fisher, 1930]. It states the effect of monetary policy on the nominal interest rate more directly, by defining the nominal interest rate as the sum of the expected real interest rate and the expected inflation. Denote by $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s, s < t)$ the sigma algebra of events which are observable before time t. Then the Fisher effect states

$$r_N(t) = \mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{r}_R(t)|\mathcal{F}_{t-}] + \mathbb{E}^{\tilde{\mathbb{P}}}[i(t)|\mathcal{F}_{t-}],$$

where $r_N(t)$ is the nominal interest rate, $\tilde{r}_R(t)$ is the real interest rate and $\tilde{\mathbb{P}}$ is some risk neutral measure.

Note that contrary to the nominal interest rate r_N, both the real interest rate and the inflation rate are not progressively measurable. This is a consequence of the nominal interest rate being set in advance, while the real interest rate and the inflation rate are not set, but simply observed a posteriori. For the rest of this paper, we therefore define the expected real interest rate process $\big(r_R(t)\big)_{t\in[0,T]}$ as the predictable projection of the real interest rate, i.e.

$$r_R(t) = \mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{r}_R(t)|\mathcal{F}_{t-}], \qquad \text{for all } t \in [0,T].$$

The calculation of indices that measure the average level of prices is very involved. As the theory behind this calculation does not offer any particular insight into the problems at hand, we assume to be given an inflation index $\big(I(t)\big)_{t\in[0,T]}$, which follows the dynamics

$$\begin{aligned} dI(t) &= I(t)\Big(\big(r_N(t) - r_R(t) + \sigma_I(t)\theta_I(t)\big)dt + \sigma_I(t)dW_I(t)\Big), \\ I(0) &= 1, \end{aligned} \tag{5.1}$$

where $\big(W_I(t)\big)_{t\in[0,T]}$ is a $\mathbb{P}$-Brownian motion and $\sigma_I(t)$ denotes the volatility of the inflation index and is assumed to be a progressively measurable process, satisfying $\int_0^T \sigma_I^2(t)dt < \infty$, $\mathbb{P}$-a.s. Furthermore, $\theta_I(t)$ denotes the market price of inflation risk and is also assumed to be progressively measurable.

Similarly to Section 2.1, by including a bank account to the market model, we may assume that the nominal interest rate process $r_N(t)$ is $\mathcal{F}_t$-progressively measurable for all $t \in [0,T]$. By including an additional market element, a so-called inflation-linked bond, and by the above discussion, we will see that we can also assume that the real interest rate process $r_R(t)$ is $\mathcal{F}_t$-progressively measurable for all $t \in [0,T]$.

5.1.2 Inflation-Linked Bonds

In order to be able to perform inflation hedging, we introduce an inflation-linked zero coupon bond, denoted by $(B^*(t, I(t)))_{t\in[0,T]}$, to the market. This bond pays real notional F at maturity T, i.e. the nominal payment consists of $FI(T)$. Henceforth, we set $F = 1$, so $B^*(t, I(t))$ denotes the unit value of the inflation-linked zero coupon bond at time t.

In order to price this inflation-linked bond, recall the state price deflator $\xi(t)$, defined as

$$\xi(t) = \exp\Big(- \int_0^t \theta(s)' dW(s) - \int_0^t \big(r_N(s) + \frac{1}{2}\|\theta(s)\|^2\big) ds\Big),$$

where $\theta(t)$ denotes the market price of risk. Contraty to Part I, by adding the inflation-linked bond to the market, $\theta(t)$ includes the market price of inflation risk $\theta_I(t)$ and $W(t) = \big(W_1(t), \dots, W_n(t), W_I(t)\big), t \in [0, T]$.

Lemma 5.1.1. *Assuming that the real-interest rate process $r_R(t)$ is uniformly bounded, an inflation-linked zero-coupon bond satisfies*

$$dB^*(t, I(t)) = B^*(t, I(t))\Big(r_R(t)dt + \frac{dI(t)}{I(t)}\Big). \tag{5.2}$$

Therefore, the inflation-linked bond can perfectly replicate the inflation index.

Proof. Since the payout of an inflation-linked zero-coupon bond is $I(T)$, by the same no-arbitrage arguments as in (3.1), its price needs to satisfy

$$B^*(t, I(t)) = \xi(t)^{-1}\mathbb{E}^{\mathbb{P}}\big[\xi(T)I(T)\big|\mathcal{F}_t\big], \quad \text{for all } t \in [0, T].$$

Inserting the definition of $\xi(T)$ and $I(T)$ we obtain

$$\xi(T)I(T) = \exp\Big(- \int_0^T (r_R(s) + \frac{1}{2}\|\theta(s)\|^2 - \sigma_I(s)\theta_I(s) + \frac{1}{2}\sigma_I^2(s))ds \\ - \int_0^T \theta(s)' dW(s) + \int_0^T \sigma_I(s) dW_I(s)\Big).$$

Let $B_R(t, T)$ denote the time t expected real value of one unit paid at maturity, i.e.

$$B_R(t, T) = \mathbb{E}^{\mathbb{P}}\big[e^{-\int_t^T r_R(s)ds}\big|\mathcal{F}_t\big].$$

Then, the price of an inflation-linked zero-coupon bond is given by

$$
\begin{aligned}
B^*(t, I(t)) =& \xi(t)^{-1}\mathbb{E}^{\mathbb{P}}\big[\xi(T)I(T)\big|\mathcal{F}_t\big] \\
=& \exp\big(\int_0^t (r_N(s) - r_R(s) + \sigma_I(s)\theta_I(s) - \frac{1}{2}\sigma_I^2(s))ds \\
&+ \int_0^t \sigma_I(s)dW_I(s)\big)\mathbb{E}^{\mathbb{P}}\big[\exp\big(-\int_t^T r_R(s)ds\big)\big|\mathcal{F}_t\big] \\
=& I(t)B_R(t, T). \qquad (5.3)
\end{aligned}
$$

Since the real interest rate is assumed to be uniformly bounded, we can apply the dominated convergence theorem together with Itô's formula on $f(x, y) = xy$ to obtain

$$
\begin{aligned}
dB^*(t, I(t)) &= I(t)dB_R(t, T) + B_R(t, T)dI(t) \\
&= I(t)r_R(t)B_R(t, T)dt + B_R(t, T)I(t)\frac{dI(t)}{I(t)} \\
&= B^*(t, I(t))\Big(r_R(t)dt + \frac{dI(t)}{I(t)}\Big).
\end{aligned}
$$

□

In practice, inflation-linked bonds usually pay coupons as well as the notional. Similarly to the case of nominal bonds, the coupon inflation bond can be replicated as the sum of zero coupon bonds. Denoting by ILB the coupon inflation bond, which pays the real coupons c_i, at predetermined times $t_1, \dots, t_n$, we have

$$
\begin{aligned}
\text{ILB}(t) &= \Big(\sum_{i=1}^n c_i I(t)e^{-\int_{t_i}^t r_R(s)ds} + I(t)e^{-\int_t^T r_R(s)ds}\Big) \\
&= I(t)\Big(\sum_{i=1}^n c_i B_R(t, t_i) + B_R(t, T)\Big),
\end{aligned}
$$

such that $t < t_1 \leq t_n < T$.

Remark. Note that Treasury Inflation Protected Securities (TIPS) are protected against deflation, unlike UK inflation-linked, gilt-edged securities. The particular bond added to the market model (2.2) is an inflation-linked zero-coupon bond whose principal is not protected against deflation.

5.2 The Market Model

Choose a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we have an $n+1$-dimensional Brownian motion $W(t) = \big(W_1(t), \dots, W_n(t), W_I(t)\big), t \in [0, T]$, for a given, finite time horizon T. Suppose the financial market contains $n+2$ tradable assets. One of the assets is a risk-free bond $(B(t))_{t\in[0,T]}$ and n are stocks $(S_i(t))_{t\in[0,T]}$. Moreover, an inflation-linked zero-coupon bond denoted by $(B^*(t, I(t)))_{t\in[0,T]}$ is used to hedge the risk of inflation. The price dynamics are given by

$$dB(t) = r_N(t)B(t)dt, \tag{5.4}$$

$$dS_i(t) = S_i(t)\Big[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{i,j}(t)dW_j(t) + \sigma_{i,I}(t)dW_I(t)\Big], \tag{5.5}$$

$$dB^*(t, I(t)) = B^*(t, I(t))\Big[\big(r_N(t) + \sigma_I(t)\theta_I(t)\big)dt + \sigma_I(t)dW_I(t)\Big], \tag{5.6}$$

where $B(0) = 1$ and $B^*(0, I(0)) = \mathbb{E}^{\mathbb{P}}\big[e^{-\int_0^T r_R(t)dt}\big]$, $\mathbb{P}$-a.s. Denote the dispersion matrix of the market by

$$\sigma(t) = \begin{bmatrix} \sigma_{1,1}(t) & \cdots & \sigma_{1,n}(t) & \sigma_{1,I}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n,1}(t) & \cdots & \sigma_{n,n}(t) & \sigma_{n,I}(t) \\ 0 & \cdots & 0 & \sigma_I(t) \end{bmatrix}. \tag{5.7}$$

Assumption 5.2.1. The *real interest rate* process $r_R(t)$, the *nominal interest rate* process $r_N(t)$, the vector of *mean rates of return* $\mu(t)$ and the *dispersion matrix* $\sigma(t)$ are uniformly bounded and $\mathcal{F}_t$-progressively measurable processes on $[0, T]$, with $r_R(t), r_N(t) \in \mathbb{R}, \mu(t) \in \mathbb{R}^{n+1}$ and $\sigma(t) \in \mathbb{R}^{(n+1)\times(n+1)}$. Furthermore, $\sigma(t)\sigma(t)'$ shall be positive definite for all $t \in [0, T]$.

Consider an investor who starts with a fixed, positive wealth x at time 0, who invests in the various securities and whose actions do not affect the market prices. The amount that is invested in the i'th stock at time t is denoted by $\pi_i(t)$, whereas the amount invested in the inflation-linked bond is denoted by $\pi_I(t)$. At time $t \in [0, T]$ we denote the total wealth of this investor by $X(t)$. We replace the portfolio process $\pi(t)$ in Definition 2.1.2 by $\pi(t) = \big(\pi_1(t), \dots, \pi_n(t), \pi_I(t)\big)'$ and retain the concept of admissibility.

Definition 5.2.2. Given a portfolio process π, the solution $X = X^\pi$ to

$$\begin{aligned} dX^\pi(t) &= \sum_{i=1}^{n} \pi_i(t) \frac{dS_i(t)}{S_i(t)} + \pi_I(t) \frac{dB^*(t, I(t))}{B^*(t, I(t))} + \big(X^\pi(t) - \pi(t)'\mathbb{I}\big) \frac{dB(t)}{B(t)}, \\ X^\pi(0) &= x, \end{aligned} \tag{5.8}$$

is called the *wealth process* corresponding to the portfolio process π and the initial capital $x > 0$.

Similar to Proposition 2.1.4, such a solution $X = X^\pi$ is unique. In order to find conditions on the portfolio process to guarantee that the corresponding wealth process exists and is non-negative, we proceed in a similar manner as in Chapter 3.

Under Assumption 5.2.1, the matrix $\sigma(t)$ is invertible and we can define the risk premium process

$$\theta(t) = \begin{pmatrix} \theta_1(t) \\ \vdots \\ \theta_n(t) \\ \theta_I(t) \end{pmatrix} = \sigma(t)^{-1} \begin{pmatrix} \mu(t) - r_N(t)\mathbb{1} \\ \sigma_I(t)\theta_I(t) \end{pmatrix},$$

which exists and is bounded, measurable and adapted to $\mathcal{F}_t$ due to Assumption 5.2.1. Under this notation we have

$$\mu_i(t) - r_N(t) = \sum_{j=1}^{n} \sigma_{i,j}(t)\theta_j(t) + \sigma_{i,I}(t)\theta_I(t),$$

for all $i \in \{1, \ldots, n\}$ and all $t \in [0, T]$. We rewrite the price dynamics of the stocks (5.5) by

$$\frac{dS_i(t)}{S_i(t)} = \big(r_N(t) + \sigma_{i,I}(t)\theta_I(t)\big)dt + \sum_{j=1}^{n} \sigma_{ij}(t)\big(dW_j(t) + \theta_j(t)dt\big) + \sigma_{i,I}(t)dW_I(t). \tag{5.9}$$

Substituting (5.4), (5.6) and (5.9) in the stochastic differential equation of the wealth process (5.8), we obtain

$$\begin{aligned} dX^\pi(t) &= \Big(r_N(t)X^\pi(t) + \pi(t)'\Gamma(t)\Big)dt + \pi(t)'\sigma(t)dW(t), \\ X^\pi(0) &= x, \end{aligned} \tag{5.10}$$

where

$$\Gamma(t) = \sigma(t)\theta(t) = \begin{pmatrix} \sum_{j=1}^{n} \sigma_{1,j}(t)\theta_j(t) + \sigma_{1,I}(t)\theta_I(t) \\ \vdots \\ \sum_{j=1}^{n} \sigma_{n,j}(t)\theta_j(t) + \sigma_{n,I}(t)\theta_I(t) \\ \sigma_I(t)\theta_I(t) \end{pmatrix}.$$

The stochastic differential equation (5.10) strongly resembles the stochastic differential equation for the wealth process in absence of inflation, given by (2.3) in Part I. However, the optimization problem in the current market will be different from the Problem 3.1.2 we have analyzed so far and hence we expect to obtain a different optimal terminal wealth.

5.3 Change of Measure

As we follow along the same lines as in Section 2.2, we only give a summary of the derivation of the risk neutral measure $\mathbb{Q}$ in the financial market of (5.6). Define

$$Z(t) = \exp\Big(- \int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\Big).$$

Then, the measure $\mathbb{Q} : \Omega \to [0,1]$, defined by $\mathbb{Q}[A] = \mathbb{E}[Z(T)\mathbb{1}_A]$ for all $A \in \mathcal{F}$, is an equivalent probability measure to $\mathbb{P}$ and $\hat{W}(t) = W(t) + \int_0^t \theta(s)ds$ is a $\mathbb{Q}$-Brownian motion. With this we can rewrite the stochastic differential equation (5.10) of the wealth process by

$$\begin{aligned} dX^\pi(t) &= \Big(r_N(t)X^\pi(t) + \pi(t)'\Gamma(t)\Big)dt + \pi(t)'\sigma(t)dW(t) \\ &= r_N(t)X^\pi(t)dt + \pi(t)'\sigma(t)d\hat{W}(t). \end{aligned}$$

Recalling the bank account numéraire $\beta(t) = \exp(-\int_0^t r_N(s)ds)$ from (2.7), we write

$$X^\pi(t)\beta(t) = x + \int_0^t \beta(s)\pi(s)'\sigma(s)d\hat{W}(s),$$

and the measure $\mathbb{Q}$ is an equivalent martingale measure to $\mathbb{P}$. Recall the notion of the state price deflator

$$\xi(t) = \beta(t)Z(t) = \exp\Big(-\int_0^t \theta(s)'dW(s) - \int_0^t (r_N(s) + \frac{1}{2}\|\theta(s)\|^2)ds\Big), \quad (5.11)$$

for $t \in [0,T]$.

Remark. Although having similar definitions, the measure $\mathbb{Q}$ and the corresponding Brownian motion $\hat{W}(t)$ differ from their repective partners in Section 2.2, due to the presence of the inflation-linked bond in the market. The processes are the same, only if inflation is zero.

In light of the structure of the inflation-linked bond in (5.6), we define an additional measure $\mathbb{Q}^T$. Define the inflation adjusted risk premium process

$$\tilde{\theta}(t) = \big(\theta_1(t), \dots, \theta_n(t), \theta_I(t) - \sigma_I(t)\big)',$$

which exists and is bounded, measurable and adapted to $\mathcal{F}_t$ due to Assumption 5.2.1. Hence, we can apply Girsanov's theorem utilizing the Doléan-Dade exponential

$$\tilde{Z}(t) = \exp\Big(-\frac{1}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)\Big). \tag{5.12}$$

Now by Itô's formula, $\tilde{Z}(t)$ is a continuous local martingale with $\mathbb{E}[\tilde{Z}(T)] = 1$. Hence we can define the measure $\mathbb{Q}^T$ by $\mathbb{Q}^T[A] = \mathbb{E}[\tilde{Z}(T)\mathbb{1}_A]$ for all $A \in \mathcal{F}$ and by a similar argument to (2.5) $\mathbb{P}$ and $\mathbb{Q}^T$ agree on the same null sets. We will see that $\mathbb{Q}^T$ is in fact also a martingale measure, where the discount factor is the inflation-linked numéraire $\big(\tilde{\beta}(t)\big)_{t\in[0,T]}$, defined by

$$\tilde{\beta}(t) = \frac{1}{B^*(t, I(t))}, \tag{5.13}$$

for $t \in [0, T]$.

Lemma 5.3.1. *The process* $(\tilde{W}(t))_{t\in[0,T]}$ *defined by* $\tilde{W}(t) = W(t) + \int_0^t \tilde{\theta}(s)ds$ *is an* $(n+1)$*-dimensional Brownian motion under* $\mathbb{Q}^T$*.*

Proof. This proof follows the same approach as Lemma 2.2.1. By Itô's formula applied to $f(t, x) = \exp(-x - \frac{1}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds)$, we have

$$d\tilde{Z}(t) = -\tilde{Z}(t)\tilde{\theta}(t)' dW(t). \tag{5.14}$$

Using the product rule, we obtain

$$\begin{aligned} d\big(\tilde{Z}(t), \tilde{W}(t)\big) &= \tilde{Z}(t)d\tilde{W}(t) + \tilde{W}(t)d\tilde{Z}(t) + d[\tilde{Z}, \tilde{W}](t) \\ &= \tilde{Z}(t)dW(t) - \tilde{W}(t)\tilde{Z}(t)\tilde{\theta}(t)' dW(t), \end{aligned}$$

and $(\tilde{W}(t))_{t\in[0,T]}$ is a continuous martingale under $\mathbb{Q}^T$. By Lévy's characterization theorem of Brownian motion, $(\tilde{W}(t))_{t\in[0,T]}$ is also a $\mathbb{Q}^T$-Brownian motion. ☐

Chapter 6

The Constrained Optimal Strategy

6.1 Problem Formulation

6.1.1 The Constrained Portfolio Problem

Before formally stating the optimization problem, we note that once again we may focus on the non-negativity of the terminal wealth, instead of the non-negativity of the full price process. This follows by Proposition 3.1.1.

Problem 6.1.1. Let the family of all admissible portfolio processes that lead to non-negative terminal wealth be denoted by

$$\mathcal{A}(x) = \Big\{\pi \in \Pi \mid X^\pi(0) \leq x \text{ and } X^\pi(T) \geq 0, \quad \mathbb{P}\text{-a.s.}\Big\}.$$

Given a constant C, we consider the problem of finding a portfolio process $\tilde{\pi} \in \mathcal{A}(x)$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(C - \frac{X^{\tilde{\pi}}(T)}{I(T)}\Big)^2\Big] = \inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\Big(C - \frac{X^{\pi}(T)}{I(T)}\Big)^2\Big], \tag{6.1}$$

and the pair $\big(X^{\tilde{\pi}}(t), \tilde{\pi}(t)\big)$ satisfies the stochastic differential equation (5.10),

We are unable to use the results of Section 3.2 as the admissibility condition of (3.3) does not include the correction for inflation $I(T)$. Hence, we first reduce (5.10) to a problem more similar to Problem 3.1.2 by defining a new stochastic process $Y^\pi(t) = X^\pi(t)\tilde{\beta}(t)$ for all $t \in [0, T]$ and note that the numéraire $\tilde{\beta}(t)$ does not affect the optimal portfolio.

Remark. As the investor cares about the utility of the real wealth level, since the purchasing power of nominal wealth decreases with inflation, we write Problem 6.1.1 in terms of the real wealth process $\big(Y^\pi(t)\big)_{t\in[0,T]}$. Instead we could inflate the constant C, and study the problem

$$\inf_{\pi\in\mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\Big(CI(T) - X^\pi(T)\Big)^2\Big].$$

By [Korn, 1997], after slight adjustments, the results for Part I are still valid when replacing the constant C by the random function $\tilde{C} = CI(T)$. Note, that then the expected inflation over the period of investment needs to be estimated at the beginning. We study the mathematically more interesting problem proposed in (6.1) instead, for which the terminal value of the inflation index does not need to be estimated.

Proposition 6.1.2. *The real wealth process $Y^\pi(t) = X^\pi(t)\tilde{\beta}(t)$ satisfies the stochastic differential equation*

$$\begin{aligned} dY^\pi(t) =& Y^\pi(t)\big(\sigma_I^2(t) - \sigma_I(t)\theta_I(t)\big)dt - Y^\pi(t)\sigma_I(t)dW_I(t) \\ &+ \tilde{\beta}(t)\pi(t)'\big(\Gamma(t) - \sigma_{n+1}(t)\sigma_I(t)\big)dt + \tilde{\beta}(t)\pi(t)'\sigma(t)dW(t), \\ Y^\pi(0) =& \tilde{\beta}(0)x, \end{aligned} \tag{6.2}$$

where $\sigma_{n+1}(t)$ denotes the $(n+1)^{th}$ column of $\sigma(t)$.

Proof. Due to the initial nominal wealth $X^\pi(0) = x$, the initial wealth of the real wealth process follows immediately. Now let $f(x,y) = \frac{x}{y}$. Then, by Itô's formula, we have

$$df(x,y) = \frac{1}{y}dx - \frac{x}{y^2}dy - \frac{1}{y^2}d[x,y](t) + \frac{x}{y^3}d[y,y](t).$$

Using the definition of the inflation-numéraire (5.13) and the stochastic differential equations (5.6) and (5.10), we plug in $f\big(X^\pi(t), B^*(t,I(t))\big)$ and obtain

$$
\begin{aligned}
dY^\pi(t) &= \tilde{\beta}(t)\bigg(dX^\pi(t) - X^\pi(t)\frac{dB^*(t,I(t))}{B^*(t,I(t))} - \pi(t)'\sigma_I(t)\sigma_{n+1}(t)dt \\
&\qquad + X^\pi(t)\sigma_I^2(t)dt\bigg) \\
&= \tilde{\beta}(t)\bigg(r_N(t)X^\pi(t)dt + \pi(t)'\Gamma(t)dt + \pi(t)'\sigma(t)dW(t) - r_N(t)X^\pi(t)dt \\
&\qquad - X^\pi(t)\sigma_I(t)\theta_I(t)dt - X^\pi(t)\sigma_I(t)dW_I(t) + X^\pi(t)\sigma_I^2(t)dt \\
&\qquad - \pi(t)'\sigma_I(t)\sigma_{n+1}(t)dt\bigg) \\
&= \tilde{\beta}(t)\bigg(\Big(\pi(t)'\Gamma(t) + X^\pi(t)(\sigma_I^2(t) - \sigma_I(t)\theta_I(t)) - \pi(t)'\sigma_I(t)\sigma_{n+1}(t)\Big)dt \\
&\qquad + \pi(t)'\sigma(t)dW(t) - X^\pi(t)\sigma_I(t)dW_I(t)\bigg) \\
&= Y^\pi(t)\Big(\big(\sigma_I^2(t) - \sigma_I(t)\theta_I(t)\big) - \sigma_I(t)dW_I(t)\Big) \\
&\qquad + \tilde{\beta}(t)\pi(t)'\Big(\big(\Gamma(t) - \sigma_I(t)\sigma_{n+1}(t)\big)dt + \sigma(t)dW(t)\Big).
\end{aligned}
$$

□

Note that by non-negativity of the inflation index, we have

$$X^\pi(t) \geq 0 \iff Y^\pi(t) \geq 0, \quad \mathbb{P}\text{-a.s.},$$

for all $t \in [0,T]$, and in order to obtain a non-negative nominal wealth process, we may focus on the non-negativity of the terminal wealth of the real wealth process instead. Furthermore, by Lemma 5.1.1 we have $B^*(T,I(T)) = I(T)$, which allows us to restate Problem 6.1.1.

Problem 6.1.3. Given a fixed claim C, we consider the problem of finding a portfolio process $\tilde{\pi} \in \mathcal{A}(x)$ such that

$$\mathbb{E}^{\mathbb{P}}\big[\big(C - Y^{\tilde{\pi}}(T)\big)^2\big] = \inf_{\pi\in\mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[\big(C - Y^\pi(T)\big)^2\big]. \tag{6.3}$$

and the pair $\big(Y^{\tilde{\pi}}(t), \tilde{\pi}(t)\big)$ satisfies the stochastic differential equation (6.2).

Now similarly to the approach in Sections 3.2 and 3.4 we can use the martingale approach and separate Problem 6.1.1 into two sub problems. First we find the terminal wealth that optimizes the quadratic utility, before solving the corresponding backward stochastic differential equation.

6.1.2 Conditions on Admissibility

In order to solve Problem 6.1.3, we express the condition $\pi \in \mathcal{A}(x)$ by an inequality, which then allows us to use Lagrangian techniques to find the optimal terminal wealth. Due to the presence of inflation, we do not use the bank-account numéraire $\beta(t)$ to discount the wealth process, but rather use the inflation-linked numéraire $\tilde{\beta}(t)$ and the measure $\mathbb{Q}^T$, introduced in Section 5.3.

Lemma 6.1.4. *The process* $(Y^\pi(t))_{t\in[0,T]}$ *is a continuous local martingale under the measure* $\mathbb{Q}^T$*.*

Proof. Inserting the Brownian motion $\tilde{W}(t)$ from Lemma 5.3.1 into (6.2), we obtain

$$\begin{aligned} dY^\pi(t) =& Y^\pi(t)\big(\sigma_I^2(t) - \sigma_I(t)\theta_I(t)\big)dt - Y^\pi(t)\sigma_I(t)dW_I(t) \\ &+ \tilde{\beta}(t)\pi(t)'\big(\Gamma(t) - \sigma_{n+1}(t)\sigma_I(t)\big)dt + \tilde{\beta}(t)\pi(t)'\sigma(t)dW(t) \\ =& - Y^\pi(t)\sigma_I(t)d\tilde{W}_I(t) + \tilde{\beta}(t)\pi(t)'\sigma(t)d\tilde{W}(t). \end{aligned} \tag{6.4}$$

From this it is apparent that the process $(\tilde{M}(t))_{t\in[0,T]}$ defined by $\tilde{M}(t) = \tilde{\beta}(t)X^\pi(t)$ is a continuous local martingale with respect to the measure $\mathbb{Q}^T$. □

Now as $Y^\pi(t) \geq 0$, a.s., the continuous local martingale $(Y^\pi(t))_{t\in[0,T]}$ is a supermartingale under $\mathbb{Q}^T$ by Lemma 3.1.3. Therefore, we must have

$$\mathbb{E}^{\mathbb{Q}^T}[Y^\pi(T)] \leq Y^\pi(0) = \tilde{\beta}(0)x. \tag{6.5}$$

We want to rewrite the admissibility constraint (3.3) in terms of the real wealth process $Y^\pi(t)$. In order to do this, define the process $\big(\tilde{\xi}(t)\big)_{t\in[0,T]}$ by

$$\tilde{\xi}(t) = \frac{\xi(t)}{\tilde{\beta}(t)}. \tag{6.6}$$

Remark. Note that $\tilde{\xi}(t)$ is not the state price deflator under the measure $\mathbb{Q}^T$ as it is neither normalized, nor serves as the discount factor under $\mathbb{Q}^T$.

Lemma 6.1.5. *The admissibility constraint* (6.5) *is equivalent to the admissibility condition* $\mathbb{E}^{\mathbb{P}}\Big[\xi(T)X^\pi(T)\Big] \leq x$*, given by* (3.3)*.*

Proof. By applying Bayes' rule, we obtain from (6.5) that

$$\mathbb{E}^{\mathbb{P}}[\tilde{Z}(T)\tilde{\beta}(T)X^{\pi}(T)] \leq \tilde{\beta}(0)x,$$

which is equivalent to

$$\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)Y^{\pi}(T)] \leq x.$$

Moreover, by inserting the definition of $\tilde{\xi}(t)$, it immediately follows that this is equivalent to

$$\mathbb{E}^{\mathbb{P}}\Big[\xi(T)X^{\pi}(T)\Big] \leq x.$$

□

Lemma 6.1.6. *For every non-negative, $\mathcal{F}_T$-measurable Ψ which satisfies $\mathbb{E}^{\mathbb{Q}^T}[\Psi] = \tilde{\beta}(0)x$, there exists a unique $\pi \in \mathcal{A}(x)$ such that the corresponding real wealth process satisfies $Y^{\pi}(T) = \Psi$, a.s.*

Proof. Similarly to Theorem 3.1.4, by [El Karoui et al., 1997, Theorem 2.1], the linear backward stochastic differential equation

$$\begin{aligned} dY(t) &= -Y(t)\sigma_I(t)d\tilde{W}_I(t) + \tilde{\beta}(t)P(t)'d\tilde{W}(t), \\ Y(T) &= \Psi, \end{aligned}$$

admits a unique, square integrable, $\mathcal{F}_t$-adapted solution (Y, P). Define

$$\pi(t) = (\sigma(t)')^{-1}P(t),$$

which is square integrable due to the uniform boundedness of $(\sigma(t)')^{-1}$ and since $P(t)$ is square integrable. Moreover, by Assumption 5.2.1, the $\mathbb{Q}^T$-local martingale $Y(t)$ is uniformly integrable and therefore

$$Y(0) = \mathbb{E}^{\mathbb{Q}^T}[Y(T)] = x\tilde{\beta}(0).$$

Hence, $\pi \in \mathcal{A}(x)$ and $\big(Y(t), \pi(t)\big)$ satisfies the dynamics of (6.2). □

6.1.3 Feasibility

Similar to Section 3.1.3, we determine the conditions under which a solution to Problem 6.1.3 exists ans is unique. Due to the discussion in Section 6.1.2, we may study the feasibility of the following problem instead.

$$\begin{aligned} &\text{Minimize} \quad \mathbb{E}\big[(C-\Psi)^2\big], \\ &\text{subject to} \quad \mathbb{E}\big[\tilde{\xi}(T)\Psi\big] = x \text{ and } \Psi \geq 0, \ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{6.7}$$

over all $\mathcal{F}_T$-measurable processes Ψ.

Proposition 6.1.7. *Define*

$$\tilde{a} = \inf_{\Psi \geq 0, \mathbb{P}\text{-}a.s.} \mathbb{E}\big[\tilde{\xi}(T)\Psi\big],$$
$$\tilde{b} = \sup_{\Psi \geq 0, \mathbb{P}\text{-}a.s.} \mathbb{E}\big[\tilde{\xi}(T)\Psi\big]. \tag{6.8}$$

If $\tilde{a} < x < \tilde{b}$, then there must be a feasible solution to Problem (6.7) *and hence to Problem 6.1.1.*

Proof. The proof is identical to the proof of Proposition 3.1.6 and therefore omitted. □

To make sure of the existence of a solution to Problem 6.1.1, we once again assume the risk premium process to be deterministic. In addition, similar to Assumption 3.4.1, in order to be able to determine the portfolio process explicitly, we also assume the real interest rate to be deterministic.

Assumption 6.1.8. The real interest rate process $r_R(t)$ and the inflation adjusted risk premium process $\tilde{\theta}(t)$ are deterministic and satisfy

$$\int_0^T \|\tilde{\theta}(s)\|^2 ds \neq 0.$$

Remark. Note that in the case of deterministic interest rates, the process $B_R(t,T)$ is given by $B_R(t,T) = e^{-\int_t^T r_R(s)ds}$ and that by (5.3), we have

$$\begin{aligned}\tilde{\xi}(t) =& \xi(t)B_R(t,T)I(t)\\ =& \exp\big(-\int_0^t \tilde{\theta}(s)'dW(s) - \frac{1}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds\big)\exp\big(-\int_0^T r_R(s)ds\big)\\ =& \frac{\tilde{Z}(t)}{\tilde{\beta}(0)}.\end{aligned}$$

6.2 Optimization of Terminal Wealth

We give a summary of the derivations in Section 3.2 and 3.3 in order to obtain the Lagrange multiplier corresponding to the optimization Problem 6.1.3. After obtaining a solution for the optimal terminal wealth we provide a more explicit form for the Lagrange multiplier.

Recall the notation of a generalized utility function from Definition 3.2.1 and denote by $\tilde{z}$ the solution to $U'(z) = 0$. Define $\tilde{\mathcal{H}}(y) = \mathbb{E}^{\mathbb{P}}\big[\tilde{\xi}(T)I(y\tilde{\xi}(T))\big]$ for all $y \in (0,\infty)$, where I denotes the truncated inverse function of U', given by (3.6).

Lemma 6.2.1. *Assume* $\tilde{\mathcal{H}}(y) < \infty$ *for all* $y \in (0, \infty)$. *Under Assumptions 5.2.1 and 6.1.8,* $\tilde{\mathcal{H}}$ *is continuous and strictly decreasing. Furthermore,*

$$\tilde{\mathcal{H}}(\infty) = \lim_{y\to\infty} \tilde{\mathcal{H}}(y) = 0,$$

$$\tilde{\mathcal{H}}(0) = \lim_{y\to 0} \tilde{\mathcal{H}}(y) = \begin{cases} \infty & \textit{if } \lim_{z\to\infty} U'(z) = 0, \\ \frac{\tilde{z}}{\tilde{\beta}(0)} & \textit{else.} \end{cases}$$

Proof. The proof follows in the same way as the proof of Lemma 3.2.4 and is therefore omitted. □

Theorem 6.2.2. *Under Assumptions 5.2.1 and 6.1.8, there exists a portfolio process* $\pi \in \mathcal{A}(x)$, *such that the corresponding real wealth process attains the optimal terminal wealth, given by*

$$Y^{\pi}(T) = \begin{cases} C & \textit{if } x\tilde{\beta}(0) \geq C, \\ \big(C - \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big)^{+} & \textit{else,} \end{cases} \tag{6.9}$$

where $\tilde{\mathcal{Y}} : (0, \tilde{\mathcal{H}}(0)) \to (0, \infty)$ *denotes the inverse of* $\tilde{\mathcal{H}}$.

Proof. By using the generalized utility function $U(x) = -\frac{1}{2}\big(C - x\big)^2$, the claim follows once we have shown that the optimal terminal wealth is given by

$$\Psi = \begin{cases} \tilde{z} & \text{if } x \geq \tilde{\mathcal{H}}(0), \\ I(\tilde{\mathcal{Y}}(x)\tilde{\xi}(T)) & \text{else .} \end{cases} \tag{6.10}$$

By Lemma 6.1.6, there exists a portfolio process $\pi \in \mathcal{A}(x)$ such that $Y^{\pi}(T) = \Psi$. Using the same methods as in the proof of Theorem 3.2.5, (6.10) is obtained immediately. □

In order to obtain a more explicit solution for $\tilde{\mathcal{Y}}(x)$, we analyze the admissibility condition (6.5). Inserting the solution for the optimal terminal wealth, yields

$$\mathbb{E}^{\mathbb{Q}^T}\left[\Big(C - \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\Big)^{+}\right] = \tilde{\beta}(0)x. \tag{6.11}$$

Define $V(T) = \log\tilde{\xi}(T)$ and recall the notation $\tilde{\theta}(t) = \big(\theta_1(t), \ldots, \theta_n(t), \theta_I(t) - \sigma_I(t)\big)'$.

Proposition 6.2.3. $V(T)$ *is normally distributed under* $\mathbb{P}$ *with* $V(T) \sim \mathcal{N}(a, b^2)$ *and normally distributed under* $\mathbb{Q}^T$ *with* $V(T) \sim \mathcal{N}(\tilde{a}, b^2)$. *Here,*

$a, \tilde{a}$ and b are given by

$$\begin{aligned} a &= -\log(\tilde{\beta}(0)) - \frac{1}{2}b^2, \\ \tilde{a} &= -\log(\tilde{\beta}(0)) + \frac{1}{2}b^2, \\ b^2 &= \int_0^T \|\tilde{\theta}(t)\|^2 dt. \end{aligned} \tag{6.12}$$

Proof. We will show that the moment generating functions of $V(T)$ under $\mathbb{P}$ and $\mathbb{Q}^T$ take the form of moment generating functions of normal distributions with the corresponding mean and variance. Recall that under the assumption of deterministic real interest rates we have $\tilde{\xi}(t) = \frac{\tilde{Z}(t)}{\tilde{\beta}(0)}$ and hence

$$\begin{aligned} M_V^{\mathbb{P}}(t) &= \mathbb{E}^{\mathbb{P}}\big[\exp(tV(T))\big] = \mathbb{E}^{\mathbb{P}}\big[\exp(t\log\tilde{\xi}(T))\big] \\ &= \mathbb{E}^{\mathbb{P}}\Big[\exp\Big(t\big(\int_0^T \big(r_R(t) - \frac{1}{2}\|\tilde{\theta}(t)\|^2\big)dt - \int_0^T \tilde{\theta}(t)'dW(t)\big)\Big)\Big] \\ &= \exp\Big(t\big(\int_0^T \big(r_R(t) - \frac{1}{2}\|\tilde{\theta}(t)\|^2\big)dt\big)\Big)\mathbb{E}^{\mathbb{P}}\Big[\exp\big(-t\int_0^T \tilde{\theta}(t)'dW(t)\big)\Big] \\ &= \exp\big(ta + \frac{1}{2}t^2b^2\big), \end{aligned}$$

since $W(t) \sim \mathcal{N}(0,t)$. Similarly,

$$\begin{aligned} M_V^{\mathbb{Q}^T}(t) &= \mathbb{E}^{\mathbb{Q}^T}\big[\exp(tV(T))\big] = \mathbb{E}^{\mathbb{P}}\big[\frac{\tilde{Z}(T)^{t+1}}{\tilde{\beta}(0)^t}\big] \\ &= \exp\big(\int_0^T \big(tr_R(t) - \frac{(t+1)}{2}\|\tilde{\theta}(t)\|^2\big)dt + \frac{(t+1)^2}{2}\int_0^T \|\tilde{\theta}(t)\|^2 dt\big) \\ &= \exp\big(t\tilde{a} + \frac{1}{2}t^2b^2\big). \end{aligned}$$

□

Similarly to the mean-variance optimization problem under inflation, outlined in [Liang and Sheng, 2015], we cannot give a general explicit solution for $\tilde{\mathcal{Y}}(x)$. Therefore, in a similar fashion as for the function $\mathcal{Y}(x)$, defined in (3.2.5), $\tilde{\mathcal{Y}}(x)$ needs to be determined numerically.

Proposition 6.2.4. *Assume $b > 0$ and $C > \tilde{\beta}(0)x$. Then $\tilde{\mathcal{Y}}(x)$ satisfies*

$$\tilde{\mathcal{Y}}(x)e^{\int_0^T \|\tilde{\theta}(t)\|^2 dt}\Phi\Big(\frac{\log(\frac{C}{\tilde{\mathcal{Y}}(x)}) - \tilde{a} - b^2}{b}\Big) = \tilde{\beta}(0)C\Phi\Big(\frac{\log(\frac{C}{\tilde{\mathcal{Y}}(x)}) - \tilde{a}}{b}\Big) - x\tilde{\beta}(0)^2, \tag{6.13}$$

for a, $\tilde{a}$ and b given by (6.12).

Proof. For ease of notation, denote $y = \tilde{\mathcal{Y}}(x)$. Then the admissibility condition reads

$$\mathbb{E}^{\mathbb{Q}^T}\big[\big(\frac{C}{y} - e^V\big)^+\big] = x\frac{\tilde{\beta}(0)}{y},$$

and by classical methods of European option pricing, we compute

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T}\big[\big(\frac{C}{y} - e^{V(T)}\big)^+\big] &= \frac{C}{y}\mathbb{Q}^T\big[\frac{C}{y} > e^{V(T)}\big] - \mathbb{E}^{\mathbb{Q}^T}\big[e^{V(T)}\mathbb{1}\{\frac{C}{y} > e^{V(T)}\}\big] \\
&= \frac{C}{y}\Phi\Big(\frac{\log(\frac{C}{y}) - \tilde{a}}{b}\Big) - \frac{1}{\tilde{\beta}(0)}\mathbb{E}^{\mathbb{Q}^T}\big[\tilde{Z}(T)\mathbb{1}\{\frac{C}{y} > e^{V(T)}\}\big] \\
&= \frac{C}{y}\Phi\Big(\frac{\log(\frac{C}{y}) - \tilde{a}}{b}\Big) \\
&\quad - \frac{1}{\tilde{\beta}(0)}e^{-\frac{1}{2}\int_0^T \|\tilde{\theta}(t)\|^2 dt}\mathbb{E}^{\mathbb{Q}^T}\Big[e^{-\int_0^T \tilde{\theta}(t)' dW(t)}\mathbb{1}\{\frac{C}{y} > e^{V(T)}\}\Big] \\
&= \frac{C}{y}\Phi\Big(\frac{\log(\frac{C}{y}) - \tilde{a}}{b}\Big) - \frac{e^{\int_0^T \|\tilde{\theta}(t)\|^2 dt}}{\tilde{\beta}(0)}\Phi\Big(\frac{\log(\frac{C}{y}) - b^2 - \tilde{a}}{b}\Big).
\end{aligned}$$

To obtain the last equality, we need to define an additional measure, $\mathbb{Q}_2$. In order to do this, define the Doléan-Dade exponential

$$\tilde{Z}_2(T) = \exp\big(-\frac{1}{2}\int_0^T \|\tilde{\theta}(t)\|^2 dt - \int_0^T \tilde{\theta}(t)' d\tilde{W}(t)\big).$$

Due to similar arguments as in Lemma 2.2.1 and Lemma 5.3.1, the process $\tilde{W}_2(t) = \tilde{W}(t) + \int_0^t \tilde{\theta}(s)ds$, is a $\mathbb{Q}_2$-Brownian motion and hence

$$\tilde{\xi}(t)\tilde{\beta}(0) = \exp\big(\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' d\tilde{W}_2(s)\big).$$

Therefore, we have

$$\mathbb{E}^{\mathbb{Q}^T}\Big[e^{-\int_0^T \tilde{\theta}(t)' dW(t)}\mathbb{1}\{\frac{C}{y} > e^{V(T)}\}\Big] = e^{\frac{3}{2}\int_0^T \|\tilde{\theta}(t)\|^2 dt}\mathbb{E}^{\mathbb{Q}_2}\Big[\mathbb{1}\{\frac{C}{y} > \tilde{\xi}(T)\}\Big].$$

By the arguments of Proposition 6.2.3, we see that $\log \tilde{\xi}(T)$ is normally distributed under $\mathbb{Q}_2$ with mean $\tilde{a} + b^2$ and variance b^2. □

Recall the notion of the ruin probability in Definition 3.3.2 and note that the ruin probability for the nominal and for the real terminal wealth are the same, as the inflation index is always positive. In order to reflect the presence of inflation in the market, we define the probability of success under inflation.

Definition 6.2.5. The *probability of success under inflation* is defined by

$$\mathbb{P}[X^{\pi}(T) > xI(T)],$$

or, equivalently, by

$$\mathbb{P}[Y^{\pi}(T) > x].$$

Example 6.2.6. Similarly to Example 3.3.3, we plot the empirical distribution of the optimal terminal wealth for both the restricted process (6.9) as well as the unrestricted process, where the non-negativity constraint is dropped.

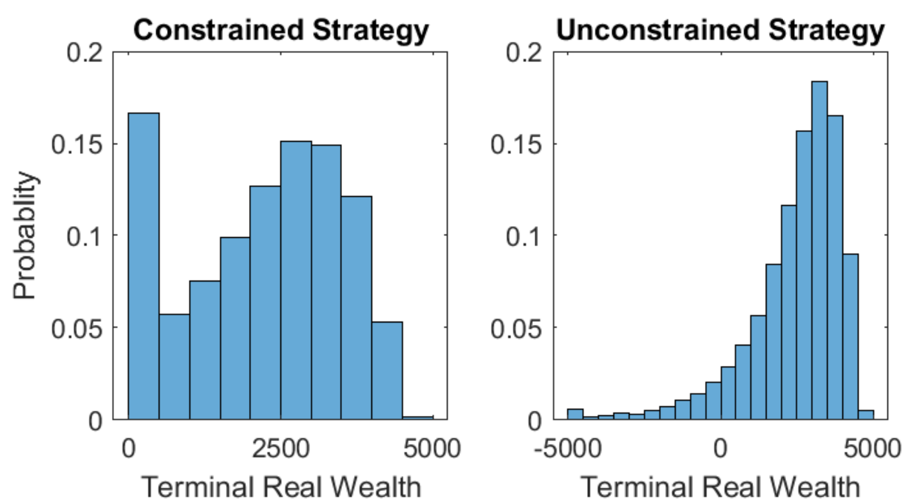


Figure 6.1: Histograms of the real terminal wealth distribution for the constrained and unconstrained portfolio problem.

For this example we suppose that all parameters are constant over time and that there is only one stock in the market. We set the market parameters as $r_N = 0.05$, $\mu = 0.08$ and $\sigma_S = 0.15$. The investor starts with an initial wealth

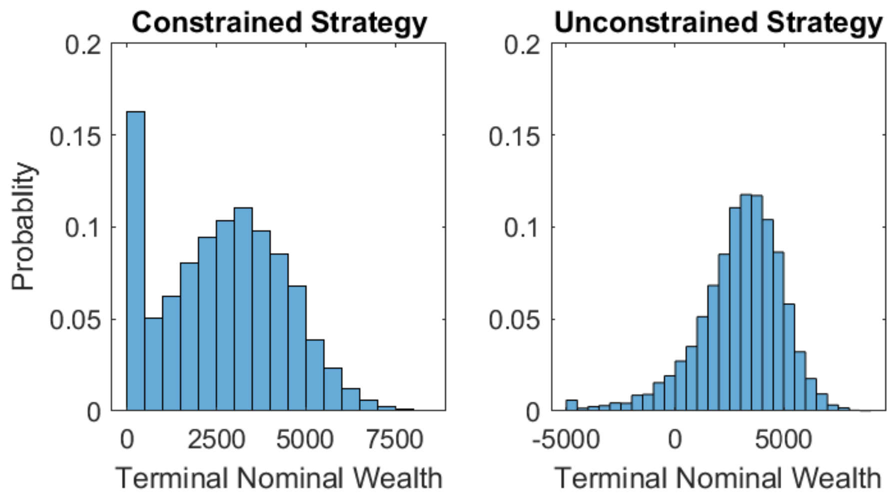


Figure 6.2: Histograms of the nominal terminal wealth distribution for the constrained and unconstrained portfolio problem.

$x = 1000$ and tries to reach $C = 5000$ over a time horizon of $T = 10$ years. The inflation parameters are given by $r_R = 0.04$, $\sigma_I = 0.05$, $\theta_I = 0.12$ and the volatility of the stock with respect to the inflation is given by $\sigma_{IS} = 0.04$.

In Figures 6.1 and 6.2, we plot the empirical terminal wealth distributions for 10'000 realizations. Note that the unconstrained distribution is cut-off at -5'000, in order to increase readability of the plot. For both strategies, the target C acts as an upper bound of the terminal real wealth. This follows directly from (6.9), as both $\tilde{\xi}(T)$ and $\tilde{\mathcal{Y}}(x)$, are strictly positive functions. In contrast to the findings of Example 3.3.3, the nominal terminal wealth can in fact be higher than the target for the real wealth C.

In order to study the advantages and disadvantages more thoroughly, we report some statistics of the final wealth out of 10'000 realizations in Table 6.1. On average, the constrained strategy leads to a lower terminal wealth, as expected. Comparing the properties in Table 6.1 to the properties of the terminal wealth in the market without inflation, in Table 3.1, we note that the presence of inflation has considerably broadened the distribution of the terminal nominal wealth. This comes as no surprise, as the terminal inflation $I(T)$ is usually bigger than one and acts as a factor on the terminal real wealth.

We note that including an inflation-linked bond as an additional market instrument has increased both the L2-distance and the ruin probability of

	Unconstrained	**Constrained**
2.5% Quantile	-1'770	0
Mean	2'970	2'660
97.5% Quantile	6'160	5'910
$\sqrt{\textbf{L2}}$**-Distance**	3'020	3'100
Median Rate of Return	11.7%	10.1%
Ruin Probability	6.9%	0.0%
Success Probability	86.2%	78.3%

Table 6.1: Properties of the empirical terminal nominal wealth distribution for the constrained and unconstrained portfolio problem.

the unconstrained optimal portfolio. On the other hand, the median rate of return and the success probability for both optimal portfolios are higher than the corresponding probabilities in Example 3.3.3. Hence, it is not immediately apparent, if including an additional market element with the index-linked bond leads to a better performance of the portfolio. We extend this discussion in Chapter 16.

6.3 The Optimal Portfolio Process

Similarly to Section 3.4, we explicitly derive the optimal portfolio process and the corresponding wealth process with and without the non-negativity requirement. Without the additional requirement, we solve the stochastic differential equation (6.2) directly by the dynamic programming approach. On the other hand, the optimal portfolio process leading to the non-negative payoff (6.9) can be seen as a put option on a certain type of underlying, which enables us to use the martingale approach to find an explicit solution.

At the end of this Section, we briefly compare the optimal portfolio processes for the optimization Problem 6.1.1 to those obtained in Section 3.4, in the market without inflation. The proper analysis of the different portfolio processes follows in Chapter 11.

6.3.1 Without Bankruptcy Prohibition

For notational purposes, define

$$\begin{aligned} A(t) &= \sigma_I^2(t) - \sigma_I(t)\theta_I(t), \\ M(t) &= \tilde{\beta}(t)\big(\Gamma(t) - \sigma_{n+1}(t)\sigma_I(t)\big), \\ \bar{\sigma}_I(t) &= \big(0, \dots, 0, \sigma_I(t)\big)', \\ D(t) &= \tilde{\beta}(t)\sigma(t). \end{aligned} \tag{6.14}$$

With this notation, the stochastic differential equation (6.2) of the real wealth process becomes

$$\begin{aligned} dY^\pi(t) &= Y^\pi(t)A(t)dt + \pi(t)'M(t)dt + \big(\pi(t)'D(t) - \bar{\sigma}_I(t)'Y^\pi(t)\big)dW(t), \\ Y^\pi(0) &= x\tilde{\beta}(0). \end{aligned} \tag{6.15}$$

Theorem 6.3.1. *Define*

$$\varphi(t) = \big(D(t)D(t)'\big)^{-1}\big(M(t) - D(t)\bar{\sigma}_I(t)\big), \tag{6.16}$$

and let $\big(a(t)\big)_{t\in[0,T]}$ *and* $\big(b(t)\big)_{t\in[0,T]}$ *satisfy the Riccati equations given by*

$$\begin{aligned} 0 &= a_t(t) + \Big[2A(t) + \bar{\sigma}_I(t)'\bar{\sigma}_I(t) - \varphi(t)'\big(D(t)D(t)'\big)\varphi(t)\Big]a(t), \\ a(T) &= -\frac{1}{2}, \end{aligned} \tag{6.17}$$

$$\begin{aligned} 0 &= b_t(t) + \Big[A(t) - \frac{1}{2}M(t)'\varphi(t) - \frac{1}{2}\varphi(t)M(t)\Big]b(t), \\ b(T) &= C, \end{aligned} \tag{6.18}$$

for all $t \in [0,T]$*, where we use the notation* $a_t(t) = \frac{\partial}{\partial t}a(t)$ *and* $b_t(t) = \frac{\partial}{\partial t}b(t)$*. Then, under Assumptions 5.2.1 and 6.1.8, the optimal portfolio process to the unconstrained optimization problem is given by*

$$\hat{\pi}(t) = \big(D(t)D(t)'\big)^{-1}\Big(\big(D(t)\bar{\sigma}_I(t) - M(t)\big)Y^{\hat{\pi}}(t) - M(t)h(t)\Big), \tag{6.19}$$

for all $t \in [0,T]$*, where* $h(t) = \frac{b(t)}{2a(t)}$*.*

Proof. See Appendix II.A. □

In order to obtain the corresponding optimal wealth process to the unconstrained optimization problem, we first find an explicit solution for $h(t)$ in (6.19). Inserting the optimal portfolio process in the stochastic differential equation (6.2) then yields the optimal real wealth process. Finally, using the definition $Y^\pi(t) = X^\pi(t)\tilde{\beta}(t)$, we obtain the optimal wealth process.

Corollary 6.3.2. *Under the same assumptions and notation as Theorem 6.3.1, we have*

$$h(t) = -C, \qquad \text{for all } t \in [0,T]. \tag{6.20}$$

Proof. Inserting the explicit solutions for $a(t)$ and $b(t)$ from (7.11) and (7.12), we obtain

$$\begin{aligned}
h(t) &= -C \exp\Big(- \int_t^T \big(A(s) + \frac{1}{2}M(s)'\varphi(s) + \frac{1}{2}\varphi(s)'M(s) \\
&\qquad\qquad + \sigma_I(s)'\sigma_I(s) - \varphi(s)'\big(D(s)D(s)'\big)\varphi(s)\big)ds\Big) \\
&= -C \exp\Big(- \int_t^T \big(A(s) + \frac{1}{2}M(s)'\big(D(s)D(s)'\big)^{-1}D(s)\bar{\sigma}_I(s) \\
&\qquad\qquad + \frac{1}{2}\bar{\sigma}_I(s)'D(s)'\big(D(s)D(s)'\big)^{1}M(s)\big)ds\Big) \\
&= -C \exp\Big(- \int_t^T \big(A(s) + \bar{\sigma}_I(s)'(D(s))^{-1}M(s)\big)ds\Big),
\end{aligned}$$

where the last step follows from Lemma II.A.1 and because $\bar{\sigma}_I(s)$ only has a single non-zero entry. Now, we insert the definitions of $D(t)$ and $M(t)$ from (6.14) and see that

$$\begin{aligned}
D(t)^{-1}M(t) &= \sigma(t)^{-1}\big(\Gamma(t) - \sigma_{n+1}(t)\sigma_I(t)\big) \\
&= \sigma(t)^{-1}\big(\sigma(t)\theta(t) - \sigma(t)\bar{\sigma}_I(t)\big) = \tilde{\theta}(t).
\end{aligned} \tag{6.21}$$

Similarly, using the definition of $A(t)$ from (6.14),

$$\begin{aligned}
A(t) + \bar{\sigma}_I(t)'D(t)^{-1}M(t) &= \sigma_I^2(t) - \sigma_I(t)\theta_I(t) + \bar{\sigma}_I(t)'\tilde{\theta}(t) \\
&= \sigma_I^2(t) - \sigma_I(t)\theta_I(t) + \sigma_I(t)\theta_I(t) - \sigma_I^2(t) = 0.
\end{aligned} \tag{6.22}$$

Therefore, we have that for all $t \in [0,T]$,

$$h(t) = -C.$$

□

By (6.21) and (6.22), the portfolio process (6.19) can also be written as

$$\hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}\big(\sigma(t)'\big)^{-1}\big(\bar{\sigma}_I(t)Y^{\hat{\pi}}(t) - \big(Y^{\hat{\pi}}(t) - C\big)\tilde{\theta}(t)\big),$$

for all $t \in [0,T]$.

Theorem 6.3.3. *Under the same assumptions as Theorem 6.3.1 and for $x\tilde{\beta}(0) \leq C$, the optimal real wealth process to the unconstrained optimization problem is given by*

$$Y^{\hat{\pi}}(t) = \big(x\tilde{\beta}(0) - C\big)e^{-\int_0^t \frac{3}{2}\|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)} + C, \tag{6.23}$$

for all $t \in [0, T]$.

Proof. Inserting the optimal portfolio process (6.19) into the stochastic differential equation (6.2), we obtain

$$\begin{aligned} dY^{\hat{\pi}}(t) =& \big(Y^{\hat{\pi}}(t)A(t) + \hat{\pi}(t)'M(t)\big)dt + \big(\hat{\pi}(t)'D(t) - Y^{\hat{\pi}}(t)\bar{\sigma}_I(t)\big)dW(t) \\ =& \Big(\bar{\sigma}_I(t)D(t)'\big(D(t)D(t)'\big)^{-1}M(t)Y^{\hat{\pi}}(t) \\ & - M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\big(Y^{\hat{\pi}}(t) + h(t)\big) + Y^{\hat{\pi}}(t)A(t)\Big)dt \\ & + \Big(\bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)Y^{\hat{\pi}}(t) \\ & - M(t)'(D(t)')^{-1}\big(Y^{\hat{\pi}}(t) + h(t)\big) - Y^{\hat{\pi}}(t)\bar{\sigma}_I(t)\Big)dW(t) \\ =& \Big(Y^{\hat{\pi}}(t)\big(A(t) + \bar{\sigma}_I(t)'D(t)^{-1}M(t)\big) \\ & - M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\big(Y^{\hat{\pi}}(t) + h(t)\big)\Big)dt \\ & - M(t)'(D(t)')^{-1}\big(Y^{\hat{\pi}}(t) + h(t)\big)dW(t). \end{aligned}$$

Now, by Corollary 6.3.2 and by (6.21) and (6.22), we can express the optimal real wealth process for the unconstrained portfolio problem by

$$dY^{\hat{\pi}}(t) = -\|\tilde{\theta}(t)\|^2\big(Y^{\hat{\pi}}(t) - C\big)dt - \tilde{\theta}(t)'\big(Y^{\hat{\pi}}(t) - C\big)dW(t).$$

Introducing the auxiliary process $Z(t) = Y^{\hat{\pi}}(t) - C$ with initial value $Z(0) = x\tilde{\beta}(0) - C$, we obtain by Itô's lemma

$$dZ(t) = -\|\tilde{\theta}(t)\|^2 Z(t)dt - \tilde{\theta}(t)'Z(t)dW(t),$$

which is the expression for a geometric Brownian motion, with solution

$$Z(t) = Z(0)e^{-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)}.$$

Hence, we can write the optimal wealth process corresponding to the portfolio process $\hat{\pi}$ of (6.19) as

$$Y^{\hat{\pi}}(t) = \big(x\tilde{\beta}(0) - C\big)e^{-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)} + C. \tag{6.24}$$

□

Corollary 6.3.4. *Under the same assumptions as in Theorem 6.3.1 and for* $x\tilde{\beta}(0) \leq C$, *the unconstrained optimization problem has an optimal solution pair* $\big(X^{\hat{\pi}}(t), \hat{\pi}(t)\big)$, *given by*

$$X^{\hat{\pi}}(t) = \frac{1}{\tilde{\beta}(t)}\big(x\tilde{\beta}(0) - C\big)e^{-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)} + \frac{C}{\tilde{\beta}(t)}, \tag{6.25}$$

$$\hat{\pi}(t) = \big(\sigma(t)'\big)^{-1}\nu(t), \tag{6.26}$$

where

$$\nu_i(t) = -\big(X^{\hat{\pi}}(t) - \frac{C}{\tilde{\beta}(t)}\big)\theta_i(t), \quad \text{for } i = 1, \ldots, n,$$
$$\nu_I(t) = X^{\hat{\pi}}(t)\sigma_I(t) - \big(X^{\hat{\pi}}(t) - \frac{C}{\tilde{\beta}(t)}\big)(\theta_I(t) - \sigma_I(t)),$$

for all $t \in [0, T]$.

Proof. Noting that $X^{\pi}(t)\tilde{\beta}(t) = Y^{\pi}(t)$, the optimal wealth process is directly obtained from the previous discussion. The expression of the optimal portfolio process follows from Theorem 6.3.1, by (6.21) and (6.22). □

6.3.2 With Bankruptcy Prohibition

In order to find an explicit solution for the optimal portfolio process and the corresponding wealth process, we proceed in a similar manner as in Section 3.4.2, by using the martingale approach. We first find an explicit solution to the BSDE

$$dY^{\pi}(t) = -Y^{\pi}(t)\sigma_I(t)d\tilde{W}_I(t) + \tilde{\beta}(t)\pi(t)'\sigma(t)d\tilde{W}(t),$$
$$Y^{\pi}(T) = \big(C - \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big)^+,$$

which the optimal wealth process must satisfy, due to Theorem 6.2.2. The optimal portfolio process is then obtained immediately by applying Itô's lemma to $Y^{\pi}(t)$.

Lemma 6.3.5. *Define* $y(t) = \tilde{\mathcal{Y}}(x)\mathbb{E}^{\mathbb{Q}^T}\big[\tilde{\xi}(T)|\mathcal{F}_t\big]$. *Then* $y(t)$ *satisfies the stochastic differential equation*

$$dy(t) = y(t)\big(- \|\tilde{\theta}(t)\|^2 dt - \tilde{\theta}(t)' dW(t)\big),$$
$$y(0)\tilde{\beta}(0) = \tilde{\mathcal{Y}}(x)\exp\big(\int_0^T \|\tilde{\theta}(s)\|^2 ds\big), \quad y(T) = \tilde{\mathcal{Y}}(x)\tilde{\xi}(T).$$

Proof. By applying Bayes' rule, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T}[\tilde{\xi}(T)|\mathcal{F}_t] =& \frac{1}{\tilde{Z}(t)}\mathbb{E}^{\mathbb{P}}\big[\tilde{Z}(T)\tilde{\xi}(T)|\mathcal{F}_t\big] \\ =& \frac{1}{\tilde{Z}(t)\tilde{\beta}(0)}e^{-\int_0^T \|\tilde{\theta}(s)\|^2 ds}\mathbb{E}^{\mathbb{P}}\big[e^{-2\int_0^T \tilde{\theta}(s)' dW(s)}|\mathcal{F}_t\big].\end{aligned}$$

Now for the term inside the conditional expectation, we use that the integral up to time t is $\mathcal{F}_t$-measurable and that Brownian motion has independent increments to write

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[e^{-2\int_0^T \tilde{\theta}(s)' dW(s)}|\mathcal{F}_t\Big] &= e^{-2\int_0^t \tilde{\theta}(s)' dW(s)}\mathbb{E}^{\mathbb{P}}\Big[e^{-2\int_t^T \tilde{\theta}(s)' dW(s)}|\mathcal{F}_t\Big] \\ &= e^{-2\int_0^t \tilde{\theta}(s)' dW(s)}e^{2\int_t^T \|\tilde{\theta}(s)\|^2 ds}.\end{aligned}$$

Crossing out and reworking the terms then yields

$$\mathbb{E}^{\mathbb{Q}^T}[\tilde{\xi}(T)|\mathcal{F}_t] = e^{-\int_0^T \left(r_R(s)-\|\tilde{\theta}(s)\|^2\right)ds-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds-\int_0^t \tilde{\theta}(s)dW(s)}.$$

Applying Itô's lemma finishes the proof. □

Similarly to the proof ot Theorem 3.4.8, the process $(y(t))_{t\in[0,T]}$ takes on the role of the underlying on which a European put option is written. Instead of relying on the Black Scholes equation for European put options, we solve the problem explicitly.

Lemma 6.3.6. *$\tilde{\xi}(T)$ is conditional on $\mathcal{F}_t$ log-normally distributed and we have*

$$\mathbb{E}^{\mathbb{Q}^T}\big[\mathbb{1}\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\}\big|\mathcal{F}_t\big] = \Phi\Bigg(\frac{\log\frac{C}{\tilde{\mathcal{Y}}(x)} - \big(\log\tilde{\xi}(t) + \frac{1}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds\big)}{\sqrt{\int_t^T \|\tilde{\theta}(s)\|^2 ds}}\Bigg), \tag{6.27}$$

for all $t \in [0,T]$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{v^2}{2}}dv$ is the cumulative distribution function of the standard normal distribution.

Proof. Note that using the $\mathbb{Q}^T$-Brownian motion $\tilde{W}$ from Lemma 5.3.1, we have

$$\tilde{\beta}(0)\tilde{\xi}(t) = \exp\big(\frac{1}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' d\tilde{W}(s)\big).$$

As the marginal distributions of Brownian motion are normal and its increments are independent, a Brownian motion $W(s)$ is also, conditional on

$\mathcal{F}_t$, normally distributed for $s > t$. Therefore, we need only prove that the conditional mean and variance of $\log \tilde{\xi}(T)$ have the correct values.

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T}[\log \tilde{\xi}(T)\big|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}^T}\big[\log \tilde{\xi}(T) - \log \tilde{\xi}(t)\big|\mathcal{F}_t\big] + \log \tilde{\xi}(T)\\ &= \mathbb{E}^{\mathbb{Q}^T}\big[\frac{1}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds - \int_t^T \tilde{\theta}(s)' d\tilde{W}(s)\big|\mathcal{F}_t\big] + \log \tilde{\xi}(t)\\ &= \frac{1}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds + \log \tilde{\xi}(t). \end{aligned} \tag{6.28}$$

Similarly, the conditional variance is given by

$$\begin{aligned}\mathbb{V}^{\mathbb{Q}^T}\big[\log \tilde{\xi}(T)\big|\mathcal{F}_t\big] &= \mathbb{V}^{\mathbb{Q}^T}\big[\log \tilde{\xi}(T) - \log \tilde{\xi}(t)\big|\mathcal{F}_t\big]\\ &= \mathbb{V}^{\mathbb{Q}^T}\big[-\int_t^T \tilde{\theta}(s)' d\tilde{W}(s)\big|\mathcal{F}_t\big]\\ &= \int_t^T \|\tilde{\theta}(s)\|^2 ds. \end{aligned} \tag{6.29}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T}\big[\mathbb{1}\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\}\big|\mathcal{F}_t\big] &= \mathbb{Q}^T\big[C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big|\mathcal{F}_t\big]\\ &= \mathbb{Q}^T\big[\log \tilde{\xi}(T) < \log \frac{C}{\tilde{\mathcal{Y}}(x)}\big|\mathcal{F}_t\big].\end{aligned}$$

Using that $\log \xi(T)$ is normally distributed with mean and variance given by (6.28) and (6.29), the claim follows. □

Lemma 6.3.7. *Using the same notation as in Lemma 6.3.6, we have*

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^T}\big[\tilde{\xi}(T)\mathbb{1}\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\}\big|\mathcal{F}_t\big]\\ &\qquad = y(t)\Phi\Bigg(\frac{\log \frac{C}{\tilde{\mathcal{Y}}(x)} - \big(\log \tilde{\xi}(t) + \frac{3}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds\big)}{\sqrt{\int_t^T \|\tilde{\theta}(s)\|^2 ds}}\Bigg).\end{aligned} \tag{6.30}$$

Proof. Define a new measure $\mathbb{Q}_2$ by $\mathbb{Q}_2[A] = \mathbb{E}^{\mathbb{Q}^T}[\tilde{Z}_2(T)\mathbb{1}\{A\}]$ for all $A \in \mathcal{F}$, with

$$\tilde{Z}_2(t) = \exp\big(-\frac{1}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' d\tilde{W}(s)\big),$$

for all $t \in [0, T]$. Due to similar arguments as in Lemma 2.2.1 and Lemma 5.3.1, the process $\big(\tilde{W}_2(t)\big)_{t\in[0,T]}$ defined by $\tilde{W}_2(t) = \tilde{W}(t) + \int_0^t \tilde{\theta}(s)ds$ is a

$\mathbb{Q}_2$-Brownian motion. We have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T}\big[\tilde{\xi}(T)\mathbb{1}\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\}\big|\mathcal{F}_t\big] &= y(t)\mathbb{E}^{\mathbb{Q}^T}\big[\frac{y(T)}{y(t)}\mathbb{1}\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\}\big|\mathcal{F}_t\big] \\ &= y(t)\mathbb{E}^{\mathbb{Q}^T}\big[\frac{\tilde{Z}_2(T)}{\tilde{Z}_2(t)}\mathbb{1}\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\}\big|\mathcal{F}_t\big] \\ &= y(t)\mathbb{Q}_2\big[C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big|\mathcal{F}_t\big],\end{aligned}$$

by Bayes' rule. Under $\mathbb{Q}_2$,

$$\tilde{\xi}(t) = \exp\big(\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' d\tilde{W}_2(s)\big),$$

and hence

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_2}\big[\log\tilde{\xi}(T)\big|\mathcal{F}_t\big] &= \mathbb{E}^{\mathbb{Q}_2}\big[\log\tilde{\xi}(T) - \log\tilde{\xi}(t)\big|\mathcal{F}_t\big] + \log\tilde{\xi}(t) \\ &= \mathbb{E}^{\mathbb{Q}_2}\big[\frac{3}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds - \int_t^T \tilde{\theta}(s)' d\tilde{W}_2(s)\big|\mathcal{F}_t\big] + \log\tilde{\xi}(t) \\ &= \log\tilde{\xi}(t) + \frac{3}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds.\end{aligned}$$

Similarly, we obtain $\mathbb{V}^{\mathbb{Q}_2}\big[\log\tilde{\xi}(T)\big|\mathcal{F}_t\big] = \int_t^T \|\tilde{\theta}(s)\|^2 ds$. Therefore, $\log\tilde{\xi}(T)$ is conditionally normal distributed under $\mathbb{Q}_2$ and the claim follows. □

We are now able to prove the equivalent of Theorem 3.4.8 in the market model (5.6) under the presence of inflation.

Theorem 6.3.8. *Under Assumptions 5.2.1 and 6.1.8 and for* $x\tilde{\beta}(0) \le C$*, the optimal real wealth process is given by*

$$Y^{\hat{\pi}}(t) = C\Phi\big(-d_-(t, y(t))\big) - y(t)\Phi\big(-d_+(t, y(t))\big), \tag{6.31}$$

for all $t \in [0, T]$*, where* $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{v^2}{2}} dv$ *is the cumulative distribution function of the standard normal distribution and*

$$\begin{aligned}d_+(t, y) &= \frac{\log(\frac{y}{C}) + \frac{1}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds}{\sqrt{\int_t^T \|\tilde{\theta}(s)\|^s ds}}, \\ d_-(t, y) &= d_+(t, y) - \sqrt{\int_t^T \|\tilde{\theta}(s)\|^2 ds}.\end{aligned}$$

Furthermore, the process y is given by

$$y(t) = \tilde{\mathcal{Y}}(x) \exp\Big(- \int_0^T (r_R(s) - \|\tilde{\theta}(s)\|^2) ds\Big)$$
$$\exp\Big(- \frac{3}{2} \int_0^t \|\tilde{\theta}(s)\|^2) ds - \int_0^t \tilde{\theta}(s)' dW(s)\Big). \tag{6.32}$$

Proof. We know that the $\mathbb{Q}^T$-local martingale $Y^{\hat{\pi}}(t)$ is uniformly integrable and therefore satisfies

$$\begin{aligned} Y^{\hat{\pi}}(t) =& \mathbb{E}^{\mathbb{Q}^T}\big[\big(C - \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big)^+ \big| \mathcal{F}_t\big] \\ =& \mathbb{E}^{\mathbb{Q}^T}\big[\big(C - \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big)\mathbb{1}\big\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big\} \big| \mathcal{F}_t\big] \\ =& C\mathbb{E}^{\mathbb{Q}^T}\big[\mathbb{1}\big\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big\} \big| \mathcal{F}_t\big] \\ & - \tilde{\mathcal{Y}}(x)\mathbb{E}^{\mathbb{Q}^T}\big[\tilde{\xi}(T)\mathbb{1}\big\{C > \tilde{\mathcal{Y}}(x)\tilde{\xi}(T)\big\} \big| \mathcal{F}_t\big], \end{aligned}$$

where the two terms are given by Lemma 6.3.6 and Lemma 6.3.7. Note that we have

$$\begin{aligned} & - \log \tilde{\xi}(t) - \frac{1}{2} \int_t^T \|\tilde{\theta}(s)\|^2 ds \\ = & \int_0^T \big(r_R(s) - \frac{1}{2}\|\tilde{\theta}(s)\|^2\big) ds + \int_0^t \|\tilde{\theta}(s)\|^2 ds + \int_0^t \tilde{\theta}(s)' dW(s) \\ = & - \Big(- \int_0^T \big(r_R(s) - \|\tilde{\theta}(s)\|^2\big) ds - \int_0^t \tilde{\theta}(s)' dW(s) - \frac{3}{2} \int_0^t \|\tilde{\theta}(s)\|^2 ds\Big) \\ & + \frac{1}{2} \int_t^T \|\tilde{\theta}(s)\|^2 ds, \end{aligned}$$

and therefore

$$\log \frac{C}{\tilde{\mathcal{Y}}(x)} - \log \tilde{\xi}(t) - \frac{1}{2} \int_t^T \|\tilde{\theta}(s)\|^2 ds = \log \frac{C}{y(t)} + \frac{1}{2} \int_t^T \|\tilde{\theta}(s)\|^2 ds.$$

We have shown that

$$-d_-(t, y(t)) = \log \frac{C}{\tilde{\mathcal{Y}}(x)} - \big(\log \tilde{\xi}(t) + \frac{1}{2} \int_t^T \|\tilde{\theta}(s)\|^2 ds\big).$$

Similar calculations for $-d_+(t, y(t))$ finish the proof. □

Corollary 6.3.9. *Under the same notations and assumptions as in Theorem 6.3.8, the optimal portfolio process to Problem 6.1.1 is given by*

$$\hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}(\sigma(t)')^{-1}\nu(t), \tag{6.33}$$

where

$$\nu_i(t) = y(t)\Phi\big(-d_+(t,y(t))\big)\theta_i(t), \qquad \text{for } i = 1,\dots,n$$
$$\nu_I(t) = Y^{\hat{\pi}}(t)\sigma_I(t) + y(t)\Phi\big(-d_+(t,y(t))\big)(\theta_I(t) - \sigma_I(t)).$$

Proof. Let $f(t,y(t)) = Y^{\hat{\pi}}(t)$ and assume that $f \in \mathcal{C}^{1,2}$. Then by Itô's lemma,

$$df(t,y) = yf_y(t,y)\big(-\tilde{\theta}(t)'d\tilde{W}(t)\big) + f_t(t,y)dt + \frac{1}{2}f_{yy}(t,y)y^2\|\tilde{\theta}(t)\|^2dt.$$

Comparing the volatility terms with those in (6.4) yields

$$\hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}\big(\sigma(t)'\big)^{-1}\big(Y^{\hat{\pi}}(t)\bar{\sigma}_I(t) - y(t)f_y(t,y(t))\tilde{\theta}(t)\big).$$

Comparing the drift terms, we have

$$f_t(t,y) + \frac{1}{2}\|\tilde{\theta}(t)\|^2y^2f_{yy}(t,y) = 0, \quad f(T,y) = \big(C-y\big)^+. \tag{6.34}$$

But (6.34) is exactly the Black Scholes equation for a European put option and hence allows for the explicit solution

$$f(t,y) = C\Phi(-d_-(t,y)) - \Phi(-d_+(t,y))y.$$

Similarly as in Corollary 3.4.9 we can show that

$$f_y(t,y) = -\Phi(-d_+(t,y)),$$

and the claim follows. □

Note that since we have

$$y(t)\Phi\big(-d_+(t,y(t))\big) = -\big(Y^{\hat{\pi}}(t) - C\Phi\big(-d_-(t,y(t))\big)\big),$$

by Theorem 6.3.8, we can also write

$$\nu_i(t) = -\big(Y^{\hat{\pi}}(t) - C\Phi\big(-d_-(t,y(t))\big)\big)\theta_i(t), \qquad \text{for } i = 1,\dots,n,$$
$$\nu_I(t) = Y^{\hat{\pi}}(t)\sigma_I(t) - \big(Y^{\hat{\pi}}(t) - C\Phi\big(-d_-(t,y(t))\big)\big)(\theta_I(t) - \sigma_I(t)).$$

Remark. If $C = \tilde{\beta}(0)x$, the corresponding optimal portfolio process is risk-free. By Theorem 6.2.2 we have $Y^{\hat{\pi}}(t) = C$ for all $t \in [0,T]$ and hence $dY^{\hat{\pi}}(t) = 0$, which indicates that the corresponding portfolio is

$$\hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}(\sigma(t)')^{-1}\psi,$$

where $\psi = (0,\ldots,0,C\sigma_I)$.

Corollary 6.3.10. *Under Assumptions 5.2.1 and 6.1.8, for $b > 0$ and $x\tilde{\beta}(0) \leq C$, the optimization Problem 6.1.1 has an optimal solution pair $\big(X^{\hat{\pi}}(t), \hat{\pi}(t)\big)$, given by*

$$X^{\hat{\pi}}(t) = \frac{1}{\tilde{\beta}(t)}\Big(C\Phi(-d_-(t,y(t))) - y(t)\Phi(-d_+(t,y(t)))\Big), \tag{6.35}$$

$$\hat{\pi}(t) = (\sigma(t)')^{-1}\nu(t), \tag{6.36}$$

where $y(t)$ is defined in Lemma 6.3.5, $\tilde{\mathcal{Y}}(x)$ is given by (6.13) *and where*

$$\nu_i(t) = -\big(X^{\hat{\pi}}(t) - \frac{C}{\tilde{\beta}(t)}\Phi\big(-d_-(t,y(t))\big)\big)\theta_i(t), \qquad \text{for } i = 1,\ldots,n,$$

$$\nu_I(t) = X^{\hat{\pi}}(t)\sigma_I(t) - \big(X^{\hat{\pi}}(t) - \frac{C}{\tilde{\beta}(t)}\Phi\big(-d_-(t,y(t))\big)\big)(\theta_I(t) - \sigma_I(t)),$$

for all $t \in [0,T]$.

Proof. Noting that $X^{\hat{\pi}}(t)\tilde{\beta}(t) = Y^{\hat{\pi}}(t)$, the conclusion is directly obtained from the previous discussion. □

Example 6.3.11. To gain a general impression of the behavior of the portfolio processes, we repeat the analysis of Example 3.4.10, adding an inflation-linked bond to the market. We use the same underlying parameters of the stock process as in Example 3.4.10, i.e. $r_N = 5\%$, $\mu = 8\%$ and $\sigma_S = 0.15$. The parameters for the inflation-linked bond are calculated empirically and we obtain $r_R = 4\%$, $\sigma_I = 0.05$, $\theta_I = 0.12$ and the volatility of the stock with respect to the inflation is $\sigma_{IS} = 0.04$. The investor starts with $x = 1000$ and the target wealth is set to $C = 5000$.

Comparing Figure 6.4 to Figure 3.3 in Example 3.4.10, we note that the wealth processes follow very similar paths, ending slightly higher for both strategies. The additional wealth spent on the inflation-linked bond is mainly borrowed from the bank account and only reduces the amount invested in the stock slightly. The biggest difference can be seen during the

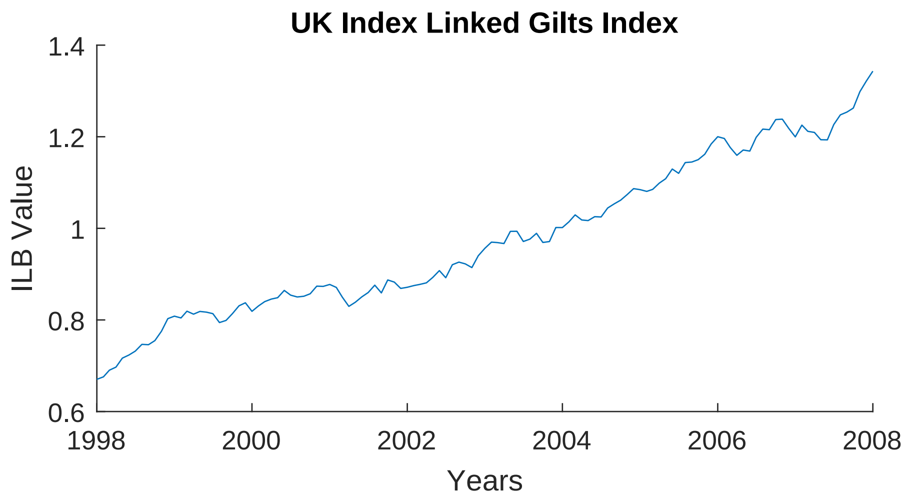


Figure 6.3: Normalized monthly historical time series of the UK Index Linked Gilts Total Returns Index between January 1998 and December 2007.

bear market of 2000 to 2002, where less money is invested in the stock and both wealth processes stay positive with the inclusion of the inflation-linked bond.

Remark. It is quickly apparent that if there is no inflation in the market, i.e. $I(t) \equiv 1$ for all $t \in [0, T]$, both the unconstrained optimal portfolio (6.26) and the corresponding wealth process (6.25), as well as the constrained optimal portfolio process (6.33) and the corresponding wealth process (6.35) reduce to their respective counterparts in Section 3.4. Note that even though the porftolio process for the stocks seems to be the same even under the presence of inflation, it is dependent on the market price of risk for the market model in Section 5.2, which is different to the market price of risk for the market model in Section 2.1. For a complete analysis of the different portfolios, we refer to Chapter 11.

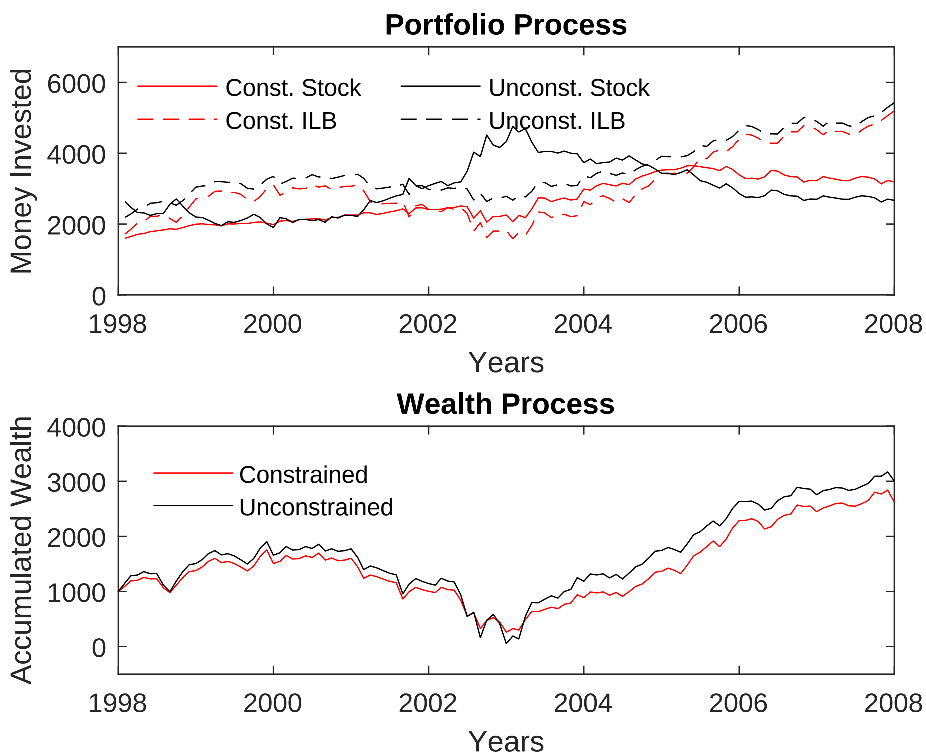


Figure 6.4: In the upper graph, the amount invested in the stock and the index-linked bond can be seen for both the constrained and the unconstrained optimal portfolio. The resulting wealth process is plotted below.

Chapter 7

The Constrained Optimal Strategy: Lower Bound

As a next step we introduce a lower constraint to limit the terminal wealth from below. We give a summary of Chapter 4 and utilize the same techniques to find the optimal terminal wealth and the corresponding portfolio process. We retain all notation from Chapter 6.

7.1 Problem Formulation

Problem 7.1.1. Given a constant C and a real number $K > 0$, we consider the problem of finding a portfolio process $\hat{\pi}^l \in \mathcal{A}(x)$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(C - \frac{X^{\hat{\pi}^l}(T)}{I(T)}\Big)^2\Big] = \inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\Big(C - \frac{X^{\pi}(T)}{I(T)}\Big)^2\Big],$$
$$\text{subject to} \qquad \frac{X^{\hat{\pi}^l}(T)}{I(T)} \geq K, \quad \text{a.s. }, \tag{7.1}$$

and the pair $\big(X^{\hat{\pi}^l}(t), \hat{\pi}^l(t)\big)$ satisfies the stochastic differential equation (5.8).

Note that we could instead set the lower constraint on the nominal wealth directly, without the inflation adjustment. We will see in the following discussion, that in this case, the approach of Chapter 4 does not produce a usable portfolio strategy.

Recall the numéraires corresponding to the bank account and the inflation linked bond as well as the auxiliary process $\tilde{\xi}(t)$, given by

$$\beta(t) = \frac{1}{B(t)} = \exp(-\int_0^t r_N(s)ds),$$
$$\tilde{\beta}(t) = \frac{1}{B^*(t, I(t))} = \frac{1}{B_R(t,T)I(t)},$$
$$\tilde{\xi}(t) = \frac{\xi(t)}{\tilde{\beta}(t)} = \frac{\tilde{Z}(t)}{\tilde{\beta}(0)},$$

and the real wealth process $Y^\pi(t) = \tilde{\beta}(t)X^\pi(t)$. Similar to Section 4.1 we also note that the non-negativity constraint is non-binding if $K > 0$ and that the set of optimal portfolio processes for Problem 7.1.1 is empty unless we have $K \le x\tilde{\beta}(T)$. For the rest of this chapter, we will hence assume that

$$K < \tilde{\beta}(T)x.$$

7.2 Solution of the Constrained Problem

In order to obtain a similar result to Corollary 4.2.2 we utilize Proposition 4.2.1 with the current utility function. The optimal terminal wealth then consists of an unconstrained wealth process plus a put option. Using the explicit solution for the optimal wealth process for the unconstrained optimization problem (6.23) at maturity, we find the initial wealth used for the unconstrained wealth process.

Corollary 7.2.1. *Under Assumptions 5.2.1 and 6.1.8, the optimal terminal real wealth for Problem 7.1.1 is of the form*

$$Y^{\hat{\pi}^l}(T) = \hat{Y}^{\hat{\pi}}(T) + \left(K - \hat{Y}^{\hat{\pi}}(T)\right)^+, \tag{7.2}$$

where $\hat{Y}^{\hat{\pi}}(t)$ is the optimal wealth process from (6.23) *with*

$$\hat{x}_0 = C\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)] - y\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)^2],$$

where y is chosen in such a way that the terminal wealth given by (7.2) *satisfies the admissibility constraint $\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)Y^{\hat{\pi}^l}(T)] = x$.*

Proof. By Proposition 4.2.1, we know that the optimal terminal wealth is given by

$$Y^{\hat{\pi}^l}(T) = \max(K, I(y\tilde{\xi}(T))).$$

Now for Problem 7.1.1, we have $U(x) = -\frac{1}{2}\big(C - x\big)^2$ and hence

$$\hat{I}(y\tilde{\xi}(T)) = C - y\tilde{\xi}(T).$$

By (6.23), for $t = T$, we have

$$\hat{Y}^{\hat{\pi}}(T) = (\hat{x}_0 - \frac{C}{\tilde{\beta}(0)})\tilde{\beta}(0)^2\tilde{\xi}(T)e^{-\int_0^T \|\tilde{\theta}(t)\|^2 dt} + C.$$

In order for the terminal wealth to be of the form (7.2), we determine $\hat{x}_0$ such that $\hat{I}(y\tilde{\xi}(T)) = \hat{Y}^{\hat{\pi}}(T)$, which is the case if and only if

$$C - y\xi(T) = (\hat{x}_0 - \frac{C}{\tilde{\beta}(0)})\tilde{\beta}(0)^2\xi(T)e^{-\int_0^T \|\tilde{\theta}(t)\|^2 dt} + C,$$

which in turn is equivalent to

$$\hat{x}_0 = \frac{C}{\tilde{\beta}(0)} - \frac{y}{\tilde{\beta}(0)^2}e^{\int_0^T \|\tilde{\theta}(t)\|^2 dt} = C\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)] - y\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)^2].$$

□

Due to the previous corollary, we can state the equation for the nominal wealth, which is of the form

$$X^{\hat{\pi}^l}(T) = \hat{X}^{\hat{\pi}}(T) + \big(KI(T) - \hat{X}^{\hat{\pi}}(T)\big)^+,$$

where $\hat{Y}^{\hat{\pi}}(t)$ is the optimal wealth process from (6.25) and $\hat{x}_0$ is chosen as in Corollary 7.2.1.

Remark. If the lower constraint in the Problem 7.1.1 was instead of the form $K \geq X^{\hat{\pi}}(T)$, then the optimal nominal terminal wealth would be

$$X^{\hat{\pi}^l}(T) = \hat{X}^{\hat{\pi}}(T) + \big(K - \hat{X}^{\hat{\pi}}(T)\big)^+.$$

7.3 The Optimal Portfolio Process

We again use (7.2), in order to obtain the optimal portfolio process in terms of the constrained optimal portfolio process (6.33). Namely, rewriting (7.2), we have

$$Y^{\hat{\pi}^l}(T) = \begin{cases} C & \text{if } x\tilde{\beta}(0) \geq C, \\ \max\big(C - y\tilde{\xi}(T), K\big) & \text{else,} \end{cases} \tag{7.3}$$

where y is chosen such that

$$\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)\big((C-K)-y\tilde{\xi}(T)\big)^{+}] = x - K\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)].$$

Hence we state the optimal wealth process as well as the optimal portfolio process in terms of the optimal processes under the non-negativity constraint. Define

$$\hat{x} = x - Ke^{-\int_0^T r_R(s)ds},$$
$$\hat{C} = C - K.$$

Theorem 7.3.1. *Denote by* $\hat{Y}^{\hat{\pi}}(t;\hat{x},\hat{C})$ *the optimal wealth process* (6.31) *at time* t *with initial wealth* $\hat{x}\tilde{\beta}(0)$ *and fixed claim* $\hat{C}$*. Under Assumptions 5.2.1, 6.1.8 and for* $x\tilde{\beta}(0) \leq C$ *the optimal wealth process to Problem 7.1.1 is given by*

$$Y^{\hat{\pi}^l}(t) = \hat{Y}^{\hat{\pi}}(t;\hat{x},\hat{C}) + K. \tag{7.4}$$

Similarly, denoting by $\hat{\pi}(t;\hat{x},\hat{C})$ *the optimal portfolio process* (6.33) *at time* t*, the optimal portfolio process to Problem 7.1.1 is given by*

$$\hat{\pi}^l(t) = \hat{\pi}\big(t;\hat{x},\hat{C}\big) + K. \tag{7.5}$$

The proof follows in the same way as the proof of Theorem 4.3.1 and is therefore omitted.

The nominal wealth process can therefore be expressed as

$$X^{\hat{\pi}^l}(t) = \hat{X}^{\hat{\pi}}(t;\hat{x},\hat{C}) + \frac{K}{\tilde{\beta}(t)},$$

with the same notation as in Theorem 7.3.1.

Remark. If the lower constraint in the Problem 7.1.1 was instead of the form $K \geq X^{\hat{\pi}}(T)$, then the auxillary processes $\hat{x}$ and $\hat{C}$ would instead be of the form

$$\hat{x} = x - K\beta(T),$$
$$\hat{C} = C - K\tilde{\beta}(T).$$

It is clear, that $\hat{C}$, and therefore also $\hat{\pi}^l(t)$, is then not $\mathcal{F}_t$-measurable for any $t < T$.

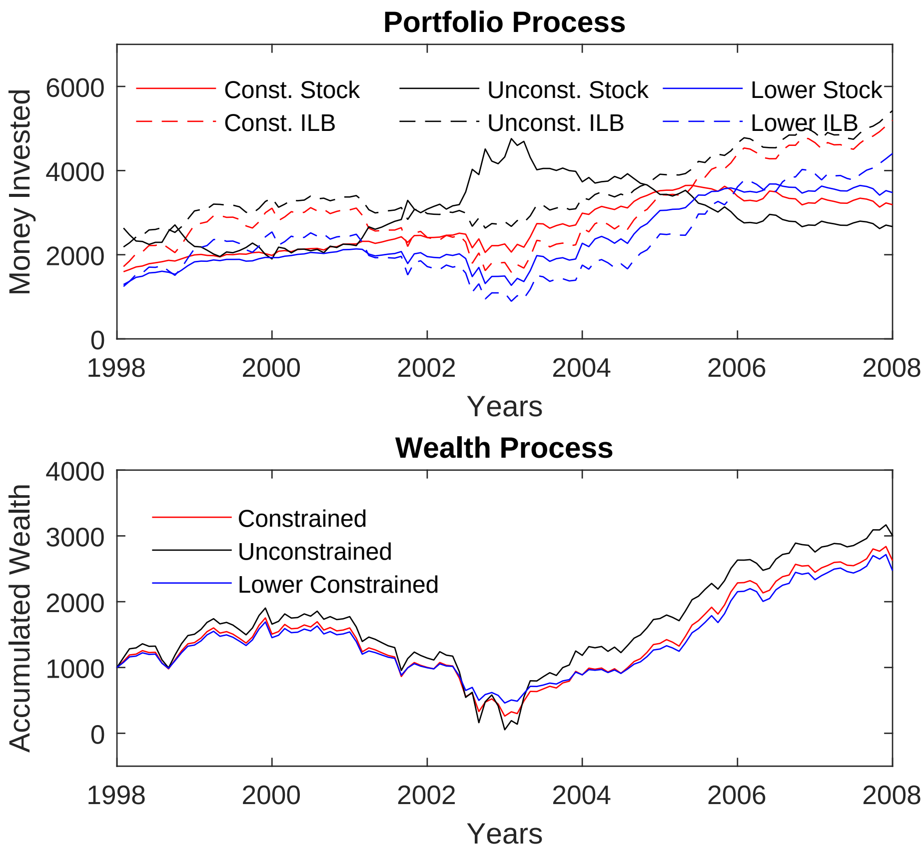


Figure 7.1: In the upper graph, the amount invested in the stock and the inflation-linked bond can be seen for the constrained portfolio, the unconstrained portfolio and the portfolio process with a lower constraint of $K = 500$. The resulting wealth process is plotted below.

Example 7.3.2. We continue the analysis of Example 6.3.11, including the portfolio and wealth process of the optimal strategy with a positive constraint. In Figure 7.1 we plot the optimal portfolio strategy with a lower bound of $K = 500$ in addition to the constrained and the unconstrained optimal portfolio processes. We observe that even though the addition of the inflation-linked bond does only slightly affect the paths of the wealth processes, the amount invested in the stock differs less between the different strategies than in Example 4.3.2 and consequently the investment looks to be much less affected by the movements in the underlying stock market.

We see in Table 7.1 that all strategies profit from the additional market

	Unconstrained	**Constrained**	**Lower Bound**
Terminal Wealth	3'000	2'620	2'470
Rate of Return	11.0%	9.6%	9.1%
Minimal Wealth	54	261	460

Table 7.1: Properties of the different portfolio processes calculated for the period 1998-2008 with the FTSE-Actuaries All Share Index as the underlying stock and the UK Index Linked Gilts Index as the inflation-linked bond.

element and show a significantly higher rate of return. We see that the lower bound of $K = 500$ only has a slight effect on the investment behavior, leading to less money invested in the risky assets at the start of the investment period. By looking at the structure of the optimal terminal wealth (7.2) it becomes apparent, that for higher K, less money is available to be invested in the risky assets.

Appendix

II.A Proof of Theorem 6.3.1

First, some preliminary results from Linear Algebra follow, which will be utilized frequently in the coming proofs.

Lemma II.A.1. *Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ be invertible matrices. Then*

i) $(AB)' = B'A'$;

ii) $B^{-1}A^{-1}$ *is the inverse of* AB;

iii) $(A^{-1})' = (A')^{-1}$.

Proof. i) Let $A = (a_{ij})$ and $B = (b_{jk})$ for $i, k = 1, \ldots, n$ and $j = 1, \ldots, m$. Then

$$(AB)'_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = \sum_{j=1}^{n} b_{jk}a_{ij} = (B'A')_{ik}.$$

ii) As matrix multiplication is associative, we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = \mathbb{1},$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = \mathbb{1},$$

where $\mathbb{1}$ denotes the identity matrix in $\mathbb{R}^{m\times m}$.

iii) Similarly,

$$A'(A^{-1})' = (A^{-1}A)' = \mathbb{1}' = \mathbb{1},$$
$$(A^{-1})'A' = (AA^{-1})' = \mathbb{1}' = \mathbb{1},$$

where we have used i). Therefore, $(A^{-1})'$ is the inverse of A'.

□

The previous lemma allows us to solve the HJB equation corresponding to the unconstrained portfolio problem corresponding to the real wealth process (6.15). Problem 6.1.3 is equivalent to maximizing

$$J(x,\pi)=\mathbb{E}^{\mathbb{P}}\big[-\frac{1}{2}\big(Y^{\pi}(T)-C\big)^2\big],$$

over all admissible strategies $\pi\in\Pi$, such that the pair $\big(Y(t),\pi(t)\big)$ satisfies the stochastic differential equation (6.2). Denote by

$$V(t,y)=\sup_{\pi\in\Pi}\mathbb{E}^{\mathbb{P}}\big[-\frac{1}{2}\big(Y^{\pi}(T)-C\big)^2\big|Y^{\pi}(t)=y\big].$$

Then, by Theorem 3.4.2, the optimal value function satisfies the HJB equations, given by

$$\begin{aligned}-\frac{1}{2}\big(y-C\big)^2=&V(T,y),\\ 0=&V_t(t,y)+A(t)yV_y(t,y)\\ &+\min_{\pi\in\Pi}\Big\{\pi(t)'M(t)V_y(t,y)\\ &+\frac{1}{2}\Big(\pi(t)'D(t)D(t)'\pi(t)-2\pi(t)'D(t)\bar{\sigma}_I(t)y\\ &+\bar{\sigma}_I(t)'\bar{\sigma}_I(t)y^2\Big)V_{yy}(t,y)\Big\}.\end{aligned}\tag{7.6}$$

Therefore, the optimal portfolio process $\hat{\pi}$ is of the form

$$\hat{\pi}(t)=\big(D(t)D(t)'\big)^{-1}\Big(D(t)\bar{\sigma}_I(t)Y^{\pi}(t)-M(t)\frac{V_y(t,Y^{\pi}(t))}{V_{yy}(t,Y^{\pi}(t))}\Big).\tag{7.7}$$

To prove Theorem 6.3.1, we proceed similarly to our approach in Section 3.4.1, by assuming that the optimal value function is of a certain quadratic form. With the help of the HJB equations (7.6) and by eliminating the dependencies on y we obtain the differential equations (6.17) and (6.18).

Lemma II.A.2. *Assume that the optimal value function is of the form*

$$V(t,y)=a(t)y^2+b(t)y+c(t).\tag{7.8}$$

Then $a(t)$ satisfies the Riccati equation (6.17) *and $b(t)$ satisfies the Riccati equation* (6.18)*.*

Proof. Note that due to the initial condition for the HJB equation (7.6), we obtain $a(T) = -\frac{1}{2}$ and $b(T) = C$ immediately. Inserting the optimal portfolio process $\hat{\pi}(t)$, given by (6.19), into the HJB equation yields

$$\begin{aligned}
0 &= V_t(t,y) + A(t)yV_y(t,y) + \frac{1}{2}\bar{\sigma}_I(t)'\bar{\sigma}_I(t)y^2V_{yy}(t,y) \\
&\quad + \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}M(t)V_y(t,y)y \\
&\quad - M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\frac{V_y(t,y)^2}{V_{yy}(t,y)} \\
&\quad + \frac{1}{2}\bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)V_{yy}(t,y)y^2 \\
&\quad + \frac{1}{2}M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\frac{V_y(t,y)^2}{V_{yy}(t,y)} \\
&\quad - \frac{1}{2}\bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}M(t)V_y(t,y)y \\
&\quad - \frac{1}{2}M(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)V_y(t,y)y \\
&\quad - \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)V_{yy}(t,y)y^2 \\
&\quad + M(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)V_y(t,y)y
\end{aligned} \tag{7.9}$$

$$\begin{aligned}
&= V_t(t,y) + A(t)yV_y(t,y) + \frac{1}{2}\bar{\sigma}_I(t)'\bar{\sigma}_I(t)y^2V_{yy}(t,y) \\
&\quad - \frac{1}{2}\Big(M(t)\frac{V_y(t,y)}{V_{yy}(t,y)} - D(t)\bar{\sigma}_I(t)y\Big)'\big(D(t)D(t)'\big)^{-1} \\
&\quad \times \Big(M(t)\frac{V_y(t,y)}{V_{yy}(t,y)} - D(t)\bar{\sigma}_I(t)y\Big)V_{yy}(t,y).
\end{aligned} \tag{7.10}$$

Note that for the value function given by (7.8), we have

$$\begin{aligned}
V_t(t,y) &= a_t(t)y^2 + b_t(t)y + c_t(t), & V_y(t,y) &= 2a(t)y + b(t), \\
V_{yy}(t,y) &= 2a(t), & \frac{V_y(t,y)^2}{V_{yy}(t,y)} &= 2a(t)y^2 + 2b(t)y + \frac{b(t)^2}{2a(t)}.
\end{aligned}$$

Inserting these into the above equation and grouping the terms depending on y^2, we obtain

$$0 = a_t(t) + \Big[2A(t) + \bar{\sigma}_I(t)'\bar{\sigma}_I(t) - \varphi(t)'\big(D(t)D(t)'\big)\varphi(t)\Big]a(t),$$

which is exactly the Riccati equation, given by (6.17). Similarly, when grouping all terms depending on y, we obtain

$$0 = b_t(T) + \Big[A(t) - \frac{1}{2}M(t)'\varphi(t) - \frac{1}{2}\varphi(t)M(t)\Big]b(t).$$

□

Lemma II.A.3. *The functions $a(t), b(t)$ and $c(t)$ in (7.8) are given by*

$$a(t) = -\frac{1}{2}\exp\Big(\int_t^T \big(2A(s) + \bar{\sigma}_I(s)'\bar{\sigma}_I(s) - \varphi(s)'\big(D(s)D(s)'\big)\varphi(s)\big)ds\Big), \tag{7.11}$$

$$b(t) = C\exp\Big(\int_t^T \big(A(s) - \frac{1}{2}M(s)'\varphi(s) - \frac{1}{2}\varphi(s)'M(s)\big)ds\Big), \tag{7.12}$$

$$c(t) = -\frac{1}{2}C^2\exp\Big(\int_t^T -M(s)'\big(D(s)D(s)'\big)^{-1}M(s)ds\Big). \tag{7.13}$$

Proof. The explicit solutions to $a(t)$ and $b(t)$ follow immediately from the Riccati equations (6.17) and (6.18). To obtain an explicit solution for $c(t)$, we group the constant terms in (7.10) and see that $c(t)$ must satisfy

$$\begin{aligned} 0 &= c_t(t) - \frac{1}{2}M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\frac{b(t)^2}{2a(t)}, \\ c(T) &= -\frac{1}{2}C^2. \end{aligned} \tag{7.14}$$

Inserting $a(t)$ and $b(t)$ yields

$$\begin{aligned} \frac{b(t)^2}{2a(t)} &= -C^2\exp\Big(\int_t^T \big(-\bar{\sigma}_I(s)'\bar{\sigma}_I(s) - M(s)'\big(D(s)D(s)'\big)^{-1}M(s) \\ &\qquad + \bar{\sigma}_I(s)'D(s)'\big(D(s)D(s)'\big)^{-1}D(s)\bar{\sigma}_I(s)\big)ds\Big) \\ &= -C^2\exp\Big(\int_t^T -M(s)'\big(D(s)D(s)'\big)^{-1}M(s)ds\Big), \end{aligned}$$

where the last step follows from Lemma II.A.1. Inserting this into (7.14) yields the claim. □

The proof of Theorem 6.3.1 now follows by inserting the explicit values into (7.7). Namely

$$\begin{aligned} \hat{\pi}(t) &= \big(D(t)D(t)'\big)^{-1}\Big(D(t)\bar{\sigma}_I(t)Y^\pi(t) - M(t)\frac{V_y(t, Y^\pi(t))}{V_{yy}(t, Y^\pi(t))}\Big) \\ &= \big(D(t)D(t)'\big)^{-1}\Big(\big(D(t)\bar{\sigma}_I(t) - M(t)\big)Y^\pi(t) - M(t)\frac{b(t)}{2a(t)}\Big). \end{aligned}$$

Part III

Introducing Contributions

Chapter 8

The Financial Market Model

In Parts I and II we analyzed the optimal portfolio problem for the quadratic utility function in markets including inflation-linked bonds, as well as in markets without such bonds. An investor was considered, who started at a fixed, given wealth $x > 0$, and who followed a self-financing strategy. This means that until maturity, no money would flow in or out of the portfolio.

Now, we view the investor as part of a pension scheme. Pension funds may be categorized either as defined benefit (DB) or defined contribution (DC), depending on who carries more of the risk. In a DB pension scheme, benefits are fixed in advance by the sponsor and contributions are adjusted throughout the duration of the accumulation phase, so that benefits can be paid in full. Therefore, the risk is generally borne by the sponsor in DB pension schemes. On the other hand, in DC pension schemes, the contributions are fixed and the benefits are the accumulated value of the contributions at the retirement date. The risk of failing to get the expected benefits is consequently borne by the contributors in a DC scheme. For most of the 20th century, the majority of pension schemes were DB schemes, but due to the increase in life expectancy in most countries, this has changed. Nowadays, most private pension schemes are based on DC.

In order to be able to compare the target-based approach to the portfolio strategies currently in use, as well as to other strategies proposed in literature, we introduce contributions to the market model.

8.1 The Market Model

The market model consists of the same tradable objects as the one of Section 5.2. Recall that we work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we have, for a given, finite time horizon T, an $n+1$-dimensional Brownian motion $W(t) = \big(W_1(t), \ldots, W_n(t), W_I(t)\big), t \in [0,T]$. A risk-free bond $(B(t))_{t\in[0,T]}$, n stocks $(S_i(t))_{t\in[0,T]}$ and an inflation-linked zero-coupon bond $(B^*(t, I(t)))_{t\in[0,T]}$, are tradable and their price dynamics are given by

$$dB(t) = r_N(t)B(t)dt, \tag{8.1}$$

$$dS_i(t) = S_i(t)\Big[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{i,j}(t)dW_j(t) + \sigma_{i,I}(t)dW_I(t)\Big], \tag{8.2}$$

$$dB^*(t, I(t)) = B^*(t, I(t))\Big[\big(r_N(t) + \sigma_I(t)\theta_I(t)\big)dt + \sigma_I(t)dW_I(t)\Big], \tag{8.3}$$

where $B(0) = 1$ and $B^*(0, I(0)) = \mathbb{E}^{\mathbb{P}}\big[e^{-\int_0^T r_R(t)dt}\big]$, $\mathbb{P}$-a.s.

In most DC schemes, contributions are determined as a fixed percentage δ of the salary of the pension member. We give an overview of possible dynamics of the salary process $\big(L(t)\big)_{t\in[0,T]}$ which have been considered in literature, before deciding which structure will be most beneficial for the current analysis.

- The simplest construction is that of a constant salary,

  $$L(t) = l,$$

  which has been used in [Korn and Krekel, 2003], [Vigna, 2014], [Di Giacinto et al., 2011] and [Gerrard et al., 2014]. Contributions can either be modeled discretely or continuously. Assuming the salary of the plan members stays the same over the very long time period of pension schemes is not very realistic. Yearly salary increases vary between work fields and countries, but have averaged over 7% during the last 40 years in the UK, as reported by the Office for National Statistics [1]. Moreover, as we consider inflation as part of the financial market, real salaries could even fall over the life time of the pension plan.

- A time-varying salary process, given by

  $$d\log L(t) = \frac{dL(t)}{L(t)} = cdt,$$

  has been proposed by [Nkeki, 2013] and [Milazzo and Vigna, 2018]. Being able to convey the trend in salary increase over the life time of

[1] `https://www.ons.gov.uk/employmentandlabourmarket/peopleinwork`

the pension plan is an improvement to the model of a constant salary process. However, salaries are rarely deterministic and are often correlated to inflation. As we consider inflation in our market model, it seems reasonable to include some randomness in the salary process.

- Introducing randomness due to inflation to the salary process yields

$$dL(t) = I(t)d(t)dt,$$

where $d(t)$ is some deterministic function of time. This has been proposed in [Yao et al., 2013]. Note that $\delta dL(t)$ denotes the distribution of the contributions, which now changes according to the price index $I(t)$. In reality, salary increases are not only correlated to inflation, but also to the general state of the market. An additional factor for randomness is needed to portray this effect.

- Recall the notation of the inflation index (5.1) and denote by $\mu_I(t) = r_N(t) - r_R(t) + \sigma_I(t)\theta_I(t)$ the instantaneous mean of inflation. Now adding both randomness due to inflation as well as randomness due to the stock market to the salary process yields

$$\frac{dL(t)}{L(t)} = \mu_L(t)dt + \sum_{i=1}^{n} \sigma_{L,i}(t)dW_i(t) + \sigma_{L,I}(t)dW_I(t). \tag{8.4}$$

A salary process of this form has been studied in [Zhang et al., 2007], [Zhang and Ewald, 2009], [Wu et al., 2015], [Okoro and Nkeki, 2013] and [Xue and Basimanebotlhe, 2015]. Here, $\mu_L(t) = \mu_I(t) + \kappa(t)$, denotes the instantaneous mean of the salary. It consists of two components, where the first part $\mu_I(t)$ is caused by expected inflation and the second part $\kappa(t)$ results from factors such as economic growth or increased welfare. The volatility factors $\sigma_{L,i}(t)$, for $i = 1, \ldots, n$, as well as $\sigma_{L,I}(t)$ measure how the stocks and the inflation affect the salary. It is apparent that the previous cases can be obtained by setting some of the parameters to zero.

Remark. In both [Battocchio and Menoncin, 2004] and [Cairns et al., 2006] an additional factor for randomness was introduced, i.e.

$$\frac{dL(t)}{L(t)} = \mu_L(t)dt + \sum_{i=1}^{n} \sigma_{L,i}(t)dW_i(t) + \sigma_{L,I}(t)dW_I(t) + \sigma_L(t)dW_L(t),$$

where $W_L(t)$ is a Brownian motion independent of $W(t)$. Note that in this case, the market becomes incomplete, as the risk from $W_L(t)$ is non-hedgeable. However, the additional factor allows to model certain risks which are not captured by (8.4). For example, it may be considered as a risk premium, compensating for the risk of redundancy, or even allowing to model the probability of working part-time, at a certain stage in life. As the incompleteness of the market introduces many problems, this is beyond the scope of this thesis and we consider the salary process given by (8.4), instead.

Assumption 8.1.1. We retain the assumptions on the market coefficients from Assumption 5.2.1 and additionally assume that the *instantaneous mean of the salary* $\mu_L(t)$ and the *volatility process of the salary* $\sigma_L(t)$, given by $\sigma_L(t) = \big(\sigma_{L,1}(t), \ldots, \sigma_{L,n}(t), \sigma_{L,I}(t)\big)$ are uniformly bounded and $\mathcal{F}_t$-progressively measurable on $[0, T]$.

Consider an investor who starts with a fixed, non-negative wealth x at time 0, who invests in the various securities and whose actions do not affect the market prices. Suppose the investor invests an additional fixed proportion δ of his or her income at each time $t \in [0, T]$. The amount that is invested in the i'th stock at time t is denoted by $\pi_i(t)$, whereas the amount invested in the inflation-linked bond is denoted by $\pi_I(t)$. Recall that a portfolio process is called admissible, if it is progressively measurable and satisfies $\int_0^T \|\pi(t)\|^2 dt < \infty$, $\mathbb{P}$-a.s. and that the family of admissible portfolio process is denoted by Π. At time $t \in [0, T]$ we denote the total wealth of this investor by $X(t)$.

Recall the notion of the risk premium process, given by

$$\theta(t) = \begin{pmatrix} \theta_1(t) \\ \vdots \\ \theta_n(t) \\ \theta_I(t) \end{pmatrix} = \sigma(t)^{-1} \begin{pmatrix} \mu(t) - r_N(t)\mathbb{1} \\ \sigma_I(t)\theta_I(t) \end{pmatrix},$$

where $\sigma(t)$ denotes the dispersion matrix, given by (5.7). The risk premium process is bounded, measurable and adapted to $\mathcal{F}_t$ due to Assumption 8.1.1.

Definition 8.1.2. Given a portfolio process π and a contribution rate δ, the solution $X = X^\pi$ to

$$\begin{aligned} dX^\pi(t) &= \Big(r_N(t)X^\pi(t) + \pi(t)'\sigma(t)\theta(t) + \delta L(t)\Big)dt + \pi(t)'\sigma(t)dW(t), \\ X^\pi(0) &= x, \end{aligned} \tag{8.5}$$

is called the *wealth process* corresponding to the portfolio process π, the initial capital x and the contribution rate δ.

Chapter 9

The Constrained Optimal Strategy

9.1 Problem Formulation

9.1.1 The Expected Future Contributions

Unlike in the self-financing case, we cannot say that the whole wealth process is non-negative a.s. if and only if the terminal wealth is non-negative a.s. This is best seen in the example of a constant, positive salary. As long as the discounted future contributions outweigh the current wealth deficit, by only trading in the bank account, the terminal wealth process will still be positive.

Define the same measure change as in Section 2.2, by

$$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[Z(T)\mathbb{1}_A], \qquad \text{for all } A \in \mathcal{F},$$

where $Z(t) = \exp\big(-\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\big)$. In order to reformulate the conditions on admissibility, we follow [Zhang et al., 2007] and define the discounted value of future contributions.

Definition 9.1.1. The discounted expected future contribution process is defined as

$$D(t) = \mathbb{E}^{\mathbb{P}}\Big[\int_t^T \frac{\xi(s)}{\xi(t)}\delta L(s)ds|\mathcal{F}_t\Big], \tag{9.1}$$

where $\xi(t) = Z(t)\beta(t)$ is the state price deflator for the bank account numéraire $\beta(t) = \exp(-\int_0^t r_N(s)ds)$, defined in (2.9) and δ denotes the contribution rate. We write $d = D(0)$ to denote the present value of all future contributions.

In order to state the admissibility conditions in Sections 3.1.2 and 6.1.2, we used that the discounted price process $\xi(t)X^\pi(t)$ was a martingale for the wealth process (5.8). Due to the presence of the contributions, this is no longer the case. The following proposition shows, that after including the future contributions and using the Martingale Representation Theorem, see e.g. [Karatzas and Shreve, 1998][Proposition 4.19], we obtain a new martingale.

Proposition 9.1.2. *The discounted process $\big(\xi(t)(X^\pi(t)+D(t))\big)_{t\in[0,T]}$ is a continuous local martingale under $\mathbb{P}$.*

Proof. Using Itô's formula for $f(x,y)=xy$, we obtain

$$\begin{aligned} d(\xi(t)X^\pi(t)) =& \xi(t)\big(r_N(t)X^\pi(t)+\pi(t)'\sigma(t)\theta(t)\big)dt+\xi(t)\pi(t)'\sigma(t)dW(t) \\ & -X^\pi(t)\xi(t)\big(r_N(t)dt+\theta(t)'dW(t)\big)-\xi(t)\pi(t)'\sigma(t)\theta(t)dt \\ & +\xi(t)\delta L(t)dt \\ =& \xi(t)\big(\pi(t)'\sigma(t)-X^\pi(t)\theta(t)'\big)dW(t)+\xi(t)\delta L(t)dt. \end{aligned} \tag{9.2}$$

Then, the conditional expectation of the discounted contributions is the sum of the contributions in the past, plus the expected future contributions, i.e.

$$\mathbb{E}^{\mathbb{P}}\big[\int_0^T \xi(s)\delta L(s)|\mathcal{F}_t\big]=\xi(t)D(t)+\int_0^t \xi(s)\delta L(s)ds,$$

which is a martingale under $\mathbb{P}$. By the Martingale Representation Theorem, there exists a square integrable process $\big(\psi(t)\big)_{t\in[0,T]}$, such that

$$d(\xi(t)D(t))=\psi(t)'dW(t)-\xi(t)\delta L(t)dt, \quad \mathbb{P}\text{-a.s.} \tag{9.3}$$

Adding (9.2) and (9.3) we obtain

$$d\big(\xi(t)(X^\pi(t)+D(t))\big)=\big(\psi(t)'+\xi(t)\big(\pi(t)'\sigma(t)-X^\pi(t)\theta(t)'\big)\big)dW(t),$$

which proves the claim. □

Now due to the uniform boundedness in Assumption 8.1.1, the local martingale $\xi(t)(X^\pi(t)+D(t))$ is uniformly integrable and therefore equal to the conditional expectation of its terminal value, i.e.

$$X^\pi(t)+D(t)=\xi(t)^{-1}\mathbb{E}^{\mathbb{P}}\big[\xi(T)X^\pi(T)\big|\mathcal{F}_t\big], \quad \text{for all } t\in[0,T].$$

Inserting $t=0$, we obtain the admissibility condition

$$x+d=\mathbb{E}^{\mathbb{P}}\big[\xi(T)X^\pi(T)\big]. \tag{9.4}$$

Hence we note that the terminal wealth is $\mathbb{P}$-a.s. non-negative if and only if the sum of the wealth process and the discounted future contributions is $\mathbb{P}$-a.s. non-negative, i.e.

$$X^\pi(T) \geq 0, \mathbb{P}\text{-a.s.} \iff X^\pi(t) + D(t) \geq 0, \text{ for all } t \in [0,T], \mathbb{P}\text{-a.s.}$$

In the following discussion we drop the requirement that the whole wealth process must be positive and instead only constrain the terminal wealth. This allows borrowing against future contributions. As this is not always possible in practice, we will introduce an alternative portfolio process in Chapter 11, where a no-shorting constraint is imposed.

For the current discussion, we define the family of all admissible portfolio processes that lead to non-negative terminal wealth by

$$\mathcal{A}(x) = \Big\{\pi \in \Pi \mid X^\pi(0) \leq x \text{ and } X^\pi(t) + D(t) \geq 0, \quad \mathbb{P}\text{-a.s.}\Big\}.$$

Proposition 9.1.3. *The expected future contributions process $D(t)$ is given by*

$$D(t) = \delta L(t) \int_t^T \exp\Big(\int_t^s \big(\mu_L(u) - r_N(u) - \sigma_L(u)'\theta(u)\big)du\Big)ds, \tag{9.5}$$

where $L(t)$ denotes the salary at time t and δ denotes the fixed percentage of the salary that is invested.

Proof. By definition, we have that

$$\begin{aligned} D(t) &= \mathbb{E}^{\mathbb{P}}\Big[\int_t^T \frac{\xi(s)}{\xi(t)}\delta L(s)ds \big| \mathcal{F}_t\Big] \\ &= \delta L(t)\mathbb{E}^{\mathbb{P}}\Big[\int_t^T \frac{\xi(s)}{\xi(t)}\frac{L(s)}{L(t)}ds \big| \mathcal{F}_t\Big] \\ &= \delta L(t)\mathbb{E}^{\mathbb{P}}\Big[\int_t^T \frac{\xi(s)}{\xi(t)}\frac{L(s)}{L(t)}ds\Big], \end{aligned}$$

where the last step follows because both the state price deflator and the salary process are geometric Brownian motion and hence $\frac{\xi(s)}{\xi(t)}\frac{L(s)}{L(t)}$ is independent of $\mathcal{F}_t$. Furthermore, inserting the Definitions (2.9) and (8.4) for $\xi(t)$, resp. for $L(t)$, we obtain

$$\begin{aligned} \frac{\xi(s)}{\xi(t)}\frac{L(s)}{L(t)} = \exp\Big(\int_t^s &\big(\mu_L(u) - r_N(u) - \frac{1}{2}\|\sigma_L(u) - \theta(u)\|^2 \\ &- \sigma_L(u)'\theta(u)\big)du + \int_t^s \big(\sigma_L(u) - \theta(u)\big)'dW(u)\Big). \end{aligned}$$

Lastly, using Fubini's theorem to switch the integral and the expectation, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\int_t^T \frac{\xi(s)}{\xi(t)}\frac{L(s)}{L(t)}dt\Big] = \int_t^T \exp\Big(\int_t^s \big(\mu_L(u) - r_N(u) \\ - \sigma_L(u)'\theta(u)\big)du\Big)ds,$$

and the claim follows. □

9.1.2 The Constrained Portfolio Problem

Recall the second measure change introduced in Section 5.3, where

$$\mathbb{Q}^T[A] = \mathbb{E}[\tilde{Z}(T)\mathbb{1}\{A\}].$$

Here, $\tilde{Z}(t)$ is given by $\tilde{Z}(t) = \exp\big(-\frac{1}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)'dW(s)\big)$, for $\tilde{\theta}(t) = \big(\theta_1(t), \dots, \theta_n(t), \theta_I(t) - \sigma_I(t)\big)'$. Recall the inflation-linked numéraire and the auxiliary process $\tilde{\xi}(t)$, given by

$$\tilde{\beta}(t) = \frac{1}{B^*(t, I(t))} \quad \text{and} \quad \tilde{\xi}(t) = \frac{\xi(t)}{\tilde{\beta}(t)}.$$

Proposition 9.1.4. *The real wealth process $Y^\pi(t) = X^\pi(t)\tilde{\beta}(t)$ satisfies the stochastic differential equation*

$$\begin{aligned} dY^\pi(t) =& Y^\pi(t)\big(\sigma_I^2(t) - \sigma_I(t)\theta_I(t)\big)dt + \delta\bar{L}(t)dt - Y^\pi(t)\sigma_I(t)dW_I(t) \\ &+ \tilde{\beta}(t)\pi(t)'\big(\sigma(t)\theta(t) - \sigma_{n+1}(t)\sigma_I(t)\big)dt + \tilde{\beta}(t)\pi(t)'\sigma(t)dW(t), \\ Y^\pi(0) =& \tilde{\beta}(0)x, \end{aligned} \tag{9.6}$$

where $\sigma_{n+1}(t)$ denotes the $(n+1)^{th}$ column of $\sigma(t)$. Here, $\bar{L}(t) = L(t)\tilde{\beta}(t)$ is the real salary level and satisfies

$$\begin{aligned} d\bar{L}(t) =& \bar{L}(t)\Big(\mu_L(t) - \big(r_N(t) + \sigma_I(t)\theta_I(t)\big) - \sigma_{L,I}(t)\sigma_I(t) + \sigma_I^2(t)\Big)dt \\ &+ \bar{L}(t)\big(\sigma_L(t)'dW(t) - \sigma_I(t)dW_I(t)\big), \\ L(0) =& l\tilde{\beta}(0). \end{aligned} \tag{9.7}$$

Proof. The derivation of the SDE (9.6) follows similarly as in Proposition 6.1.2 and is omitted here. In order to obtain the SDE for the real salary level, apply Itô's formula to $f(x, y) = xy$. □

Problem 9.1.5. Given a constant C, we consider the problem of finding a portfolio process $\tilde{\pi} \in \mathcal{A}(x)$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(C - Y^{\tilde{\pi}}(T)\Big)^2\Big] = \inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\Big(C - Y^{\pi}(T)\Big)^2\Big]. \tag{9.8}$$

and the pair $\big(Y^{\tilde{\pi}}(t), \tilde{\pi}(t)\big)$ satisfies the stochastic differential equation (9.6).

Simlar to Section 6.1.2, we can restate the admissibility condition (9.4) and Proposition 9.1.2 in terms of the real wealth process $Y^{\pi}(t)$ under the martingale measure $\mathbb{Q}^T$.

Proposition 9.1.6. *The optimal real wealth process $Y^{\pi}(t)$ satisfies the admissibility condition*

$$\mathbb{E}^{\mathbb{Q}^T}[Y^{\pi}(T)] = \tilde{\beta}(0)(x + d), \tag{9.9}$$

or equivalently

$$\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)Y^{\pi}(T)] = x + d. \tag{9.10}$$

Furthermore, the discounted process $\big((Y^{\pi}(t) + \tilde{\beta}(t)D(t))\big)_{t\in[0,T]}$ is a continuous local martingale under $\mathbb{Q}^T$.

Proof. Note that the admissibility constraint (9.10) follows immediately from the admissibility condition (9.4) of the nominal wealth process by inserting the definitions of $\tilde{\xi}(t)$ and of the real wealth process $Y^{\pi}(t)$. Moreover, the admissibility constraint (9.9) follows once the second part of the proposition is proved.

We continue as in Proposition 9.1.2. By Lemma 6.1.4 we know that the real wealth process satisfies

$$\begin{aligned} dY^{\pi}(t) =& Y^{\pi}(t)\big(\sigma_I^2(t) - \sigma_I(t)\theta_I(t)\big)dt + \delta\bar{L}(t)dt - Y^{\pi}(t)\sigma_I(t)dW_I(t) \\ &+ \tilde{\beta}(t)\pi(t)'\big(\sigma(t)\theta(t) - \sigma_{n+1}(t)\sigma_I(t)\big)dt + \tilde{\beta}(t)\pi(t)'\sigma(t)dW(t) \\ =& - Y^{\pi}(t)\sigma_I(t)d\tilde{W}_I(t) + \tilde{\beta}(t)\pi(t)'\sigma(t)d\tilde{W}(t) + \delta\bar{L}(t)dt. \end{aligned} \tag{9.11}$$

Then, we note that the conditional expectation under the new measure $\mathbb{Q}^T$ of the discounted contributions is still the sum of the contributions in the past, plus the expected future contributions, now with the numéraire $\tilde{\beta}(t)$, i.e.

$$\mathbb{E}^{\mathbb{Q}^T}\big[\int_0^T \tilde{\beta}(s)\delta L(s)|\mathcal{F}_t\big] = \tilde{\beta}(t)D(t) + \int_0^t \tilde{\beta}(s)\delta\bar{L}(s)ds,$$

which is a martingale under $\mathbb{Q}^T$. By the Martingale Representation Theorem, there exists a square integrable process $\big(\psi_2(t)\big)_{t\in[0,T]}$, such that

$$d(\tilde{\beta}(t)D(t)) = \psi_2(t)' d\tilde{W}(t) - \delta \bar{L}(t)dt, \quad \mathbb{Q}^T\text{-a.s.} \tag{9.12}$$

Adding (9.11) and (9.12) we obtain

$$d\big((Y^\pi(t) + \tilde{\beta}(t)D(t))\big) = \big(\psi_2(t)' + \tilde{\beta}(t)\pi(t)'\sigma(t)\big)d\tilde{W}(t) - Y^\pi(t)\sigma_I(t)d\tilde{W}_I(t),$$

which proves the claim. □

9.1.3 Optimization of Terminal Wealth

Due to Proposition 9.1.6, we can simply replace the initial wealth x by $x+d$ in Section 6.2, in order to find the optimal terminal real wealth under the presence of contributions and inflation. In order to guarantee the feasibility of Problem 9.1.5 and to calibrate the portfolio process explicitly, we again make the assumption of deterministic parameters.

Assumption 9.1.7. The risk premium process $\tilde{\theta}(t)$ is deterministic and satisfies

$$\int_0^T \|\tilde{\theta}(s)\|^2 ds \neq 0.$$

Furthermore, the real interest rate process $r_R(t)$ is deterministic.

Theorem 9.1.8. *Unter Assumptions 8.1.1 and 9.1.7, there exists a portfolio process $\pi \in \mathcal{A}(x)$, such that the corresponding real wealth process attains the optimal terminal wealth, given by*

$$Y^\pi(T) = \begin{cases} C & \text{if } (x+d)\tilde{\beta}(0) \geq C, \\ \big(C - \tilde{\mathcal{Y}}(x+d)\tilde{\xi}(T)\big)^+ & \text{else,} \end{cases} \tag{9.13}$$

where $\tilde{\mathcal{Y}} : (0, \tilde{\mathcal{H}}(0)) \to (0,\infty)$ denotes the inverse of $\tilde{\mathcal{H}}(y) = \mathbb{E}^{\mathbb{P}}\big[\tilde{\xi}(T)(C - y\tilde{\xi}(T))^+\big]$ for all $y \in (0,\infty)$ and is given by (6.13).

Proof. The proof follows directly from Theorem 6.2.2, by replacing x by $x+d$. □

We see that in allowing the wealth process to become negative on its path, the expected contributions are simply seen as additional initial wealth. Hence, under the restriction of non-negative terminal wealth, the investor borrows against his future contributions.

Recall the definition of the ruin probability in Definition 9.1.9. Under the presence of contributions we are now interested in receiving more money than was invested through the initial investment x and the contributions.

Definition 9.1.9. The *probability of success under contributions* is defined by

$$\mathbb{P}[Y^\pi(T) > x + d],$$

where d denotes the present value of all future cashflows.

Example 9.1.10. Similarly to the Examples 3.3.3 and 6.2.6, we plot the empirical distribution of the optimal terminal wealth for both the restricted process (9.13) as well as the unrestricted process, where the non-negativity constraint is dropped.

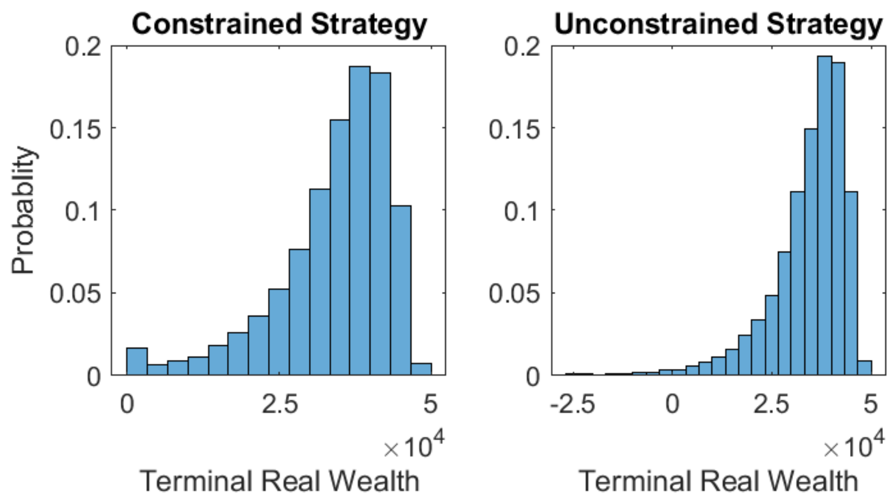


Figure 9.1: Histograms of the real terminal wealth distribution for the constrained and unconstrained portfolio problem.

For this example we suppose that all parameters are constant over time and that there is only one stock in the market. We set the market parameters as $r_N = 5\%$, $\mu = 8\%$ and $\sigma_S = 0.15$. The investor starts with an initial wealth $x = 1000$ and tries to reach $C = 50'000$ over a time horizon of $T = 10$ years. In addition to the initial investment x, an additional $\delta = 10\%$ of the stochastic salary of the plan member is invested monthly. The parameters of the stochastic salary are given by $l = 20000$, $\kappa = 0.015$, $\sigma_{LS} = 0.004$ and $\sigma_{LI} = 0.006$. Furthermore, the inflation parameters are given by $r_R = 4\%$, $\sigma_I = 0.05$, $\theta_I = 0.12$ and the volatility of the stock with respect to the inflation is given by $\sigma_{IS} = 0.04$.

In Figures 9.1 and 9.2, we plot the empirical terminal wealth distributions for 10'000 realizations. Note that the unconstrained distribution is cut-off at -25'000, in order to increase readability of the plot.

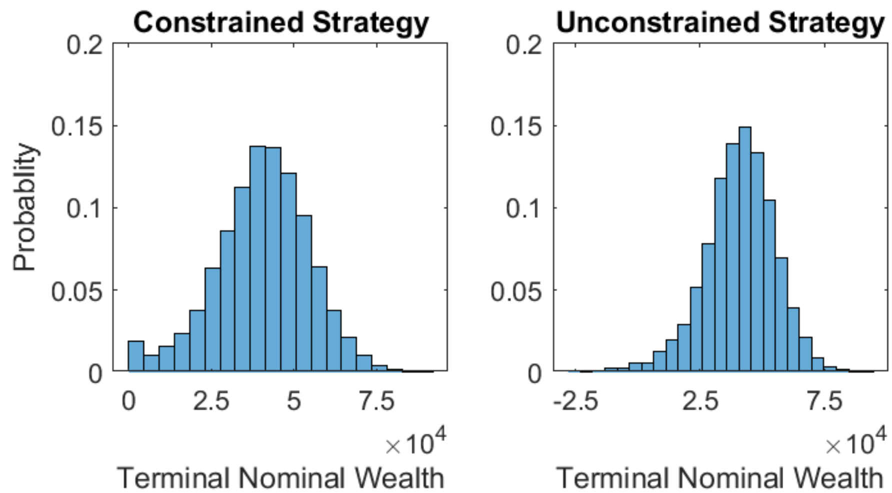


Figure 9.2: Histograms of the nominal terminal wealth distribution for the constrained and unconstrained portfolio problem.

In order to study the advantages and disadvantages more thoroughly, we report some statistics of the final wealth out of 10'000 realizations in Table 9.1. Comparing these statistics to those in Example 6.2.6, we see that the continuous investment through contributions reduces the ruin probability close to zero. This in turn reduces the difference between the unconstrained and the constrained portfolio process. We also note that contrary to the previous examples, the 2.5% quantile is lower for the constrained strategy for this set of parameters.

	Unconstrained	**Constrained**
2.5% Quantile	8'650	7'360
Mean	40'900	40'500
97.5% Quantile	67'300	67'100
$\sqrt{\textbf{L2}}$**-Distance**	18'500	18'600
Median Rate of Return	9.8%	9.6%
Ruin Probability	1.0%	0.0%
Success Probability	93.2%	92.5%

Table 9.1: Properties of the empirical terminal nominal wealth distribution for the constrained and unconstrained portfolio problem.

It is apparent from (9.13) that the higher the expected future contributions and the initial wealth x are, the lower the probability of reaching negative terminal wealth and the smaller the difference between the unconstrained and the constrained portfolio process become. We will see in Chapter 15, that the difference between the constrained and the unconstrained terminal wealth is very little for most sets of parameters. Nonetheless, negative wealth during some part of the process may happen for both strategies, due to borrowing against future contributions. For that reason we add a no-shorting constraint to the optimal portfolio in Chapter 11.

9.2 The Optimal Portfolio Process

Once more, we explicitly derive the optimal portfolio process and the corresponding wealth process with and without the non-negativity constraint. Without the additional requirement, we solve the stochastic differential equation (9.6) directly with the help of the more general Hamilton-Jacobi-Bellman equation, stated in (3.22).

9.2.1 Without Bankruptcy Prohibition

Recall the drift of the salary process, given by

$$\mu_L(t) = \kappa(t) + \mu_I(t) = \kappa(t) + r_N(t) - r_R(t) + \sigma_I(t)\theta_I(t).$$

Theorem 9.2.1. *Let $\big(a(t)\big)_{t\in[0,T]}$ be given by*

$$a(t) = -\frac{1}{2}e^{-\int_t^T \|\tilde{\theta}(s)\|^2 ds} \tag{9.14}$$

for all $t \in [0,T]$ and let $\big(f(t)\big)_{t\in[0,T]}$ satisfy the non-homogeneous partial differential equation

$$\begin{aligned} 0 &= f_t(t) + 2\delta a(t) \\ &\quad + \Big[\mu_L(t) - (r_N(t) + \sigma_I(t)\theta_I(t)) - \sigma_{LI}(t)\sigma_I(t) + \sigma_I^2(t) - \|\tilde{\theta}(t)\|^2\Big] f(t), \\ f(T) &= 0, \end{aligned} \tag{9.15}$$

for all $t \in [0,T]$. Then, under Assumptions 8.1.1 and 9.1.7, the optimal portfolio process to the unconstrained optimization problem is given by

$$\begin{aligned} \hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}\big(\sigma(t)'\big)^{-1}\Big(&\bar{\sigma}_I(t)Y^\pi(t) - \tilde{\theta}(t)\big(Y^\pi(t) - C\big) \\ &- \bar{L}(t)\frac{f(t)}{2a(t)}\big(\tilde{\theta}(t) + \sigma_L(t) - \bar{\sigma}_I(t)\big)\Big), \end{aligned} \tag{9.16}$$

for all $t \in [0,T]$*, where* $\bar{\sigma}_I(t) = \big(0,\dots,0,\sigma_I(t)\big)'$.

Proof. See Appendix III.A. □

Before we can use Theorem 9.2.1 to solve for the optimal wealth process, we need to simplify equation (9.16).

Corollary 9.2.2. *For* $f(t)$ *satisfying the non-homogeneous partial differential equation* (9.15)*, we have*

$$\frac{f(t)}{2a(t)}L(t) = D(t), \tag{9.17}$$

where $D(t)$ *is the expected value of future contributions, given by* (9.1)*.*

Proof. The solution to (9.15) is given by

$$f(t) = -\delta \int_t^T e^{-\int_t^T \|\tilde{\theta}(s)\|^2 ds + \int_t^s \big(\mu_L(u) - r_N(u) - \sigma_L(u)'\theta(u)\big)du} ds, \tag{9.18}$$

(see Proposition III.A.2 in Appendix III.A). Therefore, inserting the solution to $a(t)$ from (9.14), we have

$$\frac{f(t)}{2a(t)} = \delta \int_t^T \exp\Big(\int_t^s \big(\mu_L(u) - r_N(u) - \sigma_L(u)'\theta(u)\big)du\Big)ds,$$

and comparing this to (9.5), the claim follows. □

Proposition 9.2.3. *Define* $Z(t) = Y^{\hat{\pi}}(t) - C + D(t)\hat{\beta}(t)$*. Then*

$$dZ(t) = -\|\tilde{\theta}(t)\|^2 Z(t)dt - \tilde{\theta}(t)'Z(t)dW(t).$$

Proof. By Corollary 9.2.2, the optimal portfolio process is given by

$$\begin{aligned}\hat{\pi}(t) = \frac{1}{\hat{\beta}(t)}\big(\sigma(t)'\big)^{-1}\Big(&\sigma_I(t)Y^{\hat{\pi}}(t) - \tilde{\theta}(t)\big(Y^{\hat{\pi}}(t) - C\big) \\ &- \tilde{\beta}(t)D(t)\big(\tilde{\theta}(t) + \sigma_L(t) - \bar{\sigma}_I(t)\big)\Big).\end{aligned}$$

Inserting this process into the stochastic differential equation (9.6), we obtain

$$
\begin{aligned}
dY^{\hat{\pi}}(t) =& - \|\tilde{\theta}(t)\|^2 \big(Y^{\hat{\pi}}(t) - C\big)dt - \tilde{\theta}(t)'\big(Y^{\hat{\pi}}(t) - C\big)dW(t) \\
& - \tilde{\beta}(t)D(t)\big(\tilde{\theta}(t) + \sigma_L(t) - \bar{\sigma}_I(t)\big)'\tilde{\theta}(t)dt + \delta\tilde{\beta}(t)L(t)dt \\
& - \tilde{\beta}(t)D(t)\big(\tilde{\theta}(t) + \sigma_L(t) - \bar{\sigma}_I(t)\big)'dW(t) \\
=& - \|\tilde{\theta}(t)\|^2 \big(Y^{\hat{\pi}}(t) - C\big)dt - \tilde{\theta}(t)'\big(Y^{\hat{\pi}}(t) - C\big)dW(t) \\
& + \tilde{\beta}(t)\Big(- \|\tilde{\theta}(t)\|^2 D(t)dt - \tilde{\theta}(t)'D(t)dW(t) + \delta L(t)dt \\
& - D(t)\big((\sigma_L(t) - \bar{\sigma}_I(t)\big)'\tilde{\theta}(t)dt - D(t)\big((\sigma_L(t) - \bar{\sigma}_I(t)\big)'dW(t)\Big).
\end{aligned}
$$

Next, using Proposition 9.1.3, we can calculate $dD(t)$ directly by Leibniz' rule. Namely, denote by

$$
F(t) = \int_t^T \exp\Big(\int_t^s \big(\mu_L(u) - r_N(u) - \sigma_L(t)'\theta(u)\big)du\Big)ds.
$$

Then, $dF(t) = -dt - \big(\mu_L(t) - r_N(t) - \sigma_L(t)'\theta(t)\big)F(t)dt$, and hence

$$
\begin{aligned}
dD(t) =& \delta F(t)dL(t) + \delta L(t)dF(t) \\
=& D(t)\big(\mu_L(t)dt + \sigma_L(t)dW(t)\big) - \delta L(t)dt \\
& - \big(\mu_L(t) - r_N(t) - \sigma_L(t)'\theta(t)\big)D(t)dt.
\end{aligned}
$$

Furthermore, by Itô's formula, we calculate $d\tilde{\beta}(t)$ explicitly, as

$$
d\tilde{\beta}(t) = -\tilde{\beta}(t)\Big(\big(r_N(t) + \sigma_I(t)\theta_I(t) - \sigma_I^2(t)\big)dt + \sigma_I(t)dW_I(t)\Big).
$$

Putting everything together, we have

$$
\begin{aligned}
dZ(t) =& dY^{\hat{\pi}}(t) + d\big(\tilde{\beta}(t)D(t)\big) \\
=& dY^{\hat{\pi}}(t) + D(t)d\tilde{\beta}(t) + \tilde{\beta}(t)dD(t) + d[\tilde{\beta}, D](t) \\
=& - \|\tilde{\theta}(t)\|^2 \big(Y^{\hat{\pi}}(t) + \tilde{\beta}D(t) - C\big)dt - \tilde{\theta}(t)'\big(Y^{\hat{\pi}}(t) + \tilde{\beta}D(t) - C\big)dW(t) \\
& + \delta\tilde{\beta}(t)L(T)dt - \tilde{\beta}(t)D(t)\big((\sigma_L(t) - \bar{\sigma}_I(t))'\tilde{\theta}(t)\big)dt \\
& - \tilde{\beta}(t)D(t)(\sigma_L(t) - \bar{\sigma}_I(t))'dW(t) + \tilde{\beta}(t)D(t)\big((\sigma_L(t) - \bar{\sigma}_I(t))'\tilde{\theta}(t)\big)dt \\
& + \tilde{\beta}(t)D(t)(\sigma_L(t) - \bar{\sigma}_I(t))'dW(t) - \delta\tilde{\beta}(t)L(T)dt \\
=& - \|\tilde{\theta}(t)\|^2 \big(Y^{\hat{\pi}}(t) + \tilde{\beta}D(t) - C\big)dt - \tilde{\theta}(t)'\big(Y^{\hat{\pi}}(t) + \tilde{\beta}D(t) - C\big)dW(t) \\
=& - \|\tilde{\theta}(t)\|^2 Z(t)dt - \tilde{\theta}(t)'Z(t)dW(t).
\end{aligned}
$$

□

Theorem 9.2.4. *Under the same assumptions as in Theorem 9.2.1 and for* $(x+d)\tilde{\beta}(0) \leq C$*, the optimal real wealth process to the unconstrained optimization problem is given by*

$$Y^{\hat{\pi}}(t) = \big((x+d)\tilde{\beta}(0) - C\big)e^{-\int_0^t \frac{3}{2}\|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)} + C - \tilde{\beta}(t)D(t), \quad (9.19)$$

for all $t \in [0,T]$.

Proof. Define the auxiliary process $Z(t) = Y^{\hat{\pi}}(t) - C + D(t)\hat{\beta}(t)$ with initial value $Z(0) = (x+d)\tilde{\beta}(0) - C$. Then by Proposition 9.2.3

$$dZ(t) = -\|\tilde{\theta}(t)\|^2 Z(t)dt - \tilde{\theta}(t)' Z(t) dW(t),$$

which is the expression for a geometric Brownian motion, with solution

$$Z(t) = Z(0)e^{-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)}.$$

Hence, we can write the optimal wealth process corresponding to the portfolio process $\hat{\pi}$ of (9.16) as

$$Y^{\hat{\pi}}(t) = \big((x+d)\tilde{\beta}(0) - C\big)e^{-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)} + C - \tilde{\beta}(t)D(t). \quad (9.20)$$

□

Corollary 9.2.5. *Under the same assumptions as in Theorem 9.2.1 and for* $(x+d)\tilde{\beta}(0) \leq C$*, the unconstrained optimization problem has an optimal solution pair* $\big(X^{\hat{\pi}}(t), \hat{\pi}(t)\big)$*, given by*

$$X^{\hat{\pi}}(t) = \frac{1}{\tilde{\beta}(t)}\big((x+d)\tilde{\beta}(0) - C\big)e^{-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2 ds - \int_0^t \tilde{\theta}(s)' dW(s)} + \frac{C}{\tilde{\beta}(t)} - D(t), \quad (9.21)$$

$$\hat{\pi}(t) = \big(\sigma(t)'\big)^{-1}\nu(t), \quad (9.22)$$

where

$$\nu_i(t) = -\big(X^{\hat{\pi}}(t) + D(t) - \frac{C}{\tilde{\beta}(t)}\big)\theta_i(t) - D(t)\sigma_{L,i}(t), \qquad \text{for } i = 1,\dots,n,$$

$$\nu_I(t) = -\big(X^{\hat{\pi}}(t) + D(t) - \frac{C}{\tilde{\beta}(t)}\big)(\theta_I(t) - \sigma_I(t)) - D(t)\sigma_{L,I}(t) + \big(X^{\hat{\pi}}(t) + D(t)\big)\sigma_I(t).$$

Proof. Noting that $X^{\pi}(t)\tilde{\beta}(t) = Y^{\pi}(t)$, the optimal wealth process is directly obtained from the previous discussion. The expression of the optimal portfolio process follows from Theorem 9.2.1. □

9.2.2 With Bankruptcy Prohibition

In order to determine the optimal wealth process, we only need to adapt Section 6.3.2 slightly in order to use the martingale approach. Namely, we first find an explicit solution to the BSDE

$$\begin{aligned} dY^{\pi}(t) &= -Y^{\pi}(t)\sigma_I(t)d\tilde{W}_I(t) + \tilde{\beta}(t)\pi(t)'\sigma(t)d\tilde{W}(t) + \delta\tilde{\beta}(t)L(t), \\ Y^{\pi}(T) &= \big(C - \tilde{\mathcal{Y}}(x+d)\tilde{\xi}(T)\big)^+, \end{aligned}$$

where $\tilde{W}(t) = W(t) + \int_0^t \tilde{\theta}(s)ds$ is the $\mathbb{Q}^T$-Brownian motion defined in Lemma 5.3.1.

Theorem 9.2.6. *Under Assumptions 8.1.1 and 9.1.7 and for* $(x+d)\tilde{\beta}(0) \leq C$*, the optimal real wealth process is given by*

$$Y^{\hat{\pi}}(t) = C\Phi\big(-d_-(t,y(t))\big) - y(t)\Phi\big(-d_+(t,y(t))\big) - \tilde{\beta}(t)D(t), \qquad (9.23)$$

for all $t \in [0,T]$*, where* $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{v^2}{2}}dv$ *is the cumulative distribution function of the standard normal distribution and*

$$\begin{aligned} d_+(t,y) &= \frac{\log(\frac{y}{C}) + \frac{1}{2}\int_t^T \|\tilde{\theta}(s)\|^2 ds}{\sqrt{\int_t^T \|\tilde{\theta}(s)\|^s ds}}, \\ d_-(t,y) &= d_+(t,y) - \sqrt{\int_t^T \|\tilde{\theta}(s)\|^2 ds}. \end{aligned}$$

Furthermore, the process y *is given by*

$$\begin{aligned} y(t) = \tilde{\mathcal{Y}}(x+d)\exp\Big(&-\int_0^T (r_R(s) - \|\tilde{\theta}(s)\|^2)ds\Big) \\ &\exp\Big(-\frac{3}{2}\int_0^t \|\tilde{\theta}(s)\|^2)ds - \int_0^t \tilde{\theta}(s)'dW(s)\Big). \qquad (9.24) \end{aligned}$$

Proof. By Proposition 9.1.6, we know that the discounted process $\big((Y^{\pi}(t) + \tilde{\beta}(t)D(t)\big)_{t\in[0,T]}$ is a continuous local martingale under $\mathbb{Q}^T$. Therefore, we have

$$Y^{\hat{\pi}}(t) = \mathbb{E}^{\mathbb{Q}^T}\big[Y^{\hat{\pi}}(T)|\mathcal{F}_t\big] - \tilde{\beta}(t)D(t). \qquad (9.25)$$

By Theorem 9.1.8, the optimal terminal wealth satisfies

$$Y^{\hat{\pi}}(T) = \big(C - \tilde{\mathcal{Y}}(x+d)\tilde{\xi}(T)\big)^+,$$

for $(x+d)\tilde{\beta}(0) \leq C$ and by replacing $\tilde{\mathcal{Y}}(x)$ by $\tilde{\mathcal{Y}}(x+d)$, Theorem 6.3.8 yields

$$\mathbb{E}^{\mathbb{Q}^T}\big[Y^{\hat{\pi}}(T)|\mathcal{F}_t\big] = C\Phi\big(-d_-(t,y(t))\big) - y(t)\Phi\big(-d_+(t,y(t))\big). \tag{9.26}$$

Inserting (9.26) in (9.25) gives the claim. □

Corollary 9.2.7. *Under the same notations and assumptions as in Theorem 9.2.6 the optimal portfolio process to Problem 9.1.5 is given by*

$$\hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}(\sigma(t)')^{-1}\nu(t), \tag{9.27}$$

where

$$\begin{aligned}
\nu_i(t) =& y(t)\Phi\big(-d_+(t,y(t))\big)\theta_i(t) - \tilde{\beta}(t)D(t)\sigma_{L,i}(t), \qquad \textit{for } i = 1,\dots,n,\\
\nu_I(t) =& Y^{\hat{\pi}}(t)\sigma_I(t) + y(t)\Phi\big(-d_+(t,y(t))\big)(\theta_I(t) - \sigma_I(t))\\
&- \tilde{\beta}(t)D(t)\big(\sigma_{L,I}(t) - \sigma_I(t)\big).
\end{aligned}$$

Proof. Denote by $f(t,y) = C\Phi(-d_-(t,y)) - y\Phi(-d_+(t,y))$. Then, by Theorem 9.2.6, we know that

$$Y^{\hat{\pi}}(t) + \tilde{\beta}(t)D(t) = f(t,y(t)). \tag{9.28}$$

Similar to the proof of Corollary 3.4.9, we will group the volatility terms of (9.28). In order to obtain the volatility terms of $f(t,y)$, we have by Itô's forumla

$$df(t,y) = -y\tilde{\theta}(t)'f_y(t,y)d\tilde{W}(t) + \big(f_t(t,y) + \frac{1}{2}y^2\|\tilde{\theta}\|^2 f_{yy}(t,y)\big)dt,$$

and $f_y(t,y)$ is given by

$$\begin{aligned}
f_y(t,y) &= \frac{1}{y\sqrt{\int_t^T \|\tilde{\theta}(s)\|^2 ds}}\Big(y\phi(-d_+(t,y)) - C\phi(-d_-(t,y)\Big) - \Phi(-d_+(t,y))\\
&= -\Phi(-d_+(t,y)).
\end{aligned}$$

Now by (9.28), we know that $dY^{\hat{\pi}}(t) - d\big(\tilde{\beta}(t)D(t)\big) = df(t,y(t))$, and the volatility terms satisfy

$$Y^{\hat{\pi}}(t)\bar{\sigma}_I(t) = \tilde{\beta}(t)\big(\hat{\pi}(t)'\sigma(t) - D(t)(\sigma_L(t) - \bar{\sigma}_I(t))'\big) - y\Phi(-d_+(t,y(t)))\tilde{\theta}(t)'.$$

Solving for $\hat{\pi}(t)$ yields

$$\begin{aligned}
\hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}\big(\sigma(t)'\big)^{-1}\Big(&Y^{\hat{\pi}}(t)\bar{\sigma}_I(t) + y\Phi(-d_+(t,y(t)))\tilde{\theta}(t)\\
&- \tilde{\beta}(t)D(t)\big(\sigma_L(t) - \bar{\sigma}_I(t)\big)\Big).
\end{aligned}$$

□

Remark. Note that since we have

$$y(t)\Phi\big(-d_+(t,y(t))\big) = -Y^{\hat{\pi}}(t) - \tilde{\beta}(t)D(t) + C\Phi\big(-d_-(t,y(t))\big),$$

by Theorem 9.2.6, we can also write for $i = 1,\dots,n$,

$$\begin{aligned}\nu_i(t) &= -\Big(Y^{\hat{\pi}}(t) + \tilde{\beta}(t)D(t) - C\Phi\big(-d_-(t,y(t))\big)\Big)\theta_i(t) - \tilde{\beta}(t)D(t)\sigma_{L,i}(t),\\ \nu_I(t) &= -\Big(Y^{\hat{\pi}}(t) + \tilde{\beta}(t)D(t) - C\Phi\big(-d_-(t,y(t))\big)\Big)(\theta_I(t) - \sigma_I(t))\\ &\quad - \tilde{\beta}(t)D(t)\big(\sigma_{L,I}(t) - \sigma_I(t)\big) + Y^{\hat{\pi}}(t)\sigma_I(t).\end{aligned}$$

Corollary 9.2.8. *Under Assumptions 8.1.1 and 9.1.7 and for* $(x+d)\tilde{\beta}(0) \leq C$*, the optimization Problem 9.1.5 has an optimal solution pair* $\big(X^{\hat{\pi}}(t), \hat{\pi}(t)\big)$*, given by*

$$X^{\hat{\pi}}(t) = \frac{1}{\tilde{\beta}(t)}\Big(C\Phi\big(-d_-(t,y(t))\big) - y(t)\Phi\big(-d_+(t,y(t))\big)\Big) - D(t), \tag{9.29}$$

$$\hat{\pi}(t) = (\sigma(t)')^{-1}\nu(t), \tag{9.30}$$

where $y(t)$ *is defined in Lemma 6.3.5,* $\tilde{\mathcal{Y}}(x)$ *is given by* (6.13) *and where*

$$\begin{aligned}\nu_i(t) &= -\Big(X^{\hat{\pi}}(t) + D(t) - \frac{C}{\tilde{\beta}(t)}\Phi\big(-d_-(t,y(t))\big)\Big)\theta_i(t) - D(t)\sigma_{L,i}(t),\\ \nu_I(t) &= -\Big(X^{\hat{\pi}}(t) + D(t) - \frac{C}{\tilde{\beta}(t)}\Phi\big(-d_-(t,y(t))\big)\Big)(\theta_I(t) - \sigma_I(t))\\ &\quad - D(t)\sigma_{L,I}(t) + \big(X^{\hat{\pi}}(t) + D(t)\big)\sigma_I(t),\end{aligned}$$

for $i = 1,\dots,n$.

Proof. Noting that $X^{\pi}(t)\tilde{\beta}(t) = Y^{\pi}(t)$, the conclusion is directly obtained from the previous discussion. □

Example 9.2.9. Similar to Examples 3.4.10 and 6.3.11, we study the investment character of the two strategies, adding contributions. We use the same underlying parameters of the stock process as in Example 6.3.11, i.e. $r_N = 5\%$, $\mu = 8\%$ and $\sigma_S = 0.15$ for the stock index and $r_R = 4\%$, $\sigma_I = 0.05$, $\theta_I = 0.12$ and $\sigma_{IS} = 0.04$ for the inflation-linked bond. The parameters for the salary process are computed empirically and we obtain $\kappa = 0.015$, $\sigma_{LS} = 0.006$ and $\sigma_{LI} = 0.004$. The investor starts with $x = 1000$, an initial salary $l = 20'000$, a contribution rate $\delta = 10\%$ and the target wealth is set to $C = 50'000$.

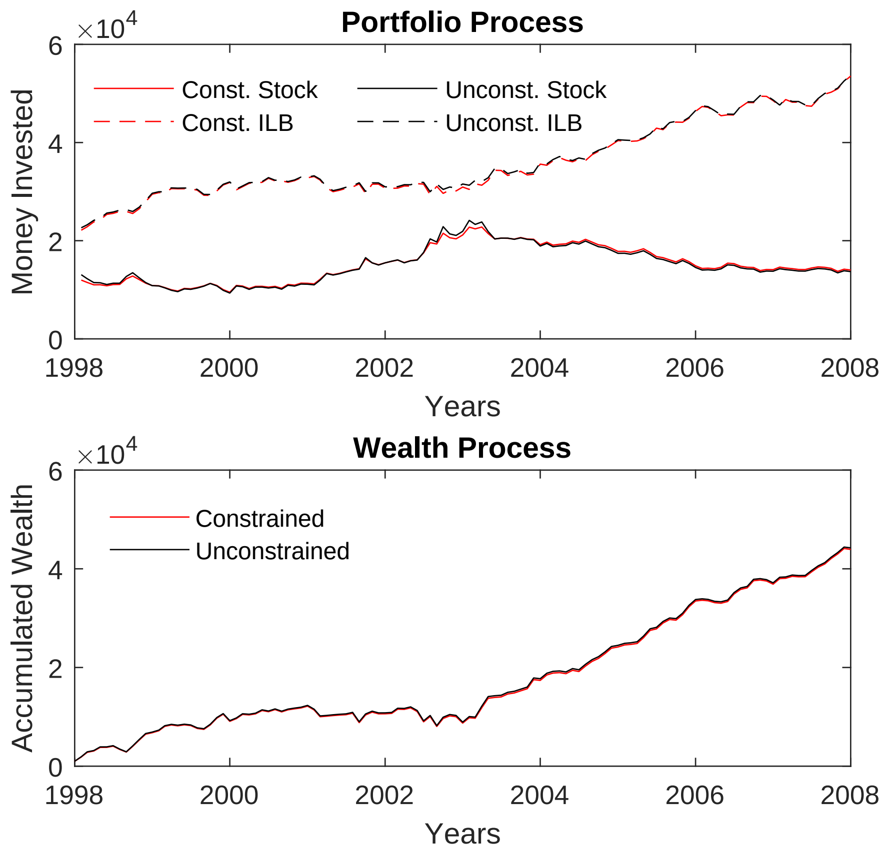


Figure 9.3: In the upper graph, the amount invested in the stock and the index-linked bond can be seen for both the constrained and the unconstrained optimal portfolio. The resulting wealth process is plotted below.

The main difference of the wealth process in Figure 9.3 to the one in the Example 6.3.11 is, that the two strategies only differ very slightly. We also see clearly, that in order to finance the heavy investment in the risky assets, a lot of money is borrowed from the bank account from the beginning of the investment period.

Chapter 10

The Constrained Optimal Strategy: Lower Bound

During the last chapter and especially in Example 9.2.9 we saw that the zero bound on the terminal wealth does not have a great impact on the portfolio strategy. This comes as no surprise, as ending with zero wealth means that not only the initial investment, but also all contributions which were invested over the duration of the pension plan have been lost. Therefore, a pension plan member may not be satisfied with a non-negativity constraint and would rather prefer a positive lower bound, similar to the ones studied in Chapters 4 and 7.

10.1 Problem Formulation

Problem 10.1.1. Given a constant C and a real number K, we consider the problem of finding a portfolio process $\hat{\pi}^l \in \mathcal{A}(x)$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(C - \frac{X^{\hat{\pi}^l}(T)}{I(T)}\Big)^2\Big] = \inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\Big(C - \frac{X^{\pi}(T)}{I(T)}\Big)^2\Big],$$

$$\text{subject to} \qquad \frac{X^{\hat{\pi}^l}(T)}{I(T)} \geq K, \qquad \text{a.s. }, \tag{10.1}$$

and the pair $\big(X^{\hat{\pi}^l}(t), \hat{\pi}^l(t)\big)$ satisfies the stochastic differential equation (8.5).

Recall the present value of all future cashflows, the numéraires corresponding to the bank account and the inflation linked bond as well as the auxiliary process $\tilde{\xi}(t)$, given by

$$\begin{aligned}
d &= \mathbb{E}^{\mathbb{P}}\big[\int_0^T \frac{\xi(s)}{\xi(t)}\delta L(s)ds\big],\\
\beta(t) &= \frac{1}{B(t)} = \exp(-\int_0^t r_N(s)ds),\\
\tilde{\beta}(t) &= \frac{1}{B^*(t,I(t))} = \frac{1}{B_R(t,T)I(t)},\\
\tilde{\xi}(t) &= \frac{\xi(t)}{\tilde{\beta}(t)} = \frac{\tilde{Z}(t)}{\tilde{\beta}(0)},
\end{aligned}$$

and the real wealth process $Y^\pi(t) = \tilde{\beta}(t)X^\pi(t)$. Similar to the discussion in Sections 9.1.3 and 9.1.2 we note that the solution to Problem 10.1.1 follows by replacing x by $x+d$ in Chapter 7. Furthermore, the non-negativity constraint is non-binding if $K>0$ and the class of optimal portfolio processes for Problem 10.1.1 is empty unless we have $K \le (x+d)\tilde{\beta}(T)$. For the rest of this chapter, we will hence assume that

$$K < \tilde{\beta}(T)(x+d).$$

10.2 Solution to the Constrained Problem

By replacing x by $x+d$, we immediately obtain the optimal terminal real wealth and the corresponding optimal portfolio process from Corollary 7.2.1 and Theorem 7.3.1. Hence, define

$$\begin{aligned}
\hat{x} &= x - Ke^{-\int_0^T r_R(s)ds},\\
\hat{C} &= C - K.
\end{aligned}$$

Theorem 10.2.1. *Denote by $\hat{Y}^{\hat{\pi}}(t;\hat{x},\hat{C})$ the optimal wealth process (9.23) at time t with initial wealth $\hat{x}\tilde{\beta}(0)$ and fixed claim $\hat{C}$. Under Assumptions 8.1.1, 9.1.7 and for $(x+d)\tilde{\beta}(0) \le C$ the optimal wealth process to Problem 10.1.1 is given by*

$$Y^{\hat{\pi}^l}(t) = \hat{Y}^{\hat{\pi}}(t;\hat{x},\hat{C}) + K. \tag{10.2}$$

Similarly, denoting by $\hat{\pi}(t;\hat{x},\hat{C})$ the optimal portfolio process (6.33) at time t, the optimal portfolio process to Problem 10.1.1 is given by

$$\hat{\pi}^l(t) = \hat{\pi}\big(t;\hat{x},\hat{C}\big) + K. \tag{10.3}$$

Note that the optimal terminal real wealth for Problem 10.1.1 is then given by

$$Y^{\hat{\pi}^l}(T) = \hat{Y}^{\hat{\pi}}(T) + \big(K - \hat{Y}^{\hat{\pi}}(T)\big)^+, \tag{10.4}$$

where $\hat{Y}^{\hat{\pi}}(t)$ is the optimal wealth process from (9.19) with

$$\hat{x}_0 = C\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)] - y\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)^2],$$

where y is chosen in such a way that the terminal wealth given by (10.4) satisfies the admissibility constraint $\mathbb{E}^{\mathbb{P}}[\tilde{\xi}(T)Y^{\hat{\pi}^l}(T)] = x + d$.

Example 10.2.2. In order to understand the behavior of the portfolio process with a general lower bound, we continue the analysis of Example 9.2.9, including the portfolio and wealth process of the optimal strategy with a positive constraint of $K = 25'000$.

	Unconstrained	**Constrained**	**Lower Bound**
Terminal Wealth	45’500	45’000	40’600
Rate of Return	10.3%	10.1%	8.4%
Minimal Wealth	1’000	1’000	1’000

Table 10.1: Properties of the different portfolio processes calculated for the period 1998-2008 on historical data.

We see in Figure 10.1 that the investment behavior of the optimal portfolio process with a lower bound is very different from the other two optimal strategies, even though the terminal wealth is comparable. The guarantee is financed from the beginning, by borrowing less money from the bank account and reduced investment in the risky assets. This leads to a lower rate of return during years of good market performance, while losing less money during bear markets, see e.g. the period of 2002 to 2004.

The costs associated with the guarantee can be seen in Table 10.1. For this set of parameters, the present value of all future cashflows is $d = 20'000$ and hence more than half of the investment is guaranteed after correcting for inflation. This reduces the rate of return by almost 2%.

Remark. From a practical point of view, a fixed lower guarantee is sub optimal for a pension plan. As noted in Section 8.1, the model for the stochastic salary used in this thesis is not able to capture the risk of redundancy. A more practical lower bound could be set as a percentage of the contributions, hence adapting in the case of a jump in the salary. Such a guarantee would clearly not be measurable at the beginning and change over the duration of the pension plan.

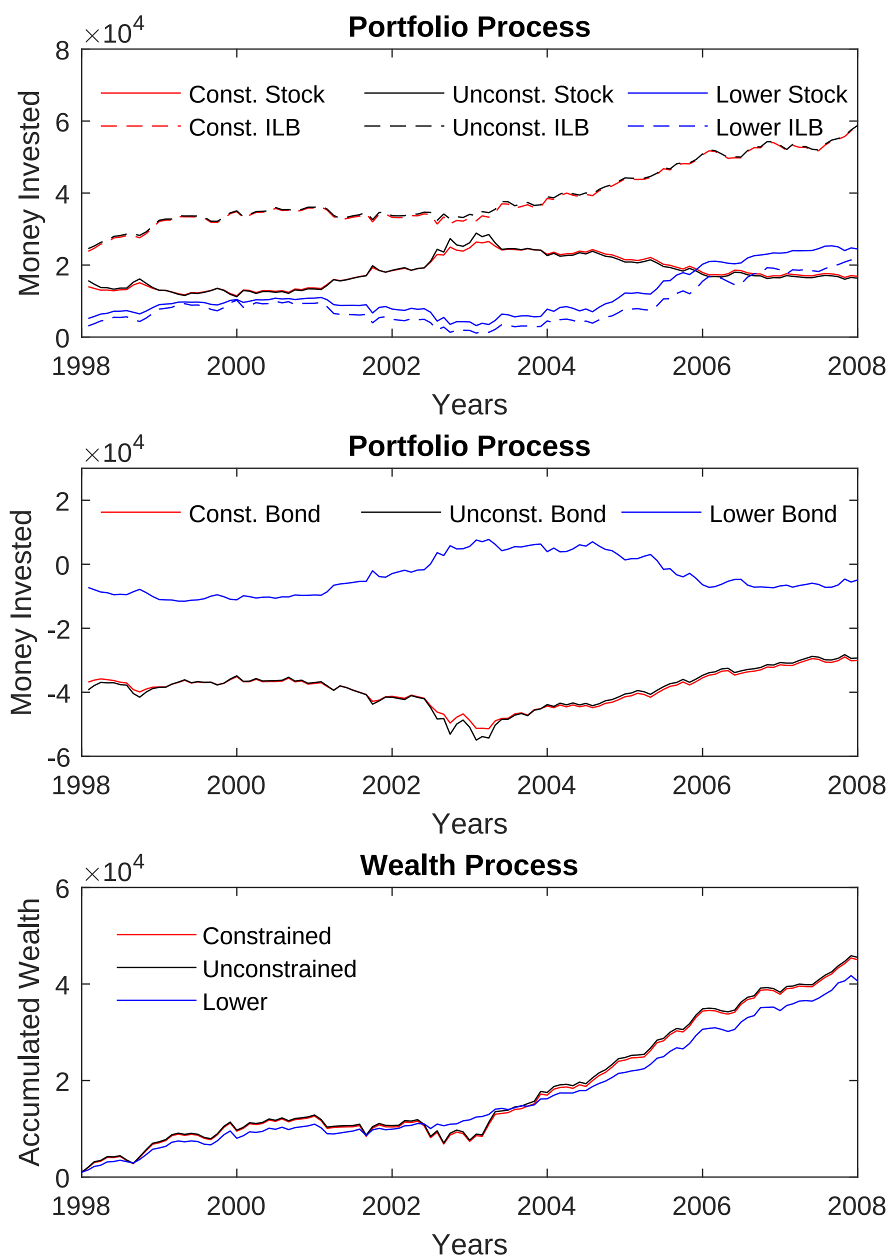


Figure 10.1: In the upper graph, the amount invested in the stock and the inflation-linked bond can be seen for the constrained portfolio, the unconstrained portfolio and the portfolio process with a lower constraint of $K = 25'000$. The investment in the bank account is shown in the middle, while the resulting wealth process is plotted below.

Chapter 11

Analysis of the Portfolio Processes

At this point, we are able to compare the solutions to the optimization problems of Parts I, II and III, namely

$$\inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\big(C - X^{\pi}(T)\big)^2\Big], \tag{11.1}$$

for Part I and

$$\inf_{\pi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\big(C - \frac{X^{\pi}(T)}{I(T)}\big)^2\Big], \tag{11.2}$$

for Parts II and III. For the following discussion, $\mathcal{A}(x)$ will either be chosen to be the set of admissible processes that lead to non-negative terminal wealth, i.e.

$$\mathcal{A}(x) = \Big\{\pi \in \Pi \mid X^{\pi}(0) \leq x \text{ and } X^{\pi}(T) \geq 0, \quad \mathbb{P}\text{-a.s.}\Big\},$$

when we discuss the constrained portfolio problem, or dropping the constraint of non-negative terminal wealth when we discuss the unconstrained portfolio problem. Furthermore, we assume all coefficients to be deterministic functions and uniformly bounded on $[0, T] \times \Omega$.

11.1 Unconstrained Portfolio Processes

We begin by examining the unconstrained optimal portfolio problem, i.e. where $\mathcal{A}(x)$ is the family of admissible portfolio processes with initial wealth

x. Denote by $\hat{\pi}_1(t)$ the optimal portfolio process for the unconstrained optimization problem 11.1 of the classical market model (2.2) and by $\hat{\pi}_2(t)$ and $\hat{\pi}_3(t)$ the optimal portfolio process for the unconstrained optimization problem 11.2 of the market model (5.6) excluding and including contributions, respectively. Then

$$\hat{\pi}_1(t) = \big(\sigma(t)'\big)^{-1}\Big(- \big(X^{\hat{\pi}_1}(t) - C\frac{\beta(T)}{\beta(t)}\big)\theta(t)\Big), \tag{11.3}$$

$$\hat{\pi}_2(t) = \big(\sigma(t)'\big)^{-1}\Big(- \big(X^{\hat{\pi}_2}(t) - \frac{C}{\tilde{\beta}(t)}\big)\tilde{\theta}(t) + X^{\hat{\pi}_2}(t)\bar{\sigma}_I(t)\Big), \tag{11.4}$$

$$\begin{aligned}\hat{\pi}_3(t) = \big(\sigma(t)'\big)^{-1}\Big(&- \big(X^{\hat{\pi}_3}(t) + D(t) - \frac{C}{\tilde{\beta}(t)}\big)\tilde{\theta}(t) \\ &+ \big(X^{\hat{\pi}_3}(t) + D(t)\big)\bar{\sigma}_I(t) - D(t)\sigma_L(t)\Big),\end{aligned} \tag{11.5}$$

for all $t \in [0,T]$, where $\beta(t)$ is the bank account numéraire (2.7), $\tilde{\beta}(t)$ is the inflation-linked numéraire (5.13), $\tilde{\theta}(t) = \theta(t) - \bar{\sigma}_I(t)$ is the adjusted market price of risk and $D(t)$ is the value of future contributions, given by Proposition 9.1.3. Moreover, the corresponding optimal wealth processes are given by

$$X^{\hat{\pi}_1}(t) = \frac{1}{\beta(t)}\Big(\big(x - C\beta(T)\big)e^{-\int_0^t \|\theta(s)\|^2 ds}Z(t) + C\beta(T)\Big), \tag{11.6}$$

$$X^{\hat{\pi}_2}(t) = \frac{1}{\tilde{\beta}(t)}\Big(\big(x\tilde{\beta}(0) - C\big)e^{-\int_0^t \|\tilde{\theta}(s)\|^2 ds}\tilde{Z}(t) + C\Big), \tag{11.7}$$

$$X^{\hat{\pi}_3}(t) = \frac{1}{\tilde{\beta}(t)}\Big(\big((x+d)\tilde{\beta}(0) - C\big)e^{-\int_0^t \|\tilde{\theta}(s)\|^2 ds}\tilde{Z}(t) + C - \tilde{\beta}(t)D(t)\Big), \tag{11.8}$$

for all $t \in [0,T]$, where $Z(t)$ is the Doléan-Dade exponential (2.4), $\tilde{Z}(t)$ is the Doléan-Dade exponential (5.12) and $d = D(0)$ is the present value of future contributions.

We notice in particular, that the optimal portfolio process (11.5) under presence of both inflation and contributions has three components.

- A speculative component, proportional to the distance between the optimal wealth increased by the discounted expected future contributions and the discounted target. This implies that if everything else stays the same, the higher the target wealth, the higher the speculative component.

- A second component, hedging the optimal wealth increased by the discounted expected future contributions against inflation.
- A third component, hedging the discounted expected future contributions against the salary risk in order to offset any shock to the stochastic salary.

In the case of no contributions, it is apparent that both the optimal wealth process (11.8) as well as the optimal portfolio process (11.5) reduce to the optimal wealth process $X^{\hat{\pi}_2}(t)$, given by (11.7), and the optimal portfolio process $\hat{\pi}_2(t)$, given by (11.4), respectively.

Similarly, in the case of no inflation, i.e. $r_N(t) = r_R(t)$ and $\sigma_I(t) \equiv \sigma_{IS}(t) \equiv 0$ for all $t \in [0, T]$, the inflation-linked numéraire $\tilde{\beta}(t)$ satisfies

$$\tilde{\beta}(t) = e^{\int_t^T r_R(s)ds} = \frac{\beta(t)}{\beta(T)}.$$

Therefore, both the optimal wealth process (11.7) as well as the optimal portfolio process (11.4) reduce to the optimal wealth process $X^{\hat{\pi}_1}(t)$, given by (11.6), and the optimal portfolio process $\hat{\pi}_1(t)$, given by (11.3), respectively. This allows us to determine the optimal portfolio process for the unconstrained optimization problem in the financial market without inflation, but introducing contributions of the form (8.4). Namely, the optimal portfolio process is then given by

$$\hat{\pi}_4(t) = \big(\sigma(t)'\big)^{-1}\Big(- \big(X^{\hat{\pi}_4}(t) + D(t) - C\frac{\beta(T)}{\beta(t)}\big)\theta(t) - D(t)\sigma_L(t)\Big), \quad (11.9)$$

for all $t \in [0, T]$, where $X^{\hat{\pi}_4}(t)$ is the corresponding optimal wealth process.

Lastly, we notice that if the discounted target cannot be reached in a risk free manner, i.e. $C > (x + d)\tilde{\beta}(0)$, then the probability of reaching it in real terms is zero. This follows, since

$$X^{\hat{\pi}_3}(t) < \frac{C}{\tilde{\beta}(t)} - D(t), \quad \mathbb{P}\text{-a.s.},$$

for all $t \in [0, T]$, and since $D(T) = 0$.

11.1.1 Cut-Shares

We note that as soon as a second tradable asset is introduced to the market, be it an inflation-linked bond or a second stock, all the optimal portfolio processes may require to have a negative amount invested in one of the assets.

Moreover, many cases require the optimal strategy to borrow considerable amounts of money, so the optimal share invested falls outside the range $[0,1]$. In practice this may not be possible, as there may exist regulatory limits on the shares, and short-selling might be forbidden. One possibility to overcome this practical issue, is to add an additional constraint on the investment strategy itself. This, however, results in a significantly higher degree of complexity. The optimization Problem 3.1.2 with an additional no-shorting constraint is analyzed in [Heunis, 2014], using the approach of convex duality. It may be possible to generalize this approach and to include inflation and contribution, however this is beyond the scope of this thesis.

An alternative way to deal with the above problem, is to enforce the no-shorting constraint manually. The so called ”cut-shares” resulting from this procedure are sub-optimal and lead to some reduction in the efficiency of the portfolio. Modifications of this type have been used by [Vigna, 2014] and [Menoncin and Vigna, 2017], who used to following procedure:

- At any time $t \in [0,T]$, if some of the shares do not belong to $[0,1]$, at least one position is short, as the shares need to sum up to 1. All the negative shares are then set to zero, while the remaining shares are adjusted such that they sum up to one and such that their ratios stay the same.

We apply the algorithm to the unconstrained portfolio process, as the non-negativity of the terminal wealth immediately follows from the no-shorting constraint. In Figure 11.1 we repeat the analysis of Example 3.3.3 and note that with a median rate of return $r_{irr} = 7.2\%$, the cut-share strategy performs much worse than the unconstrained strategy for the same target wealth. On the other hand, removing any short positions of the portfolio process significantly reduces the risk. In Chapter 15 we study the possibility of choosing different target wealths for the portfolio strategies depending on the underlying risk and compare the resulting performance in more detail.

11.2 Constrained Portfolio Processes

Let $\mathcal{A}(x)$ denote the family of admissible porfolio processes with non-negative terminal wealth, i.e.

$$\mathcal{A}(x) = \Big\{\pi \in \Pi \mid X^{\pi}(0) \leq x \text{ and } X^{\pi}(T) \geq 0, \quad \mathbb{P}\text{-a.s.}\Big\}.$$

Denote by $\hat{\pi}_1(t)$ the optimal portfolio process for the constrained optimization Problem 11.1 of the classical market model (2.2) and by $\hat{\pi}_2(t)$ and

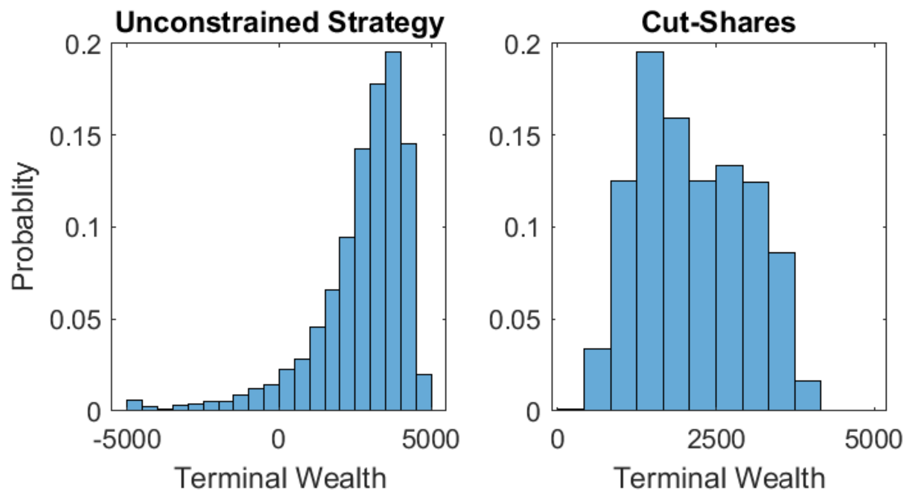


Figure 11.1: Histograms of the terminal wealth distribution for the unconstrained portfolio problem with and without cut-shares.

$\hat{\pi}_3(t)$ the optimal portfolio process for the constrained optimization Problem 11.2 of the market model (5.6) excluding and including contributions, respectively. Then

$$\hat{\pi}_1(t) = \big(\sigma(t)'\big)^{-1}\Big(- \big(X^{\hat{\pi}_1}(t) - \Phi\big(-d^1_-(t, y_1(t))\big) C \frac{\beta(T)}{\beta(t)}\big)\theta(t)\Big), \quad (11.10)$$

$$\hat{\pi}_2(t) = \big(\sigma(t)'\big)^{-1}\Big(- \big(X^{\hat{\pi}_2}(t) - \Phi\big(-d^2_-(t, y_2(t))\big) \frac{C}{\tilde{\beta}(t)}\big)\tilde{\theta}(t) + X^{\hat{\pi}_2}(t)\bar{\sigma}_I(t)\Big), \quad (11.11)$$

$$\hat{\pi}_3(t) = \big(\sigma(t)'\big)^{-1}\Big(- \big(X^{\hat{\pi}_3}(t) + D(t) - \Phi\big(-d^2_-(t, y_2(t))\big) \frac{C}{\tilde{\beta}(t)}\big)\tilde{\theta}(t) + \big(X^{\hat{\pi}_3}(t) + D(t)\big)\bar{\sigma}_I(t) - D(t)\sigma_L(t)\Big), \quad (11.12)$$

for all $t \in [0, T]$, where $\beta(t)$ is the bank account numéraire (2.7), $\tilde{\beta}(t)$ is the inflation-linked numéraire (5.13), $\tilde{\theta}(t) = \theta(t) - \bar{\sigma}_I(t)$ is the adjusted market price of risk and $D(t)$ is the value of future contributions, given by Proposition 9.1.3. The auxillary processes y_1 and d^1 are defined in Theorem 3.4.8, whereas y_2 and d^2 are defined in Theorem 6.3.8. Moreover, the corresponding optimal wealth processes are given by

$$X^{\hat{\pi}_1}(t) = \Phi\big(- d^1_-(t, y_1(t))\big)C\frac{\beta(T)}{\beta(t)} - \Phi\big(- d^1_+(t, y_1(t))\big)y_1(t), \tag{11.13}$$

$$X^{\hat{\pi}_2}(t) = \frac{1}{\tilde{\beta}(t)}\Big(\Phi\big(- d^2_-(t, y_2(t))\big)C - \Phi\big(- d^2_+(t, y_2(t))\big)y_2(t)\Big), \tag{11.14}$$

$$X^{\hat{\pi}_3}(t) = \frac{1}{\tilde{\beta}(t)}\Big(\Phi\big(- d^2_-(t, y_2(t))\big)C - \Phi\big(- d^2_+(t, y_2(t))\big)y_2(t)\Big) - D(t), \tag{11.15}$$

for all $t \in [0, T]$.

Similar to the unconstrained case, the optimal portfolio process (11.10) consists of three components. While the components used to hedge against the risk of inflation and the salary risk are the same as for the unconstrained optimal portfolio process (11.3), the speculative component is slightly different. Due to the equations (11.13)-(11.15) for the wealth processes, we see that the speculative component is still always positive, but due to the additional factor of the cumulative distribution function, it will always be smaller in the constrained case, for the same current wealth.

This explains why the difference between the portfolio processes was much larger in Example 3.4.10 than in either Example 6.3.11 or Example 9.2.9, as the only difference is the speculative component, which makes up less of the investment under the inclusion of inflation or contributions.

Due to similar arguments as in Section 11.1, $\hat{\pi}_3$ reduces to $\hat{\pi}_2$ in the case of no contributions, which in turn reduces to $\hat{\pi}_1$, if there is no inflation present in the market. The optimal portfolio process for the constrained optimization problem in the financial market without inflation, but including contributions can be derived as

$$\begin{aligned}\hat{\pi}_4(t) = \big(\sigma(t)'\big)^{-1}\Big(&- \big(X^{\hat{\pi}_4}(t) + D(t) - \Phi\big(- d^2_-(t, y_2(t))\big)C\frac{\beta(T)}{\beta(t)}\big)\theta(t) \\ &- D(t)\sigma_L(t)\Big),\end{aligned}$$

for all $t \in [0, T]$, where $X^{\hat{\pi}_4}(t)$ is the corresponding optimal wealth process.

Appendix

III.A Proof of Theorem 9.2.1

Recall the notation

$$\begin{aligned} A(t) &= \sigma_I^2(t) - \sigma_I(t)\theta_I(t), \\ M(t) &= \tilde{\beta}(t)\big(\Gamma(t) - \sigma_{n+1}(t)\sigma_I(t)\big), \\ \bar{\sigma}_I(t) &= \big(0,\dots,0,\sigma_I(t)\big)', \\ D(t) &= \tilde{\beta}(t)\sigma(t). \end{aligned} \tag{11.16}$$

Furthermore, introduce the process $\big(\varrho(t)\big)_{t\in[0,T]}$, given by

$$\varrho(t) = \mu_L(t) - (r_N(t) + \sigma_I(t)\theta_I(t)) - \sigma_{LI}(t)\sigma_I(t) + \sigma_I^2(t), \tag{11.17}$$

for all $t \in [0,T]$.

We follow the dynamic programming approach and consider Problem 9.1.5, starting from some time $t \in [0,T]$ with the initial states $Y^\pi(t) = y$ and $\bar{L}(t) = l$. That is, the dynamics of $Y^\pi(s)$ and $\bar{L}(s)$ can be written as

$$\begin{aligned} dY^\pi(s) =& \big(Y^\pi(s)A(s) + \pi(s)'M(s) + \delta\bar{L}(s)\big)ds \\ &+ \big(\pi(s)'D(s) - \bar{\sigma}_I(s)'Y^\pi(s)\big)dW(s), \\ d\bar{L}(s) =& \bar{L}(s)\varrho(s)ds + \bar{L}(s)\big(\sigma_L(s) - \bar{\sigma}_I(s)\big)'dW(s), \end{aligned}$$

with boundary conditions $Y^\pi(t) = y$ and $\bar{L}(t) = l$. Correspondingly, the value function is defined by

$$V(t,y,l) = \min_{\pi\in\Pi} \mathbb{E}^{\mathbb{P}}\Big[\big(C - Y^\pi(T)\big)^2 \big| Y^\pi(t) = y, \bar{L}(t) = l\Big].$$

As long as the choice of parameters is clear, we denote by $V = V(t,y,l)$ in order to increase the readability of the following derivations. Now, by the remark after Theorem 3.4.2, the value function satisfies the general HJB equation (3.22), i.e.

$$
\begin{aligned}
-\frac{1}{2}(y-C)^2 &= V(T,y,l)\\
0 &= V_t + \big(A(t)y+\delta l\big)V_y\\
&\quad + l\varrho(t)V_l + \frac{1}{2}l^2\big(\sigma_L(t)-\bar\sigma_I(t)\big)'\big(\sigma_L(t)-\bar\sigma_I(t)\big)V_{ll}\\
&\quad + \min_{\pi\in\Pi}\Big\{\pi(t)'M(t)V_y + \frac{1}{2}\Big(\pi(t)'D(t)D(t)'\pi(t)\\
&\qquad - 2\pi(t)'D(t)\bar\sigma_I(t)y + \bar\sigma_I(t)'\bar\sigma_I(t)y^2\Big)V_{yy}\\
&\qquad + l\big(\pi(t)'D(t)-\bar\sigma_I(t)y\big)\big(\sigma_L(t)-\bar\sigma_I(t)\big)V_{yl}\Big\}. \qquad (11.18)
\end{aligned}
$$

Suppose that $V_{yy} > 0$. Then, the first order condition for $\pi(t)$ reads

$$
\begin{aligned}
0 = M(t)V_y &+ \big(D(t)D(t)'\hat\pi(t) - D(t)\bar\sigma_I(t)y\big)V_{yy}\\
&+ lD(t)\big(\sigma_L(t)-\bar\sigma_I(t)\big)V_{yl},
\end{aligned}
$$

and the optimal portfolio strategy is given by

$$
\hat\pi(t) = \big(D(t)D(t)'\big)^{-1}\Big(D(t)\bar\sigma_I(t)y - M(t)\frac{V_y}{V_{yy}} - lD(t)\big(\sigma_L(t)-\bar\sigma_I(t)\big)\frac{V_{yl}}{V_{yy}}\Big). \qquad (11.19)
$$

Corollary III.A.1. *The optimal value function satisfies*

$$
\begin{aligned}
0 = V_t &+ \big(A(t)y + \bar\sigma_I(t)'D(t)^{-1}M(t)y + \delta l\big)V_y\\
&+ l\varrho(t)V_l + \frac{1}{2}l^2\big(\sigma_L(t)-\bar\sigma_I(t)\big)'\big(\sigma_L(t)-\bar\sigma_I(t)\big)V_{ll}\\
&- \frac{1}{2}l^2\big(\sigma_L(t)-\bar\sigma_I(t)\big)'\big(\sigma_L(t)-\bar\sigma_I(t)\big)\frac{V_{yl}^2}{V_{yy}}\\
&- \frac{1}{2}M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\frac{V_y^2}{V_{yy}}\\
&- \big(\sigma_L(t)-\bar\sigma_I(t)\big)'D(t)^{-1}M(t)l\frac{V_yV_{yl}}{V_{yy}}. \qquad (11.20)
\end{aligned}
$$

Proof. Inserting the optimal portfolio strategy (11.19) in to the different terms of (11.20) yields the following terms

$$
\begin{aligned}
&\hat{\pi}(t)'M(t) = \\
&\qquad \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}M(t)y \\
&\qquad - M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\frac{V_y}{V_{yy}} \\
&\qquad - \big(\sigma_L(t)-\bar{\sigma}_I(t)\big)'D(t)'\big(D(t)D(t)'\big)^{-1}M(t)l\frac{V_{yl}}{V_{yy}}; \\
&\hat{\pi}(t)'D(t)D(t)'\hat{\pi}(t) = \\
&\qquad \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)y^2 \\
&\qquad - M(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)y\frac{V_y}{V_{yy}} \\
&\qquad - \big(\sigma_L(t)-\bar{\sigma}_I(t)\big)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)yl\frac{V_{yl}}{V_{yy}} \\
&\qquad - \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}M(t)y\frac{V_y}{V_{yy}} \\
&\qquad - M(t)'\big(D(t)D(t)'\big)^{-1}M(t)\frac{V_y^2}{V_{yy}^2} \\
&\qquad - \big(\sigma_L(t)-\bar{\sigma}_I(t)\big)'D(t)'\big(D(t)D(t)'\big)^{-1}M(t)l\frac{V_yV_{yl}}{V_{yy}^2} \\
&\qquad - \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\big(\sigma_L(t)-\bar{\sigma}_I(t)\big)yl\frac{V_{yl}}{V_{yy}} \\
&\qquad - M(t)'\big(D(t)D(t)'\big)^{-1}D(t)\big(\sigma_L(t)-\bar{\sigma}_I(t)\big)l\frac{V_yV_{yl}}{V_{yy}^2} \\
&\qquad - \big(\sigma_L(t)-\bar{\sigma}_I(t)\big)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\big(\sigma_L(t)-\bar{\sigma}_I(t)\big)l^2\frac{V_{yl}^2}{V_{yy}^2}; \\
&\hat{\pi}(t)'D(t)\bar{\sigma}_I(t)y = \\
&\qquad \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)y^2 \\
&\qquad - M(t)0\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)y\frac{V_y}{V_{yy}} \\
&\qquad - \big(\sigma_L(t)-\bar{\sigma}_I(t)\big)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\bar{\sigma}_I(t)ly\frac{V_{yl}}{V_{yy}};
\end{aligned}
$$

and finally

$$\begin{aligned}
&\hat{\pi}(t)'D(t)\big(\sigma_L(t)-\bar{\sigma}_I(t)\big)= \\
&\qquad \bar{\sigma}_I(t)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\big(\sigma_L(t)-\bar{\sigma}_I(t)\big)yl \\
&\qquad - M(t)'\big(D(t)D(t)'\big)^{-1}D(t)\big(\sigma_L(t)-\bar{\sigma}_I(t)\big)l\frac{V_y}{V_{yy}} \\
&\qquad - \big(\sigma_L(t)-\bar{\sigma}_I(t)\big)'D(t)'\big(D(t)D(t)'\big)^{-1}D(t)\big(\sigma_L(t)-\bar{\sigma}_I(t)\big)l^2\frac{V_{yl}}{V_{yy}}.
\end{aligned}$$

Now grouping the terms with the same derivatives and simplifying yields the claim. □

In order to solve the non-linear PDE (11.20), we assume that the value function is of quadratic form, i.e.

$$V(t,y,l)=y^2a(t)+yb(t)+c(t)+l^2d(t)+le(t)+ylf(t),$$

where $a(t),b(t),c(t),d(t),e(t)$ and $f(t)$ are deterministic functions and need to be determined. Calculating the derivatives, we have

$$\begin{aligned}
V_t &= y^2a_t(t)+yb_t(t)+c_t(t)+l^2d_t(t)+le_t(t)+ylf_t(t), \\
V_l &= 2ld(t)+e(t)+yf(t), \qquad V_{ll}=2d(t), \\
V_y &= 2ya(t)+b(t)+lf(t), \qquad \frac{V_{yl}^2}{V_{yy}}=\frac{f(t)}{2a(t)}, \\
\frac{V_yV_{yl}}{V_{yy}} &= yf(t)+\frac{b(t)f(t)}{2a(t)}+l\frac{f(t)^2}{2a(t)}.
\end{aligned}$$

Inserting the derivatives back into (11.20), we obtain a bivariate polynomial in terms of y and l. In order for (11.20) to be zero for all possible choices of y and l, every polynomial term needs to be zero individually. Together with the boundary conditions of the HJB equation (11.20), we obtain a system of six simpler PDE's.

$$\begin{cases} a_t(t) + 2a(t)\big(A(t) + \bar{\sigma}_I(t)'D(t)^{-1}M(t)\big) \\ -a(t)M(t)'\big(D(t)D(t)'\big)^{-1}M(t) = 0, \\ a(T) = -\frac{1}{2}, \end{cases}$$
$$\begin{cases} b_t(t) + b(t)\big(A(t) + \bar{\sigma}_I(t)'D(t)^{-1}M(t)\big) \\ -b(t)M(t)\big(D(t)D(t)'\big)^{-1}M(t) = 0, \\ b(T) = C, \end{cases}$$
$$\begin{cases} c_t(t) - \frac{1}{2}M(t)\big(D(t)D(t)'\big)^{-1}M(t)\frac{b^2(t)}{2a(t)} = 0, \\ c(T) = -\frac{1}{2}C^2, \end{cases}$$
$$\begin{cases} d_t(t) + \delta f(t) + 2d(t)\varrho(t) + \|\sigma_L(t) - \bar{\sigma}_I(t)\|^2\big(d(t) - \frac{a(t)}{f(t)}\big) \\ -\frac{1}{2}M(t)\big(D(t)D(t)'\big)^{-1}M(t)\frac{f^2(t)}{a(t)} - \big(\sigma_L(t) - \bar{\sigma}_I(t)\big)'D(t)^{-1}M(t)\frac{f^2(t)}{2a(t)} = 0, \\ d(T) = 0, \end{cases}$$
$$\begin{cases} e_t(t) + \delta b(t) + e(t)\varrho(t) - \frac{1}{2}M(t)\big(D(t)D(t)'\big)^{-1}M(t)\frac{b(t)f(t)}{a(t)} \\ -\big(\sigma_L(t) - \bar{\sigma}_I(t)\big)'D(t)^{-1}M(t)\frac{b(t)f(t)}{2a(t)} = 0, \\ e(T) = 0, \end{cases}$$
$$\begin{cases} f_t(t) + 2\delta a(t) + f(t)\varrho(t) + f(t)\big(A(t) + \bar{\sigma}_I(t)'D(t)^{-1}M(t)\big) \\ -M(t)\big(D(t)D(t)'\big)^{-1}M(t)f(t) - \big(\sigma_L(t) - \bar{\sigma}_I(t)\big)'D(t)^{-1}M(t)f(t) = 0, \\ f(T) = 0. \end{cases} \tag{11.21}$$

In order to determine the optimal portfolio process (11.19) we solve the equations for $a(t), b(t)$ and $f(t)$ explicitly. Now the PDE's for $a(t)$ and $b(t)$ are exactly the Riccati equations in Theorem 6.3.1 and simplifying their respective solutions (7.11) and (7.12), we obtain

$$a(t) = -\frac{1}{2}e^{-\int_t^T \|\tilde{\theta}(s)\|^2 ds}, \quad b(t) = Ce^{-\int_t^T \|\tilde{\theta}(s)\|^2 ds}. \tag{11.22}$$

Note that the PDE (11.21) is no longer a homogeneous partial differential equation due to the presence of the term $2\delta a(t)$, hence the analytical solution cannot be derived using the same method as solving the Riccati equations. In order to find the analytical solution, we introduce the associated homogeneous PDE with parameter $\tau < T$. For all $t \leq \tau$ let $v(t, \tau)$

satisfy

$$0 = v_t(t,\tau) + v(t,\tau)\big(\varrho(t) - \|\tilde{\theta}(t)\|^2 - \big(\sigma_L(t) - \bar{\sigma}_I(t)\big)'\tilde{\theta}(t)\big),$$
$$v(\tau,\tau) = -\delta \exp\big(-\int_\tau^T \|\tilde{\theta}(s)\|^2 ds\big). \tag{11.23}$$

Proposition III.A.2. *Let $v(t,\tau)$ be the solution of* (11.23)*. Then the solution of* (11.21) *can be expressed as*

$$f(t) = \int_t^T v(t,\tau)d\tau.$$

Proof. For $f(t)$ defined as above, we have $f(T) = \int_T^T v(t,\tau)d\tau = 0$, which satisfies the boundary condition in (11.21). Let $\tau = t$ in the boundary condition of (11.23). Then $v(t,t) = -\delta \exp\big(\int_t^T \|\tilde{\theta}(s)\|^2 ds\big)$. Differentiating f with respect to t, we obtain by Leibniz' rule

$$\begin{aligned} f_t(t) &= -v(t,t) + \int_t^T v_t(t,\tau)d\tau \\ &= \delta \exp\big(\int_\tau^T \|\tilde{\theta}(s)\|^2 ds\big) + \int_t^T v_t(t,\tau)d\tau, \end{aligned}$$

and therefore

$$f_t(t) + f(t)\big(\varrho(t) - \|\tilde{\theta}(t)\|^2 - \big(\sigma_L(t) - \bar{\sigma}_I(t)\big)'\tilde{\theta}(t)\big) - \delta \exp\big(\int_\tau^T \|\tilde{\theta}(s)\|^2 ds\big)$$
$$= \int_t^T \Big(v_t(t,\tau) + v(t,\tau)\big(\varrho(t) - \|\tilde{\theta}(t)\|^2 - \big(\sigma_L(t) - \bar{\sigma}_I(t)\big)'\tilde{\theta}(t)\big)\Big)d\tau.$$

which is zero by the definition of $v(t,\tau)$ □

By Proposition III.A.2, the solution of (11.21) follows once the solution to (11.23) is found. In the case of $\delta \equiv 0$, it follows that $v(t,\tau) = 0$, for all $t \in [0,\tau]$ and hence $f(t) \equiv 0$. In the case where $\delta \neq 0$, we propose that the form of the solution of (11.23) is

$$v(t,\tau) = C^v(t,\tau)e^{B^v(t,\tau)},$$

for two deterministic functions $B^v = B^v(t,\tau)$ and $C^v = C^v(t,\tau)$. Inserting this back in the definition of v in (11.23), we obtain the following system of ordinary partial differential equations

$$\begin{aligned} 0 &= B_t^v + \big(\varrho(t) - \|\tilde{\theta}(t)\|^2 - \big(\sigma_L(t) - \bar{\sigma}_I(t)\big)'\tilde{\theta}(t)\big), \qquad & 0 &= C_t^v, \\ B^v(\tau,\tau) &= -\int_\tau^T \|\tilde{\theta}(s)\|^2 ds, & C^v(\tau,\tau) &= -\delta. \end{aligned}$$

Solving these equations, we obtain

$$B^v(t,\tau) = -\int_\tau^T \|\tilde{\theta}(s)\|^2 ds + \int_t^\tau \Big(\varrho(s) - \big(\sigma_L(s) - \bar{\sigma}_I(s)\big)'\tilde{\theta}(s) - \|\tilde{\theta}(s)\|^2\Big) ds,$$
$$C^v(t,\tau) = -\delta,$$

for all $t \in [0,\tau]$. By Proposition III.A.2, the solution to (11.21) is then given by

$$f(t) = -\delta \int_t^T e^{-\int_t^T \|\tilde{\theta}(s)\|^2 ds + \int_t^\tau \left(\varrho(s) - \left(\sigma_L(s) - \bar{\sigma}_I(s)\right)'\tilde{\theta}(s)\right) ds} d\tau. \tag{11.24}$$

Proof of Theorem 9.2.1. In order to obtain the optimal portfolio process, we insert the derivatives of the value function into (11.19). Inserting $y = Y^\pi(t)$ and $l = \bar{L}(t)$, this yields

$$\hat{\pi}(t) = \frac{1}{\tilde{\beta}(t)}\big(\sigma(t)\sigma(t)'\big)^{-1}\Big(\sigma(t)\bar{\sigma}_I(t)Y^\pi(t) - \bar{L}(t)\sigma(t)\big(\sigma_L(t) - \bar{\sigma}_I(t)\big)\frac{f(t)}{2a(t)}\Big)$$
$$- \big(\sigma(t)\theta(t) - \sigma(t)\bar{\sigma}_I(t)\big)\big(Y^\pi(t) - C + \bar{L}(t)\frac{f(t)}{2a(t)}\big).$$

Reordering the terms yields the claim. □

Part IV

Numerical Analysis

Chapter 12

Overview

During the current part, we analyze the portfolio processes introduced in Parts I to III with the focus on four main characteristics.

Constant or Deterministic Parameters

Contrary to most mathematical literature on DC pension plans, the optimal portfolio processes of Parts I to III have been introduced in such a way that they may utilize non-constant, deterministic time series for the underlying parameters. To analyze the difference in performance, we compare the constrained and the unconstrained portfolio process for constant and for deterministic parameters.

If the deterministic time series of the parameters is known from the beginning, using time series reflecting the path of the parameters improves the performance substantially. However, instead of maintaing that the change in the parameter values happens deterministically, one might introduce a probabilistic model, reflecting different states of the market. This leads to the theory of regime-switching models which surpasses the scope of this thesis.

In order to compare the model performance, we estimate the parameters on historical data and predict future values either as constants, through the maximum likelihood estimator, or as a deterministic time series, through variations of autoregressive models. The theory behind the parameter estimation is summarized in the Appendices IV.A-IV.C. From a practitioner's point of view, we also include the analysis of a portfolio process, where the parameters are updated annually. Note, that this breaks many of the model assumptions and does not result in an optimal portfolio.

We find that for both sets of historical data analyzed, the models utilizing deterministic parameters are a slight upgrade to those only using constant parameters, even under model uncertainty. However, as the parameter estimation models used for this thesis only use the time series itself to forecast the parameters, they are prone to large estimation errors. Hence, for the remaining numerical analysis, the use of constant parameters suffices.

Constrained or Unconstrained Portfolio Process

During Parts I to III, we have seen the optimal portfolio processes for three different optimization problems, each in three market models. We compare the performance of the different portfolio processes in the simple market model of Part I and in the most general market model of Part III. It is important to note that the non-negativity constraint for the constrained portfolio problem is less useful in the full market model (8.3), as only the non-negativity of the terminal wealth is guaranteed. For that reason, we include the analysis of the cut-shares, introduced in Section 11.2. The no-shorting constraint imposed on the cut-shares, guarantees that the whole wealth process is non-negative.

In order to compare the performance of the different models, we set the target wealth for each strategy, such that all strategies carry the same underlying risk. We choose the expected shortfall at the 95% confidence interval as the risk measure and calculate the internal rate of return as the measure of performance. We compare the portfolio processes for different parameters and include the ruin and success probabilities in the analysis.

We find that for the market model of Part I the strategy resulting from cut-shares shows the best performance on average. The higher target wealth allows to invest more heavily in the risky assets, while the no shorting constraint asures that the underlying risk stays low. For the full market model of Part III however, the strategy resulting from cut-shares is generally outperformed by the other two. This comes as no surprise, as the optimal portfolio strategy borrows against the future contributions, which is not possible under the no-shorting constraint.

The Importance of the Inflation-Linked Bond

We investigate the advantage of incorporating inflation risk and adding an inflation-linked bond to the market. During Examples 6.2.6, 6.3.11 and 7.3.2 we touched upon the advantages lightly, but have not studied the investment behavior in detail. We find that a higher median rate of return and increased upside potential and conclude that an inflation-linked bond has significant advantage to hedge inflation risk.

Various studies have produced different results on the impact of the inflation-linked bond on wealth optimization problems. While [Zhang, 2012] claims the equivalence of real and nominal portfolio choices for a power utility maximizing investor, both [Yao et al., 2013] and [Zhang and Guo, 2018] point to differences in the portfolio strategy. For the target-based approach, it is apparent from both the structure of Problem 6.1.1 and the resulting optimal portfolio process (6.26) that including inflation to the market warps the investment behavior significantly.

Comparison to other Portfolio Processes

During this chapter we will study whether the target-based strategies are viable alternatives to current strategies used for pension plans.

We compare the performance of the target-based optimal strategies to popular strategies used in practice. Lifecycle strategies currently make up most of the pension plan market in the UK and any possible alternative needs to out-perform the classical lifecycle strategy in most situations. We also include the optimal strategy for a power utility maximizing investor to the analysis, as an alternative mathematical strategy. Studying the performance for three specific stock market scenarios we see that the target-based optimal portfolio is a viable alternative to either strategy.

Nonetheless, the target-based strategy shows the weakness of heavily borrowing thorughout the duration of the pension plan which may lead to bankruptcy. As an alternative we use the target-based optimal portfolio resulting from cut-shares. Enforcing the no-shorting constraint ensures that the wealth remains positive and that regulatory restrictions are met. We find that even though the additional constraint leads to a worse performance on average, not being able to borrow is an advantage in certain market situations. Furthermore, the resulting terminal wealth distribution is narrower, leading to more certainty for the pension plan member.

Chapter 13

Parameter Estimation

In order to evaluate the performance of the target-based portfolio optimization approach, we need market consistent models to estimate the time series of the different parameters. Note that due to the assumptions of a deterministic set of parameters in Assumptions 3.4.1 and 6.1.8, the estimation is performed at the start and the parameters are estimated only once for the whole duration of the process. We are aware that this assumption is a severe restriction on the usefulness of the model in practice. However, we believe that already in this model, with suitably chosen estimations of the parameters, the results will provide good insight in the performance of the target-based portfolio optimization approach.

Aim and Scope.

We want to model the time series for the parameters of the financial markets (2.2), (5.1) and (8.3) for future dates $t \in (0, T)$ such that:

- the model is economically reasonable, e.g. stock returns should always be at least as high as the risk free interest rate;
- the model works with statistical tools and is based on historical data;
- the model allows the prediction of parameters in the very long term, i.e. up to 40 years in the future.

Due to the multitude of parameter estimation models used in academia and in practice, we do not claim that the models used in this thesis are optimal for all possible applications. During the analysis in Chapter 14, we compare the case of deterministic parameters to the one with constant parameters and show in what ways the quality of performance is increased.

The Data

- In order to estimate the nominal interest rate, monthly historical data of the Bank of England base rate is used. Data is available for the period of January 1900 to December 2017. This is the rate that the Bank of England charges other banks and financial institutions for overnight loans and is set in order to maintain monetary and financial stability.

- The stock used in the market is the FTSE Actuaries All-Share Index, currently consisting of 641 companies traded on the London Stock Exchange. This index is used by UK actuaries in order to investigate the question of investment research and is intended to reflect the average yield and volatility of stocks in the UK. Monthly historical data for the period of January 1920 to December 2017 is available for the FTSE Actuaries All Share Index.

- Monthly historical data for the UK inflation index is used to estimate the parameters for the inflation index (5.1). This index consists of the Interim Index of Retail Prices between January 1900 and June 1947 and the Retail Prices Index between July 1947 and December 2017. It measures the change in the cost of a representative sample of goods and services and is published monthly by the Office for National Statistics.

- In order to estimate the parameters for the inflation-linked bond (5.6), monthly historical data of the FTSE Actuaries UK Index-Linked Gilts Index is used. First issued in May 1982, the index is calculated on a total return basis and measures the performance of the index-linked gilts market as a whole, as well as the performance of individual maturity segments of the market.

- All information concerning the wage level is found in [ONS, 2018]. To estimate the drift and volatility terms of the wage level, the file *K54L*, from the Office of National Statistics, is used. It contains information about the UK Wage Index between January 1930 and December 2017.

We thank Professor David Wilkie for the extensive financial market data on the bank base rate, the FTSE indices and the retail index.

13.1 Estimation of the Nominal Interest Rate

As short-term risk-free zero-coupon bonds do not exist, another proxy is needed in order to estimate and predict the nominal interest rate process $r_N(t)$. We calibrate the model to the Bank of England base rate as an (almost) risk-free financial instrument, readily available in the market. The relevance of the short-term interest rate is both economically and financially immense. From a macroeconomic point of view, this base rate is set by the central bank in order to meet inflation forecasts and to maintain economic stability. From the financial perspective, the short rate is needed to construct the whole yield curve, as the yields at other maturities are risk-adjusted averages of the expected future short rate.

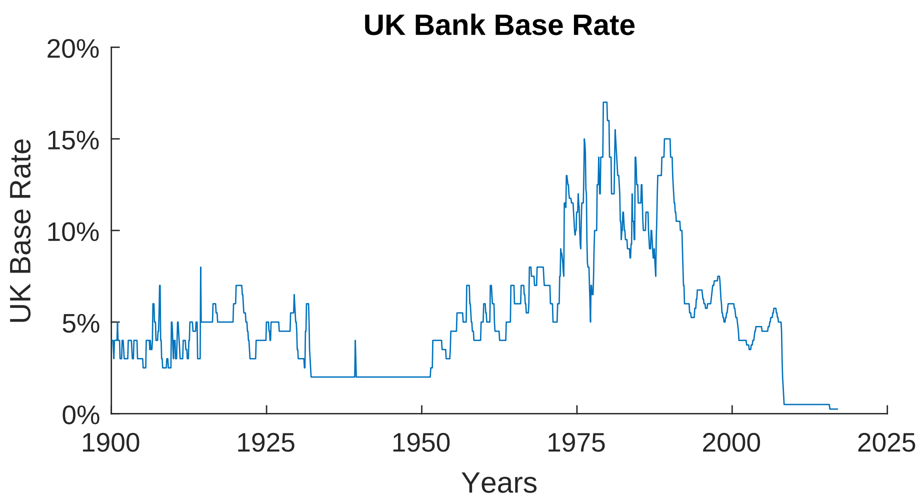


Figure 13.1: Time series of the monthly UK bank base rates for $t \in \{01/1900, \ldots, 12/2017\}$.

In order to satisfy the economical requirements, models should incorporate macroeconomic variables as predictors, e.g. inflation, the unemployment rate and GDP. On the other hand, LIBOR and spot rates, as well as the bond market can be used to construct the interest rate yield curve and hence approximate the nominal interest rate process. See e.g. [Diebold et al., 2016] and [Ang and Piazzesi, 2003] for more detail on the prediction of interest rates using latent variables. As the study of consistent yield curve prediction is very involved and needs more data, we instead utilize the tools developed in Section IV.B and view the interest rate as a time series displaying some degree of autocorrelation.

13.1.1 Estimation using the ARMA Model

In order to showcase the methodology in utilizing the ARMA model, we perform the complete analysis outlined in Appendix IV.B on the historical data between January 1955 and December 2005 with the goal of predicting the interest rate until December 2025. Similar studies can then be performed for different sets of historical data. Note that care needs to be taken for the inter-war period between 1930 and 1950, as well as in more recent times since 2008. The interest rate in those periods has been close to constant, which is a clear violation to the stationarity needed to use the ARMA model.

Model Validation.

As the UK base rate has on average only been adjusted twice every year and has been constant otherwise, we transform the interest rate into a total return index, initialized on January 1955, and measure the interest rates semi-annually. As the resulting process is still not stationary, we use the logarithm as a non-linear transformation and use the model on log-interest rates.

In Figure 13.2 we see that the autocorrelation function of this time series decreases, but slowly. Hence it is not apparent from the visual illustration alone, if the time series is stationary, or not. The econometrics toolbox

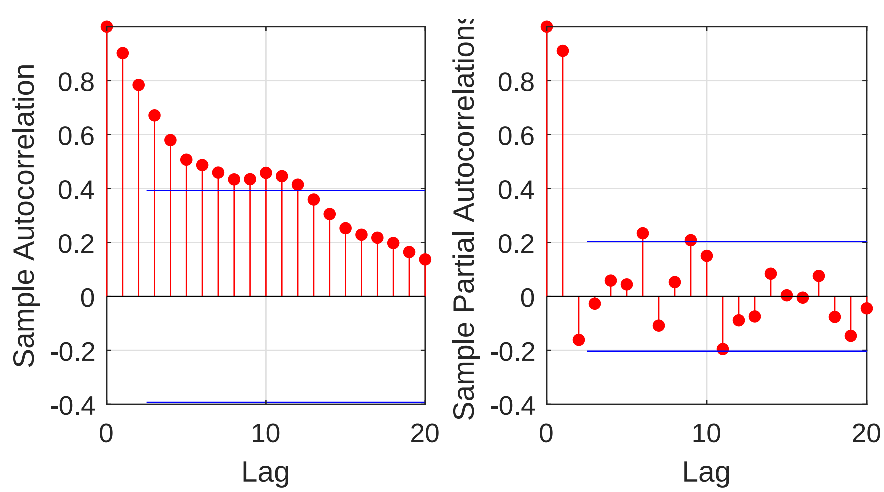


Figure 13.2: The autocorrelation function of the biannual log-interest rates can be seen on the left-hand side, while the corresponding partial autocorrelation function is displayed on the right-hand side.

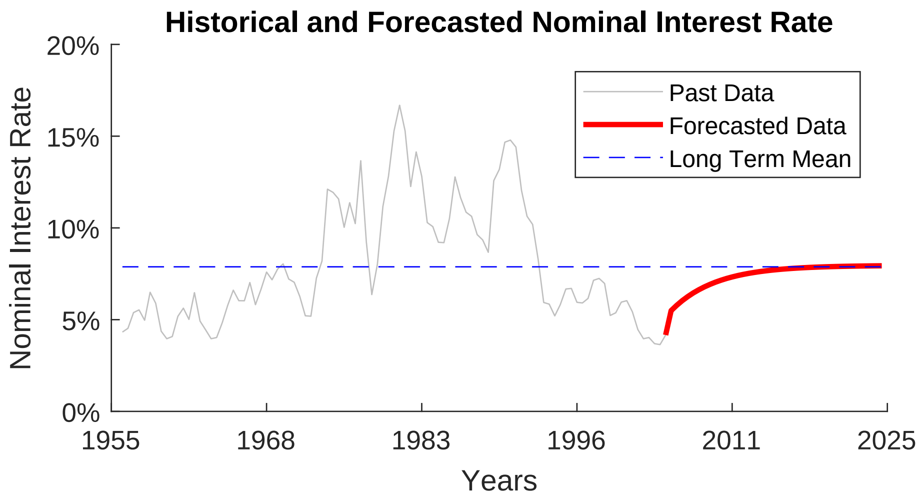


Figure 13.3: On a semi-annual grid, $\Delta = 1/2$, we see the time series of the historical interest rates for $t \in \{01/1955, \dots, 07/2004\}$ and forecasted interest rates for $t \in \{01/2005, \dots, 07/2024\}$, calculated by an ARMA$(1,1)$ model. The long term average interest rate has been plotted as a dashed, blue line.

in Matlab provides an efficient implementation of the augmented Dickey-Fuller test of Section IV.B.1 and we use it to test for a unit root at the 5% confidence level.

We find that for the period of January 1955 until December 2005, the augmented Dickey-Fuller test rejects the null hypothesis of a unit root at the 5% confidence level for lags up to 4. For the fifth lag, the null hypothesis cannot be rejected. However, as the power of the test decreases quickly with the number of lags, we choose to view the time series of the semi-annual interest rates as stationary and continue with estimating the different parameters for the ARMA model.

Model Calibration.

As the autocorrelation function and the partial autocorrelation function in Figure 13.2 are hard to interpret, we use the system identification toolbox to fit an ARMA(p, q) model to the data for $p, q \in (0, \dots, 5)$. Using the BIC to measure goodness-of-fit, the ARMA$(1,1)$ model produces the best fit, given by

$$X(t) = \phi X(t-1) + W(t) + \theta W(t-1),$$

for $\phi = 0.8835$ and $\theta = 0.1541$. With a normalized mean square error of 0.58, the fit of the model is still rather bad, which could be due to some degree of unstationarity in the data, as well as due to the fact that interest rates do not depend entirely on their past values, but also on many other underlying variables.

In Figure 13.3 we see that the prediction power of the model quickly decreases and that after ten years, the best prediction is the long-term average of the nominal interest rate.

13.2 Estimation of the Stock Parameters

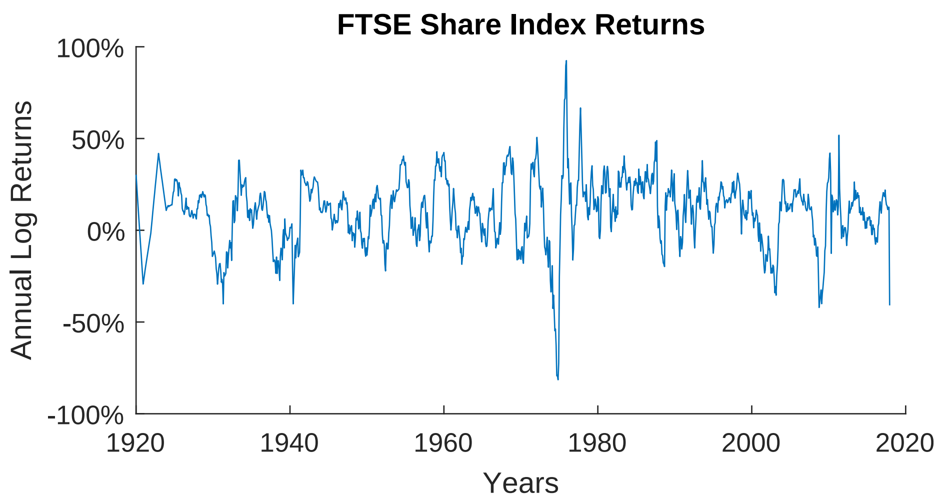


Figure 13.4: Time series of the yearly log returns of the FTSE All Share Index for $t \in \{01/1920, \ldots, 12/2017\}$.

Without loss of generality, we introduce only one stock in our model, which takes the role of a stock market index. In addition to simplicity, this also has the added benefit that a big index is more likely to satisfy the assumption of no market frictions, which would lead to difficulties in deciding on the market prices and assessing the model performance. We calibrate the model to the FTSE All Share Index.

13.2.1 Estimation using the ARMA / GARCH Model

Due to the presence of heteroskedasticity in the data of the stock returns, the GARCH model outlined in Section IV.C seems viable in order to forecast both the volatility parameter $\sigma_S(t)$, as well as the drift parameter $\mu_S(t)$. We use the data between January 1955 and December 2005 to predict both variables until December 2025. The methodology can then be repeated for different sets of historical data.

Model Validation.

Due to the exponential structure of the stock returns in the market model (2.2) we measure the annual log returns monthly in order to obtain a stationary time series.

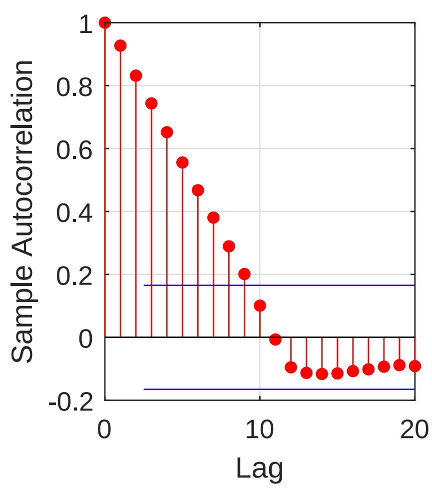


Figure 13.5: The autocorrelation function of the annual log returns can be seen on the left-hand side, while the autocorrelation function of the squared annual log returns is displayed on the right-hand side.

In Figure 13.5 we see that both the process itself, as well as the squared process show correlation. The augmented Dickey-Fuller test rejects the null hypothesis of a unit root at the 5% confidence interval and therefore we cannot reject the stationarity of the series. With this and the autocorrelation functions of Figure 13.5 in mind, we continue with estimating the different parameters for an ARMA/GARCH model.

Model Calibration.

We use the econometrics toolbox to fit an ARMA(p_A, q_A) / GARCH(p_G, q_G) model to the data for $p_A, q_A, p_G, q_G \in (0, \ldots, 3)$ and note that the standardized residuals have more large values than expected under a standard normal distribution. This suggests a Student's t-distribution might be more appropriate for the innovation distribution, for which we also fit the model. Once more, we use the BIC to measure the goodness-of fit of the different models. The ARMA$(2,3)$ / GARCH$(1,1)$ model, with Student's t-distributed innovations shows the best fit.

In Figure 13.6 we see that the prediction power for both the log returns, as well as for the conditional variance, quickly decreases and both rates converge to their long term average.

Note that so far we have only forecasted the returns, and not the drift parameter $\mu(t)$ itself. Denote the forecasted stock returns by $\hat{R}_S(t)$. Then

$$\mathbb{E}^{\mathbb{P}}\Big[\log\big(\frac{S(t)}{S(t-1)}\big)\Big] = \mu_S(t) - \frac{1}{2}\sigma_S^2(t),$$

and therefore

$$\hat{\mu}_S(t) = \hat{R}_S(t) + \frac{1}{2}\hat{\sigma}_S^2(t).$$

13.3 Estimation of the ILB Parameters

Due to the very high demand of inflation-linked bonds by pension schemes and private insurers and comparatively low supply, the assumptions of no market frictions may not portray the market correctly. In [Schroders, 2018] it was estimated that private sector pension schemes hold over 80% of the total supply of inflation-linked bonds. As pension funds are not active traders of bonds, most of these bonds are therefore not traded regularly. Therefore, prices of the inflation-linked bond carry less information about the underlying value, as the key buyers are driven mainly by risk management, rather than by speculation.

An alternative to index-linked gilts might be found in overseas inflation-linked bonds, such as the Treasury Inflation-Protected Security (TIPS) in the US. However, the underlying inflation rates in the UK and US are correlated, but still differ. Together with the denomination of the TIPS in dollars, this means that TIPS are not a clean substitute to gilts. In Figure 13.7 we see the inflation in the UK since 1982, together with the annual returns of the FTSE Gilts All Share Index. To circumvent the high inflation in the 80's, which breaks the assumption of stationarity, we use the data between January 1992 and December 2005 to predict the inflation variables until December 2025. Not using the first ten years of data for the inflation-linked bond has the added benefit of excluding the period where the supply of such bonds was at its lowest.

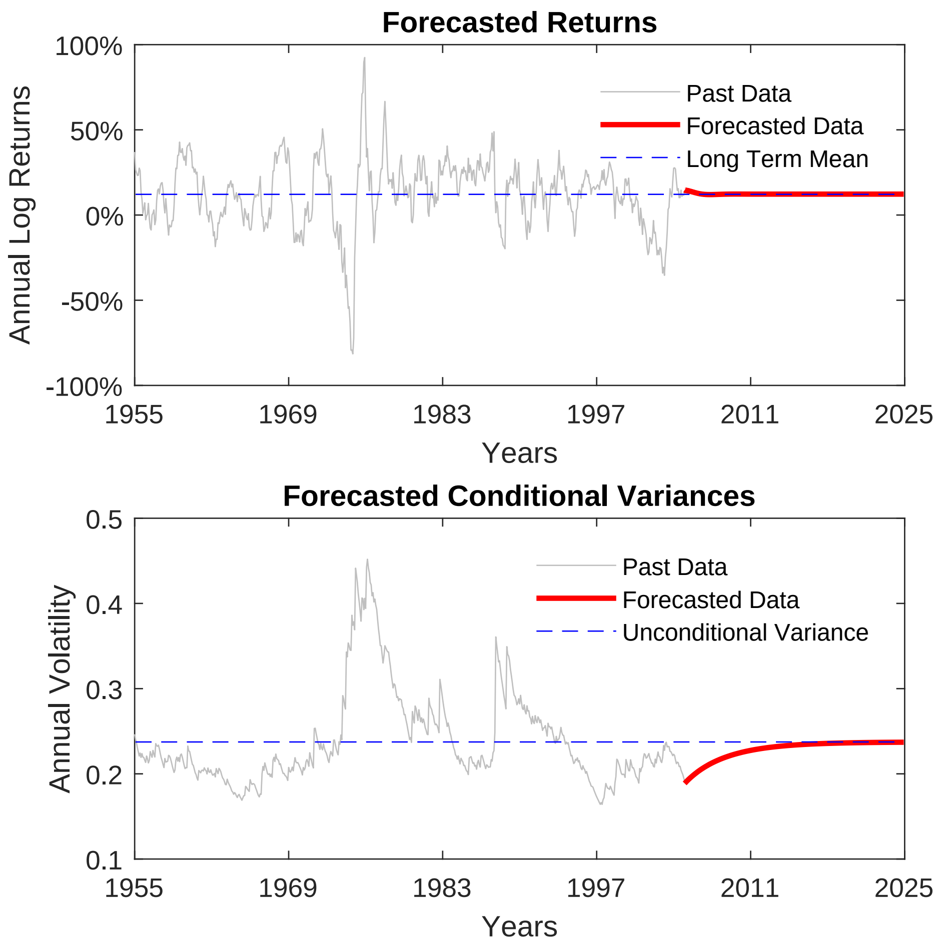


Figure 13.6: In the upper figure, the time series of the historical log-returns for $t \in \{01/1955, \ldots, 12/2004\}$ of the FTSE index can be seen. Moreover, the forecasted log returns for $t \in \{01/2005, \ldots, 12/2024\}$, calculated by an ARMA$(3, 2)$/GARCH$(1, 1)$ model, are shown. The long term average return rate has been plotted as a dashed, blue line. In the bottom figure, the time series for the conditional variance is shown in a similar fashion.

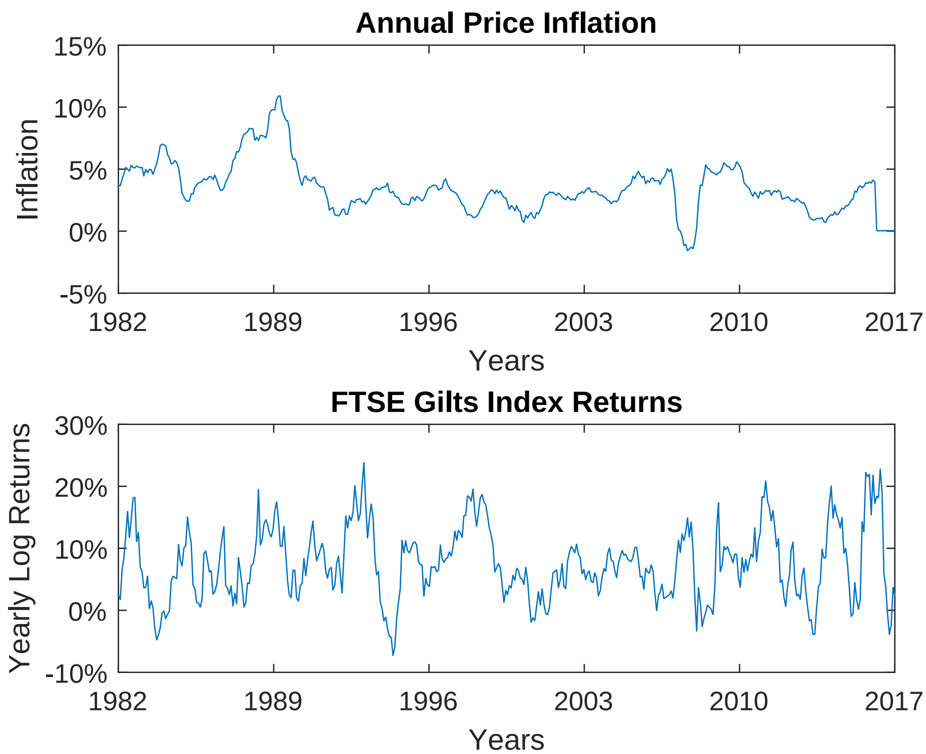


Figure 13.7: Annual price inflation measured by the retail price index is shown in the upper figure. The time series of the yearly log returns of the FTSE Gilts Index for $t \in \{05/1982, \ldots, 12/2017\}$ can be seen below.

In order to determine and predict the parameters of the inflation-linked bond, $\sigma_I(t)$ and $\theta_I(t)$, as well as the real interest rate $r_R(t)$, we need to forecast both the inflation index (5.1) and the inflation-linked bond (5.6).

13.3.1 Inflation-Linked Bond Estimation using the ARMA / GARCH Model

Similar to Section 13.2.1, the GARCH approach seems more viable than using some form of ARMA model, due to the heteroskedasticity of the data. By the same procedure as in Section 13.2.1, we find the best fit for this data period to be obtained by an ARMA(2,1)/GARCH(1,1) with normal innovations. Recall the stochastic differential equation of the inflation-linked bond

$$dB^*(t, I(t)) = B^*(t, I(t))\Big(\big(r_N(t) + \sigma_I(t)\theta_I(t)\big)dt + \sigma_I(t)dW_I(t)\Big). \quad (13.1)$$

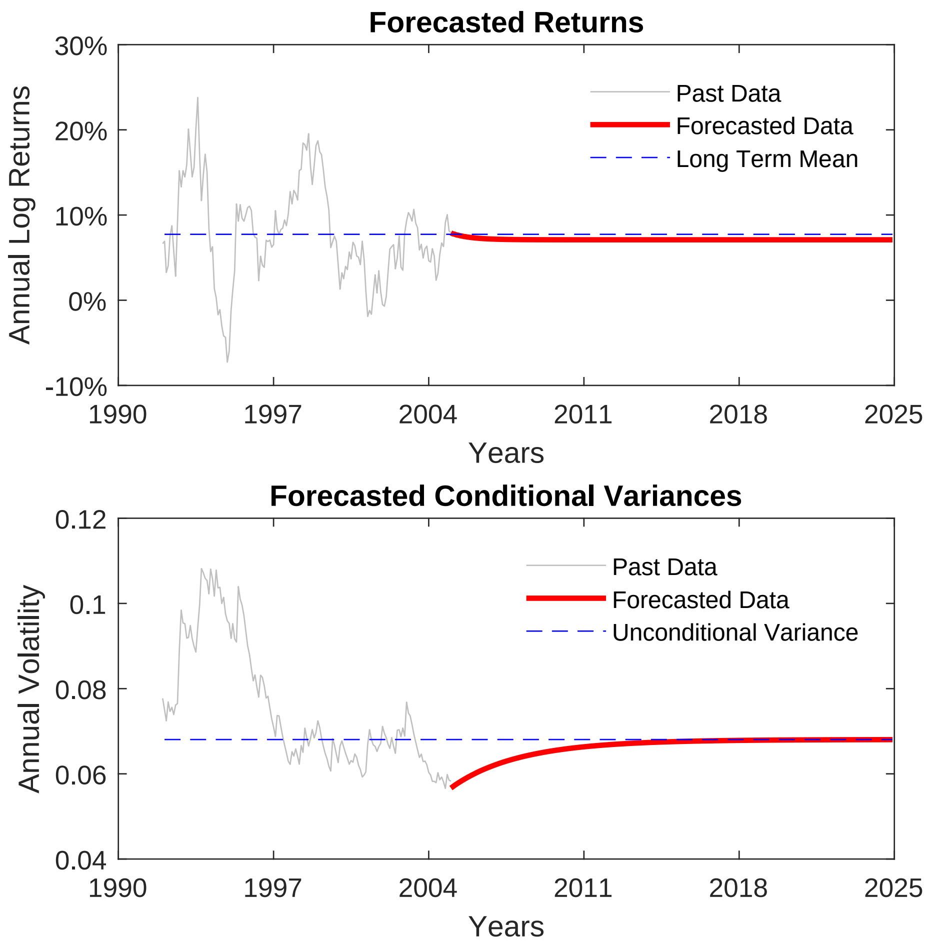


Figure 13.8: In the upper figure, the time series of the historical log-returns for $t \in \{01/1992, \ldots, 12/2004\}$ of the FTSE Gilts Index can be seen. Moreover, the forecasted log returns for $t \in \{01/2005, \ldots, 12/2024\}$, calculated by an ARMA$(2,1)$/GARCH$(1,1)$ model, are shown. The long term average return rate has been plotted as a dashed, blue line. In the bottom figure, the time series for the conditional variance is shown in a similar fashion.

The conditional variance of the fitted model provides an estimate for the volatility parameter $\sigma_I(t)$. In order to estimate the market price of inflation risk $\theta_I(t)$, we first estimate the drift term by

$$\hat{\mu}_{ILB}(t) = \hat{r}_N(t) + \hat{\sigma}_I(t)\hat{\theta}_I(t) = \hat{R}_I(t) + \frac{1}{2}\hat{\sigma}_I^2(T),$$

where $\hat{R}_I(t)$ denotes the forecasted return of the inflation-linked bond. Then

$$\hat{\theta}_I(t) = \frac{\hat{\mu}_{ILB}(t) - \hat{r}_N(t)}{\hat{\sigma}_I(t)},$$

where $\hat{r}_N(t)$ denotes the time series of nominal interest rates obtained by the method outlined in Section 13.1. Note that we have monthly estimates for the parameters of the inflation-linked bond, but only semi-annual estimates for the nominal interest rate. In order to obtain time series of the same lengths, we convert the time series of the interest rate to include monthly data, by assuming the parameters to stay constant in each semester.

13.3.2 Inflation Index Estimation using the ARMA Model

In order to obtain the real interest rate, the only remaining parameter to estimate is the trend of the inflation index. Note that in our market model, the volatility is the same for the inflation index (5.1) and the inflation-linked bond (5.6). This is not true in practice, which might be due to the supply and demand problem of the index-linked gilts, or due to the national bank's interest inflation-targeting approach, influencing the Retail Price Index directly. Note that as soon as the real interest rate is known, we could generate a theoretical inflation index by (5.2), with the same trend as the retail price index, but with the volatility of the index-linked gilts.

As the volatility of the inflation index has already been estimated in the previous section, we use an ARMA model to estimate and forecast the trend of the inflation index. In order to obtain a stationary time series, we use the square function as a non linear transformation. Using the BIC to measure goodness-of-fit, the AR(2) model produces the best fit.

13.3.3 Correlation Estimation between Stock and Inflation

In the market model 8.3, the volatility of the stock process is due to the Brownian motion of the inflation index, as well as another, independent Brownian motion. Therefore, the correlation between the stock process and the inflation process needs to be estimated in order to split the forecasted volatility of Section 13.2.1 into σ_S and σ_I. Next to a multitude of models to estimate and predict correlation between different stocks and market elements, we could also fit a multivariate ARMA/GARCH model to both the stock and the inflation simultaneously.

For the purpose of this thesis, it is sufficient to estimate the correlation as a constant, by the Pearson correlation coefficient, defined by

$$\rho_{S,I} = \frac{\sum_{k=1}^{n} \big(R_S(kh) - \hat{m}_S\big)\big(R_I(kh) - \hat{m}_I\big)}{\sqrt{\big(R_S(kh) - \hat{m}_S\big)^2 \sum_{k=1}^{n} \big(R_I(kh) - \hat{m}_I\big)^2}}, \tag{13.2}$$

where R_S and R_I denote the log-returns of the stock and the inflation-linked bond, respectively, and $\hat{m}_S$ and $\hat{m}_I$ denote the MLE of the respective sample average.

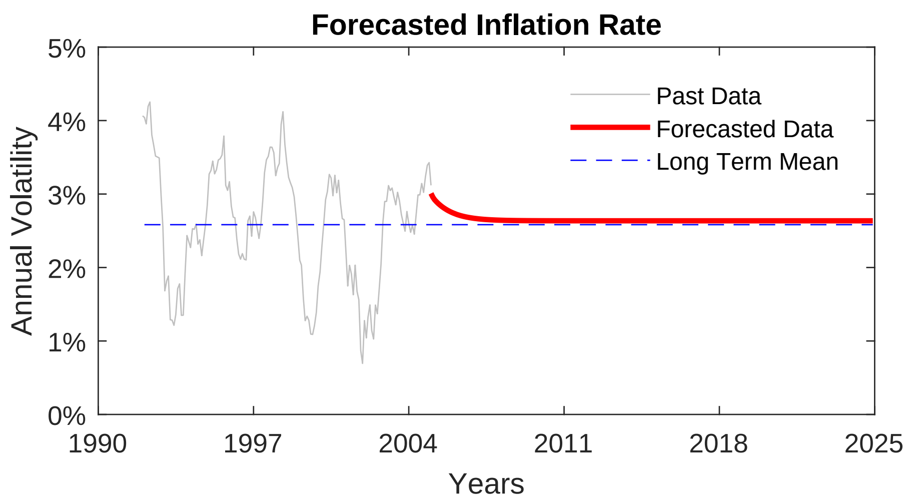


Figure 13.9: On a monthly grid, $\Delta = 1/12$, we see the time serie of the historical inflation rates for $t \in \{01/1992, \ldots, 12/2004\}$ and forecasted interest rates for $t \in \{01/2005, \ldots, 12/2024\}$, calculated by an AR(2) model. The long term average inflation rate has been plotted as a dashed, blue line.

We now have the tools to estimate the real interest rate $r_R(t)$ and the two volatility parameters $\sigma_S(t)$ and $\sigma_{IS}(t)$, for all $t \in [0, T]$. Recall the stochastic differential equation for the inflation index

$$dI(t) = I(t)\Big(\big(r_N(t) - r_R(t) + \sigma_I(t)\theta_I(t)\big)dt + \sigma_I(t)dW_I(t)\Big),$$

and denote the drift term by $\mu_I(t) = r_N(t) - r_R(t) + \sigma_I(t)\theta_I(t)$. Comparing this to (13.1), we obtain the estimate of the real interest rate by

$$\hat{r}_R(t) = \hat{\mu}_{ILB}(t) - \hat{\mu}_I(t).$$

The real interest rate is then used to normalize the inflation-linked bond, such that

$$B^*(0, I(0)) = e^{-\int_0^T r_R(t)dt}.$$

For the volatility parameters, we define

$$\hat{\sigma}_S(t) = \rho_{S,I}\hat{\sigma}(t),$$
$$\hat{\sigma}_{IS}(t) = \sqrt{1-\rho_{S,I}^2}\hat{\sigma}(t),$$

for all $t \in [0, T]$, where $\hat{\sigma}(t)$ is given by (13.5) and $\rho_{S,I}$ is given by (13.2).

13.4 Estimation of the Wage Parameters

Figure 13.10: Annual wage inflation measured by the UK inflation index for $t \in \{01/1930, \ldots, 12/2017\}$.

Four parameters need to be estimated for the stochastic contribution process. The initial wealth l and the contribution rate δ are constants, while the drift $\mu_L(t)$ and the volatility $\sigma_L(t)$ are deterministic functions. For the contribution rate, we use the average private sector, open DC rate of $\delta = 10\%$, published in [Hutton, 2011]. As the initial salary, we use $l =$

$20'000$, which corresponds closely to the average yearly earning in the UK by full-time employees under the age of 40, see [ONS, 2018]. For individual plan members, both constants can be chosen specifically for that person. Moreover, as the actual contribution rate is not constant, but depends on the underlying salary, including a salary-dependent contribution factor δ may be the focus of future work.

For the deterministic parameters, we follow the same procedure as in the previous section and use the data between January 1955 and December 2005 to predict the salary variables until December 2025.

13.4.1 Estimation using the ARMA / GARCH Model

In Figure 13.10 we see that the assumption of stationary volatility necessary to use the ARMA model is violated. Hence, utilizing the GARCH outlined in Section IV.C seems viable in order to forecast both the volatility parameter $\sigma_L(t)$, as well as the drift parameter $\mu_L(t)$. Due to the strong correlation between inflation and the wage index, we exclude data before 1990, as we did in Section 13.3. Using the BIC to measure goodness-of-fit, we observe that the best fit is obtained by the AR(1)/ARCH(1) model with normal innovations.

We see in Figure 13.11 that the the forecasted returns converge only very slowly to the long term mean. This is due to the small parameter for p_A in the ARMA structure of the model. The smaller this parameter, the quicker the correlation to previous data falls off, and hence the high returns at the start of the 1990's only gradually increase the forecasted parameters.

The estimate for $\kappa(t)$, the drift due to economic growth and increased welfare, is then

$$\hat{\kappa}(t) = \hat{\mu}_L(t) - \hat{\mu}_I(t).$$

13.4.2 Correlation Estimation to Stock and Inflation

Similar to Section 13.3.3, we estimate the correlation of the wage index to the stock and the inflation index by the Pearson correlation coefficient (13.2).

Note that for the current data set, we obtain $\rho_{I,L} = 0.879$, and $\rho_{S,L} = 0.009$. Therefore, there remains some randomness in the wage index, which cannot be explained by the inflation index, or the stock index. This is due to the weakness of the model for the stochastic wage process, on which we touched in the remark in Section 8.1. In order to circumvent this problem, we resize the correlations to eliminate the dependence on a third risk factor,

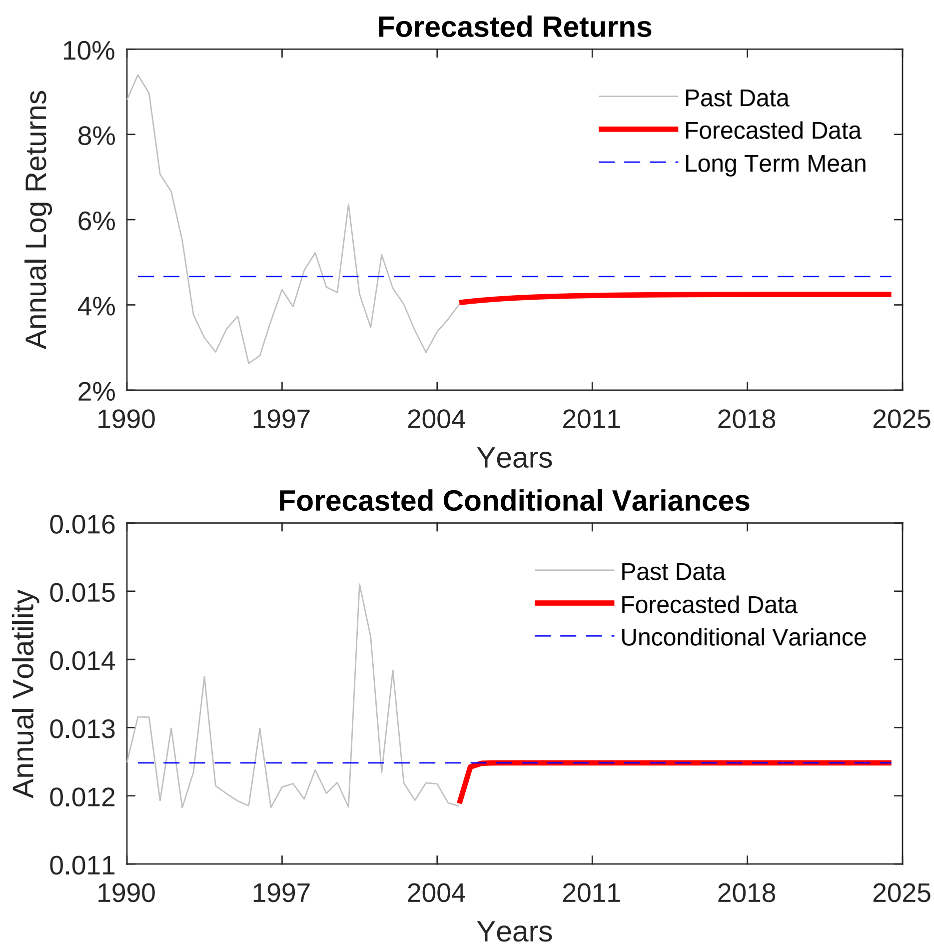


Figure 13.11: In the upper figure, the time serie of the historical log-returns for $t \in \{01/1990, \dots, 12/2004\}$ of the Wage Index can be seen. Moreover, the forecasted log returns for $t \in \{01/2005, \dots, 12/2024\}$, calculated by an AR(1)/ARCH(1) model, are shown. The long term average return rate has been plotted as a dashed, blue line. In the bottom figure, the time series for the conditional variance is shown in a similar fashion.

and obtain

$$\hat{\sigma}^2_{L,I}(t) = \frac{\rho_{I,L}}{\rho_{S,L}\sqrt{1-\rho^2_{I,L}} + \rho_{I,L}}\hat{\sigma}^2_L(t),$$

$$\hat{\sigma}^2_{L,S}(t) = \frac{\rho_{S,L}\sqrt{1-\rho^2_{I,L}}}{\rho_{S,L}\sqrt{1-\rho^2_{I,L}} + \rho_{I,L}}\hat{\sigma}^2_L(t).$$

13.4.3 Constant Parameters

Whenever we use constant, instead of deterministic parameters, we utilize the MLE defined in IV.A.2 as the estimator. There are two different model structures, and hence two different MLE estimators for the parameters.

In order to estimate the nominal interest rate and the inflation rate as a constant, we use the long term average. That is, if the constant nominal interest rate r_N is predicted from T years of data, we have

$$\hat{r}_N = \frac{1}{T}\sum_{t=1}^{T} r_N(t),$$

where $r_N(t)$ is the annual interest rate in period t.

For the parameters of a geometric Brownian motion $\big(S(t)\big)_{t\in[0,T]}$, we use the MLE defined in IV.A.2 for the parameters μ and σ. Let

$$R(kh) = \log(S(kh)) - \log(S((k-1)h)),$$

denote the deterministic log-returns, where the interval $[0,T]$ is partitioned into n equidistant sub intervals, for $h, 2h, \ldots, nh$. Define $m = \mu - \frac{1}{2}\sigma^2$ and $v^2 = \sigma^2 h$. Then $S(t) = \exp(m - vZ)$, for $Z \sim \mathcal{N}(0,1)$. Therefore, the log returns follow a log-normal distribution and the MLE for m and v are given by

$$\hat{m} = \frac{1}{n}\sum_{k=1}^{n} R(kh), \tag{13.3}$$

$$\hat{v}^2 = \frac{1}{n}\sum_{k=1}^{n}\big(R(kh) - \hat{m}\big)^2. \tag{13.4}$$

Inserting the definitions of m and v, the MLE for the drift and the volatility terms are given by

$$\hat{\sigma}^2 = \frac{1}{nh} \sum_{k=1}^{n} \big(R(kh) - \hat{m}\big)^2, \tag{13.5}$$

$$\hat{\mu} = \frac{1}{h}\hat{m} + \frac{1}{2}\hat{\sigma}^2. \tag{13.6}$$

13.5 Alternative Models

In the previous discussion we saw that autoregressive models do not produce satisfying results in predicting market parameters in the long term. For all parameters analyzed, the models lead to some form of mean-reverting time series, quickly approaching the long-term average. Various alternative models may be used to increase the quality of the parameter forecast.

An alternative to the autoregression models is the vector autoregression model (VAR), where instead of fitting a model to a single parameter, a multitude of parameters is estimated simultaneously. It extends Definition IV.B.2, by allowing $X(t)$ and $W(t)$ to be n-dimensional random variables. VAR models focus more on correlation than on structural estimates, which may lead to better performance if the value of a single parameter is of less importance than the interaction between the parameters.

Without including additional data to the model, Wilkie's model, see e.g. [Wilkie, 1984], may offer a viable alternative. The model links the realization of inflation with other variables using a cascade type approach. The inflation is modeled as an autoregressive process, driving the other economic variables, including interest rates, wages and stock returns.

Better performance may also be attained when the parameter values are modeled as stochastic process instead. Note that for those models, the portfolio processes developed in this thesis may not be optimal anymore, as the uncertainty of the parameters would need to be included in both the dynamic programming and the martingale approach. In [Xue and Basimanebotlhe, 2015] the real interest rate is modeled by an Ornstein-Uhlenbeck process, while the stock returns are assumed to be mean reverting in addition to stochastic interest rates in [Guan and Liang, 2014].

Finally, using a regime-switching model for the parameters showed potential in [Zhou and Yin, 2003]. An outlook on regime-switching models is given in Section 14.3.1.

Chapter 14

Constant vs. Deterministic Parameters

In order to evaluate the performance of constant and deterministic parameter processes, we use historical data to estimate the parameters as described in Chapter 13. Using a subsequent period of historical data, we calculate the wealth process for both the unconstrained, as well as the constrained optimal portfolio process.

We perform the analysis for two different sets of historical data. For the period of January 1985 to January 1995, we use data since 1955 to predict the parameters. As index-linked gilts were only introduced to the British market in 1982, there is not enough data to estimate the inflation parameters and similarly, due to high wage inflation in the 1970's, wage parameters are hard to predict. Therefore, we only predict the stock and the interest rate parameters to calculate the optimal portfolio processes of Part I.

For the period of January 2005 to January 2015, we use the predicted parameters of Sections 13.1 to 13.4 in order to calculate the optimal portfolio processes for the full model of Part III.

In addition to comparing the performance of constant and deterministic parameters, we also analyze a more practical approach, where the parameters are re-estimated annually. Note that this does not only break the Assumption 9.1.7 of deterministic coefficients, but also invalidates the martingale approach of Section 9.2.2. Therefore, the resulting portfolio process is not optimal and great care needs to be taken in its utilization.

14.1 Comparison on the Period 1985-1995

This period was chosen, as non of the historical indices analyzed experienced a major shock. In Table 14.1, we see the predicted parameters compared to those realized. Note that the we state the realized volatility for the whole period as the historical value for σ. We observe that the volatility is overestimated by the ARMA/GARCH model, but all estimated values are still close to reality. It is also apparent, that the range of annual log-returns is much larger in reality, than those estimated by a Black-Scholes model. This is due to the normality assumptions and the inability to model jumps and is a well-known disadvantage of the Black-Scholes model.

	MLE	**Deterministic**	**Historical**
$r(t)$	0.08	(0.08,0.10)	(0.05,0.15)
$\mu(t)$	0.22	(0.19,0.23)	(-0.15,0.50)
$\sigma(t)$	0.15	(0.15,0.24)	0.14

Table 14.1: Estimated parameters for the period 1985-1995. For the deterministic time series and the historical parameters, the range of values is stated.

In Figure 14.1 and in Table 14.2 the properties of the wealth process for an initial wealth $x = 1000$ and target wealth $C = 7500$ can be seen. We observe that for both the unconstrained portfolio process, as well as for the constrained portfolio process, the deterministically estimated parameters slightly outperform the constant parameters. Surprisingly, for both portfolio strategies, the strategy where the parameters are updated annually performs worst. Note that the wealth process is only lower than the other two after 1991, where the updated portfolio goes short in the stock, while the other portfolios do not. This may be an indication, that the rudimentary approach of updating the parameters annually may lead to non-stationarity of the time series, invalidating the models used to forecast the parameters.

We also note that the minimal wealth attained is lower for the non-constant parameters. As a very crude tool to estimate risk, this might indicate that the additional estimation uncertainty may increase the risk for the portfolio processes where the parameters are modeled as a time series.

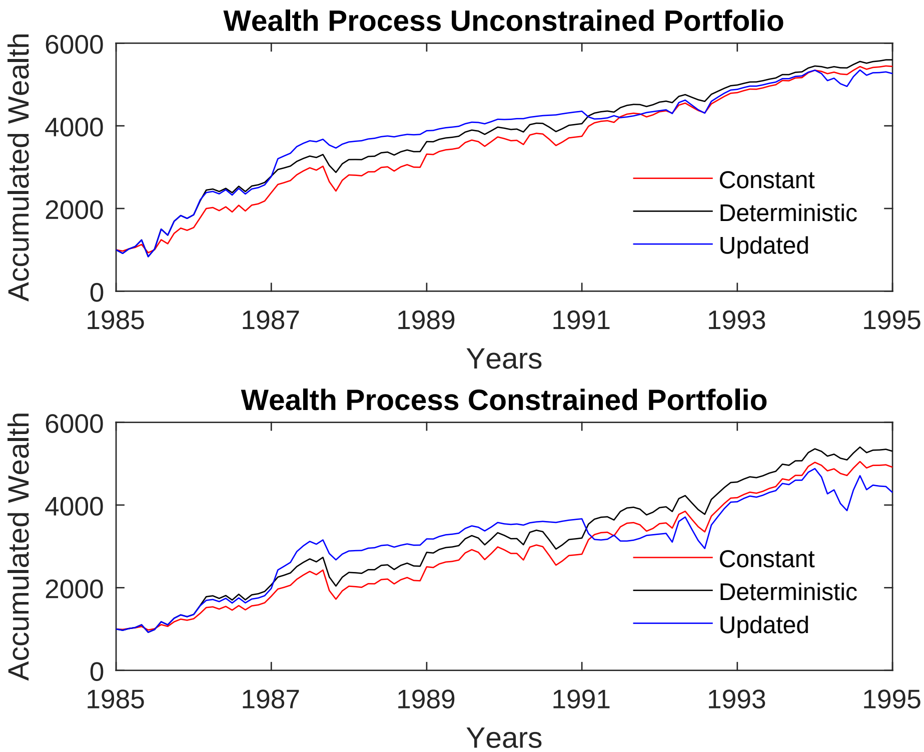


Figure 14.1: In the upper graph, the wealth process for the unconstrained optimal portfolio process (3.26) is featured for 1985-1995. The wealth process for the constrained optimal portfolio process (3.36) is seen below.

Unconstrained	**Constant**	**Deterministic**	**Updated**
Terminal Wealth	5’440	5’600	5’260
Rate of Return	16.9%	17.2%	16.6%
Minimal Wealth	930	835	835
Constrained			
Terminal Wealth	4’920	5’300	4’300
Rate of Return	15.9%	16.7%	14.6%
Minimal Wealth	977	921	921

Table 14.2: Properties of the portfolio processes calculated for the period 1985-1995.

14.2 Comparison on the Period 2005-2015

In contrast to the period 1985-1995, the stock markets experienced major upsets in the period 2005-2015. The financial crisis of 2007-2008 led to a fall of the FSTE All Share Index of over 30%. Moreover, low inflation expecations and quantitative easing programs led to a Bank Base Rate below 1% since March 2009. Therefore, we expect the parameter estimation error to be of a higher degree than in the previous section and in turn expect the differences in the properties of the portfolio processes to be higher as well.

In Table 14.1, we see the predicted parameters compared to those realized. Note that the we state the realized volatility for the whole period as the historical values for all volatility parameters. Neither the MLE, nor any of the autoregressive models are able to forecast the low nominal interest rates and in turn, the inflation parameters. In addition to the inability to model jumps for the stock returns, we also note that the correlation coefficients are overestimated for this specific period.

	MLE	**Deterministic**	**Historical**
$r_N(t)$	0.08	(0.05,0.08)	(0.005,0.06)
$\mu_S(t)$	0.14	(0.16,0.14)	(-0.40,0.50)
$\sigma_S(t)$	0.18	(0.17,0.21)	0.17
$r_R(t)$	0.05	(0.04,0.05)	(-0.02,0.1)
$\sigma_I(t)$	0.06	(0.06,0.07)	0.05
$\theta_I(t)$	-0.01	(-0.11,0.02)	(-0.2,0.8)
$\sigma_{IS}(t)$	0.09	(0.08,0.1)	0.03
$\kappa(t)$	0.02	(0.01,0.02)	(-0.004,0.02)
$\sigma_{LS}(t)$	0	0	0
$\sigma_{LI}(t)$	0.01	0.01	0

Table 14.3: Estimated parameters for the period 2005-2015. For the deterministic time series and the historical parameters, the range of values is stated.

In Figure 14.2 and in Table 14.4 the properties of the wealth process for an initial wealth $x = 1000$ and target real wealth $C = 50'000$ can be seen. We observe that the processes with updated parameters perform best. Moreover, the constrained portfolio process profits greatly from deterministic parameters, whereas in the case of the unconstrained portfolio process, the performance is slightly worse for deterministic parameters than for con-

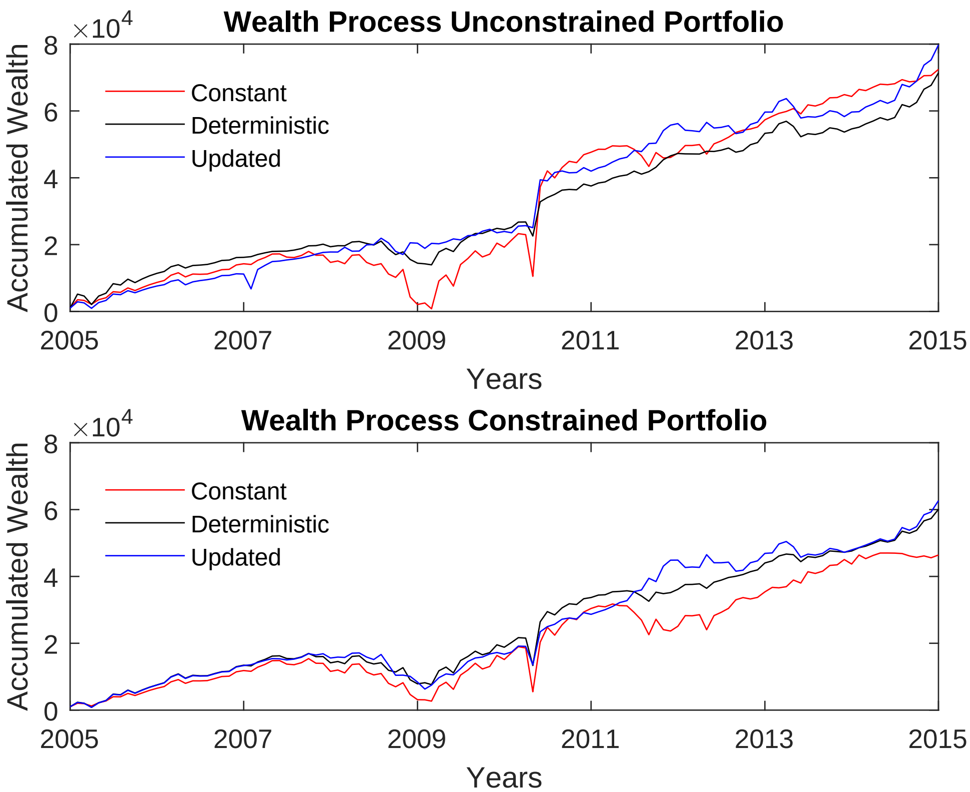


Figure 14.2: In the upper graph, the wealth process for the unconstrained optimal portfolio process (9.22) is featured for 2005-2015. The wealth process for the constrained optimal portfolio process (9.29) is seen below.

stant parameters. Once more, this only becomes apparent after the first half of the investment period, once the unconstrained optimal portfolio with constant parameters recovers.

Note that the time series of the estimated parameters varies most from the MLE during the first 5 years. Therefore, differences during that period can mainly be explained by the underlying parameter values. Since the current wealth then varies between the portfolio processes, the investment behavior continues to be different for the remainder of the investment period, which explains the growing dissimilarity between the portfolio processes.

Unconstrained	**Constant**	**Deterministic**	**Updated**
Terminal Wealth	72'400	71'400	80'000
Rate of Return	18.5%	18.3%	20.0%
Minimal Wealth	797	1000	942
Constrained			
Terminal Wealth	46'400	60'100	62'700
Rate of Return	11.5%	15.7%	16.3%
Minimal Wealth	1000	845	823

Table 14.4: Properties of the portfolio processes calculated for the period 2005-2015.

14.3 Comparison on Generated Data

During the discussion of the previous two sections, it was not clear if deterministic parameters lead to a better performance than constant parameters. Due to model uncertainty, the time series resulting from deterministic parameter estimation may be further from the realized values than the MLE, distorting the comparison. Moreover, two periods of historical data are hardly enough to evaluate the model performance in a meaningful way.

However, if model uncertainty is excluded and parameter values are known exactly, using the realized parameters clearly leads to a better outcome. This may not be applicable in practice, as the development of neither stock processes nor interest rates is ever predictable. Nonetheless, it is important to understand, that the better we can forecast the parameters, the better the portfolio processes perform.

In order to see this, we generate the stochastic processes using deterministic and predictable time series for the parameters. We assume that there are two states in the market, in which the parameters are constant. For example, the market may be roughly divided in bullish and bearish periods, for which the underlying parameters will be very different. For the moment, we maintain that the change from one state to the other one is deterministic and fully predictable.

Example 14.3.1. We return to the market model 2.2 and assume that only the drift and volatility parameters change between the two market states, while the interest rate remains constant. In Figure 14.3 we see an example of possible values for a market which starts bullish and becomes bearish in

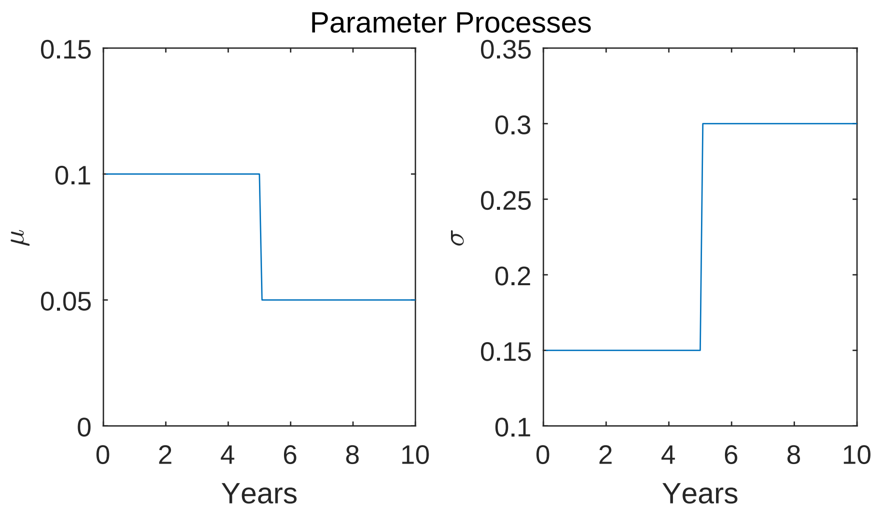


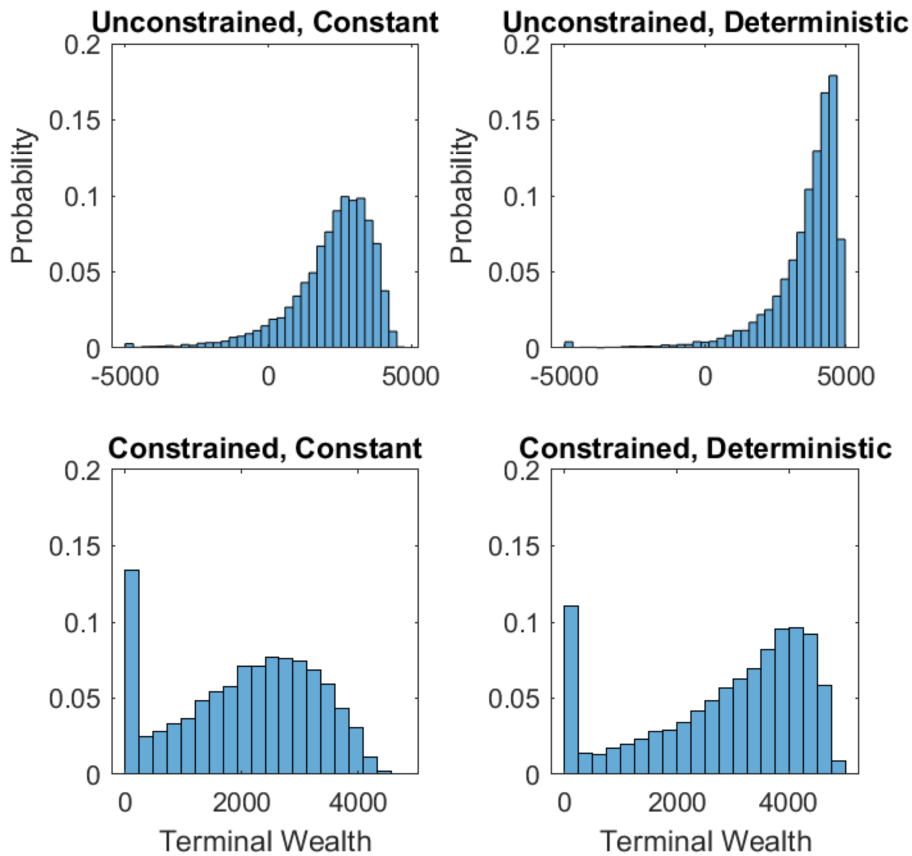
Figure 14.3: An example of $\mu(t)$ and $\sigma(t)$ taking different values in two market states.

the second half of the investment period.

Assuming that the parameters are predictable for the whole investment period, we compare the performance of the portfolio processes using the exact parameters, i.e. a time series, to those only using the average parameter value. We calculate the empirical terminal wealth distribution for both the unconstrained and the constrained optimal portfolio using constant interest rate $r = 4\%$, initial wealth $x = 1000$ and a target of $C = 5000$. In Figure 14.5 we observe that using the exact parameters clearly leads to a better performance, increasing the median rate of return by close to 3% for both portfolio processes.

The situation of Example 14.3.1 is very far from those faced in practice. Not only does the investor somehow know the exact values of the parameters for the stock price process in both market states, but is also able to predict at what time the switch happens. The first point may be remedied quickly, as market parameters may be estimated using similar procedures as in Chapter 13. Certain parameters of the models will differ between the two market states, leading to different empirical estimates.

In order to resolve the second problem, one may introduce a probabilistic model to determine the time of the switch. The market mode switches according to some underlying Markov chain, independent of the Brownian motion of the stock price process. Note that this introduces an additional



Median Rate Of Return	Constant	Deterministic
Unconstrained	10.3%	13.9%
Constrained	7.9%	11.8%

Table 14.5: Histograms of the terminal wealth distribution for the unconstrained and the constrained optimal portfolio for constant and for deterministic parameter processes. The resulting Median Rate Of Return can be seen below.

source of risk, rendering the market incomplete. Furthermore, the portfolio processes developed in this thesis are no longer optimal as the assumptions of deterministic processes are violated.

In order to resolve the second problem, one may introduce a probabilistic model to determine the time of the switch. The market mode switches according to some underlying Markov chain, independent of the Brownian motion of the stock price process. Note that this introduces an additional source of risk, rendering the market incomplete. Furthermore, the portfolio processes developed in this thesis are no longer optimal as the assumptions of deterministic processes are violated.

14.3.1 Outlook: Regime-Switching Models

The process outlined at the end of the last section is known as "Regime-Switching" and has been used to model abrupt changes in economic time series due to financial crises, or dramatic changes in fiscal and monetary policy. It is a promising extension to the problems discussed during this thesis and we summarize the approach taken in [Zhou and Yin, 2003].

In addition to the setup in the market model (8.3), we choose a continuous-time Markov chain $\alpha(t)$, taking values in a finite state space $\mathcal{S} = \{1, 2, \dots, l\}$, where the states of the chain α represent different hidden states of the underlying economy. Instead of only depending on time, some of the parameters also depend on the current state of the market. The wealth process (8.5) then reads

$$
\begin{aligned}
dX^\pi(t) =& \Big(r_N(t,\alpha(t))X^\pi(t) + \pi(t)'\sigma(t,\alpha(t))\theta(t,\alpha(t)) + \delta L(t) \Big) dt \\
&+ \pi(t)'\sigma(t) dW(t,\alpha(t)), \\
X^\pi(0) =& x, \quad \alpha(0) = i_0.
\end{aligned}
$$

Both the feasibility discussed in Section 3.1.3 and the solution for Problem 3.1.2 need to be analyzed and proved once more. To the best of our knowledge, the optimal portfolio process for a quadratic utility function in a regime-switching environment has not yet been studies in detail. [Zhou and Yin, 2003] solves the mean-variance problem without inflation or contributions, while [Korn et al., 2011] solve the problem for a CRRA utility function.

The regime-switching approach is a generalization of the methodology used in Parts I to III and is a promising option for future research.

14.4 On the Choice of Parameter Processes

The two historical examples of Sections 14.1 and 14.2 are not enough to discard parameters estimated as constants. Nevertheless, the wealth processes are very sensitive to changes in the underlying parameters and estimating future parameters as time series seem to improve the performance of the strategy. Moreover, we see in the case of a heavily changing economy, that parameters which are updated lead to a better result.

The problem of finding a closed form solution for the portfolio strategy in the case of non-deterministic, progressively measurable parameter processes has yet to be solved. The results in the current section indicate, that such a portfolio process might greatly outperform those analyzed here. As we only work on generated stock processes for the remainder of this thesis, and do not use the historical data to further extent, we will continue with constant parameter estimates and note the importance of more general parameters for future work.

Chapter 15

Constrained vs. Unconstrained Porfolio Process

We have seen in Examples 3.3.3, 6.2.6 and 9.1.10 that the no-ruin option of the constrained portfolio process has its price and results in lower probability of success and in lower expected terminal wealth. The additional guarantee of some non-negative minimal terminal wealth of the optimal portfolio processes with a lower bound increases this price further.

In this chapter we introduce an additional performance methodology with which we will compare the performance of the unconstrained and the constrained portfolio process for different parameter values. In addition to the ruin probability and the success probability, this will give us enough tools to compare which process outperforms the other under different market situations. We do not include the portfolio process with a lower bound in this discussion, as the choice of the guarantee heavily depends on the plan member and its value is hard to quantify.

We also compare the performance of the unconstrained portfolio process with the strategies resulting from cut-shares introduced in Section 11.2. Similar to the updated parameters in the last section, this does not result in an optimal portfolio process, but has the advantage of satisfying both the non-negativity constraint for the whole path of the wealth process, as well as the no-shorting constraint.

15.1 Performance Methodology

In the examples so far, the choice for the target wealth C has been very arbitrary and was simply used as an upper bound to the wealth process. However, the target wealth plays an important role for both the expected terminal wealth, as well as the amount of risk taken by the portfolio processes.

We use techniques developed for mean-variance optimizing portfolios, which face a similar problem in choosing a parameter for the risk aversion of the plan member. It has been shown in [Zhou and Li, 2000] and in [Menoncin and Vigna, 2017] that there is a one-to-one correspondence between optimal portfolios of target-based optimization problems and those of mean-variance optimizing problems. A mean-variance optimal portfolio, is a portfolio $\pi \in \mathcal{A}(x)$ that minimizes

$$\alpha \mathbb{V}[X^\pi(T)] - \mathbb{E}[X^\pi(T)],$$

where $\alpha > 0$ is a measure of risk aversion. Portfolio processes that solve mean-variance optimization problems for given parameters α result in a so-called *efficiency frontier*, i.e. the set $\big(\mathbb{V}[X^\pi(T)], \mathbb{E}[X^\pi(T)]\big)$, where a higher expected terminal wealth is paid for by having higher variance.

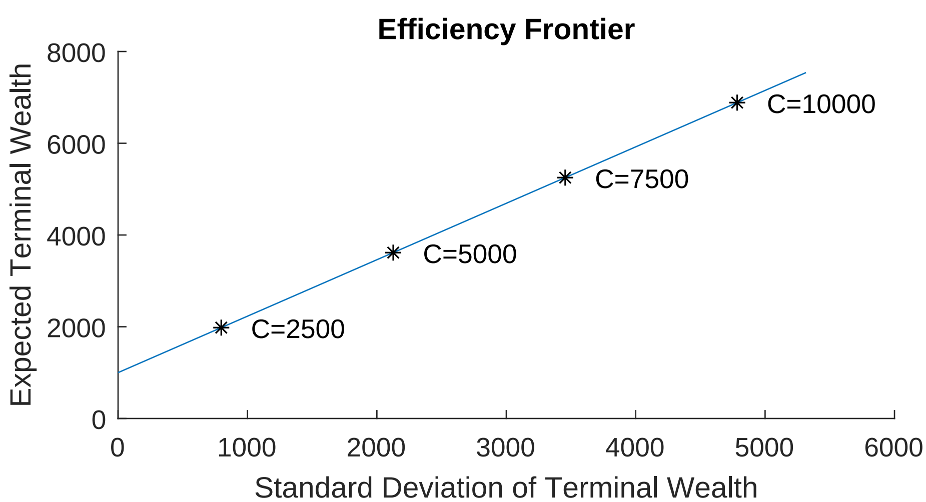


Figure 15.1: The expected terminal wealth against the variance of the terminal wealth of the unconstrained portfolio process (3.26) for different target values C.

Due to the one-to-one correspondence to target-based portfolios, we can reconstruct this efficiency frontier by varying the target wealth C. In Figure 15.1 we see such an efficiency frontier for the market model of Part I, with parameters $r = 0$, $\mu = 5\%$, $\sigma = 0.15$ and initial wealth $x = 1000$, for various values for the target wealth C. It is apparent that as C increases, the expected terminal wealth also becomes larger. This in turn is paid for by higher variance, and in turn higher downside risk.

We adapt the methodology of [Guillén et al., 2013] to compare the quality of investment strategies, which do not necessarily have the same underlying risk. We measure quality as the ability to maximize terminal wealth given the amount of risk inherent to the strategy. In addition to the median rate of return, we also use the ruin and success probabilities as indicators of the performance.

This performance evaluation will always be connected to a *benchmark strategy*. We use the strategy that invests a constant proportion of the wealth in the risky assets, at all times $t \in [0, T]$, and define the benchmark strategy by a certain fixed proportion π^b of risky investment. We then find the target wealth C, such that the target-based optimal portfolios are *equivalent* to said benchmark strategy. Here, two strategies are called equivalent if they share the same downside risk.

15.1.1 The Expected Shortfall

In this section we give an introduction to the expected shortfall as an approach to measuring downside risk exposure. For more details on risk measurement, we refer to [McNeil et al., 2005].

Definition 15.1.1. For the terminal wealth $X^\pi(T)$, the value-at-risk at the confidence level $\alpha \in (0, 1)$, henceforth VaR_α, is the α-quantile of the underlying distribution, i.e.

$$\mathrm{VaR}_\alpha(X^\pi(T)) = \sup \Big\{ l \in \mathbb{R} \mid \mathbb{P}\big[X^\pi(T) \le l\big] \le 1 - \alpha \Big\}.$$

Typical values for α are $\alpha = 0.95$ or $\alpha = 0.99$. Note that the main weakness of VaR as a risk-quantifying tool is the lack of subadditivity. This means that the sum of the VaR of two individual portfolios is not necessarily larger than the VaR of the combined portfolio. As some diversification benefit is usually observed in reality, a reasonable risk measure should satisfy the property of subadditivity.

In order to transform the VaR risk measure into a coherent risk measure, we utilize the expected shortfall.

Definition 15.1.2. For terminal wealth $X^\pi(T)$ with $\mathbb{E}\big[\ \mid X^\pi(T) \mid\ \big] < \infty$, the expected shortfall at confidence level $\alpha \in (0,1)$, henceforth ES_α is defined as

$$\mathrm{ES}_\alpha(X^\pi(T)) = \mathbb{E}\big[X^\pi(T) \mid X^\pi(T) < \mathrm{VaR}_\alpha(X^\pi(T))\big].$$

Note that whereas the VaR_α gives the $(1-\alpha)$-event on the left tail, the ES_α gives the expected value of an event which is below the $(1-\alpha)$-event. Therefore, for the same confidence level α, ES_α is always smaller or equal than VaR_α.

15.1.2 The Annual Financial Gain

We set the confidence level for the ES to $\alpha = 0.95$ and calculate the terminal wealth distribution of the benchmark strategy for different values of π^b for the proportion invested in the stock. For each π^b, we calculate the target wealth C such that the target-based optimal portfolios have the same downside risk. Note that different values for C are used for the constrained and the unconstrained portfolio processes.

We utilize three tools to evaluate the performance of the different strategies.

- The ruin probability, defined in Definition 3.3.2. This is only non-zero for the unconstrained portfolio strategy and measures the probability of negative terminal wealth.
- The success probability, defined in Definition 9.1.9. This measures the probability of achieving a net benefit, i.e. of obtaining terminal wealth higher than the sum of the initial investment and the contributions.
- The internal rate of return, i.e. the interest rate r_{irr} such that

  $$\int_0^T \frac{\delta L(t)}{(1+r_{\mathrm{irr}})^t} dt + x = \frac{X_m^\pi(T)}{(1+r_{\mathrm{irr}})^T},$$

  where $X_m^\pi(T)$ denotes the median of the terminal wealth distribution.

Definition 15.1.3. The difference between the median internal rate of return r_{irr}^P of the portfolio process and the median internal rate of return r_{irr}^b of the benchmark strategy is called the *annual financial gain/loss.*

We calculate $\mathrm{ES}_{0.95}$ and internal rate of return numerically for all strategies.

15.2 Comparison in the Market Model of Part I

In order to compare the different portfolio strategies, we first calibrate the equivalent target wealth for each process. In Figure 15.2 we see the equivalent target wealth for different values π^b of the benchmark strategy for one specific set of parameters. In order to increase readability, the interest rate is set to $r = 0$, the initial wealth to $x = 1000$ and the investment horizon to $T = 10$ years. For this example, $\mu = 5\%$ and $\sigma = 0.25$.

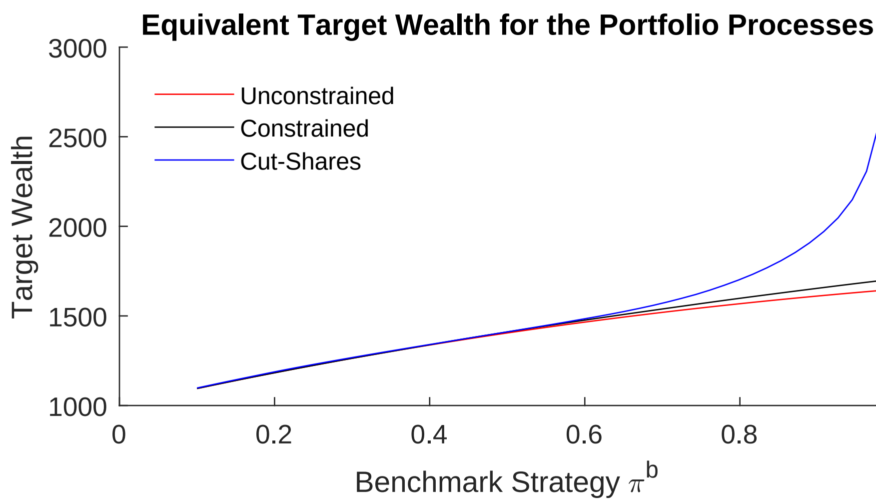


Figure 15.2: The equivalent target wealth of the portfolio processes for different benchmark strategies.

We observe that the higher the investment of the benchmark strategy, the higher the equivalent target wealth for the portfolio processes. This can be explained by looking at the structure of the portfolio process (3.36), for example, where we see that a higher target wealth leads to a higher investment in the risky asset, which in turn affects the expected shortfall.

In Figure 15.2 we also note that for higher values of π^b, i.e. for strategies which allow for a larger amount of risk, the equivalent target wealth of the strategies start to differ substantially. The unconstrained optimal portfolio process carries the most risk, which necessitates a lower target wealth than for the other two strategies. Similarly, the strategy resulting from cut-shares is less risky than even the constrained optimal portfolio process, by including a no-shorting constraint, and hence a higher target wealth can be chosen.

Note that the equivalent target wealth of the cut-shares strategy for $\pi^b = 1$ is undefined, as the only strategy which results in the same expected shortfall and satisfies the no-shorting constraint is exactly the strategy of investing everything in the asset for the whole duration. However, this strategy is then independent of the target wealth. Similarly, we do not include the equivalent strategies for $\pi^b = 0$, as all strategies would then invest everything in the bank account, resulting in the same strategy.

$\boldsymbol{\pi^b = 0.2}$	**Unconstrained**	**Constrained**	**Cut-Shares**
Equivalent Wealth	1'180	1'180	1'190
Annual Gain	-0.09%	-0.09%	-0.06%
Success Probability	82.9%	82.8%	82.8%
$\boldsymbol{\pi^b = 0.5}$			
Equivalent Wealth	1'400	1'410	1'410
Annual Gain	-0.05%	-0.05%	0.00%
Success Probability	82.9%	82.4%	82.4%
$\boldsymbol{\pi^b = 0.8}$			
Equivalent Wealth	1'570	1'600	1'700
Annual Gain	0.26%	0.28%	0.71%
Success Probability	82.9%	81.7%	80.1%

Table 15.1: The equivalent target wealth and the corresponding performance attributes for different values of π^b.

In Table 15.1 some performance parameters are seen for three specific values of the benchmark strategy. We note that the success probability of the unconstrained optimal strategy is independent of the target wealth. This can also be proved empirically by inserting the optimal terminal wealth (3.15) into the success probability (3.18). Note that for this specific set of parameters, both the constrained strategy, as well as the strategy resulting from cut-shares outperform the unconstrained optimal strategy. This shows that even though the unconstrained optimal strategy has higher annual gain for the same target wealth, this comes at the price of a much higher underlying risk.

Measuring the performance by the success probability instead, the unconstrained portfolio process still outperforms the other two. This measure does not give any indication as to how much the risk-free strategy is outperformed and how high the underlying risk is. Including the ruin probability, which lies between 0.1% and 0.8% for the unconstrained portfolio strategy, we see once again, that the spread of the terminal distribution is much higher

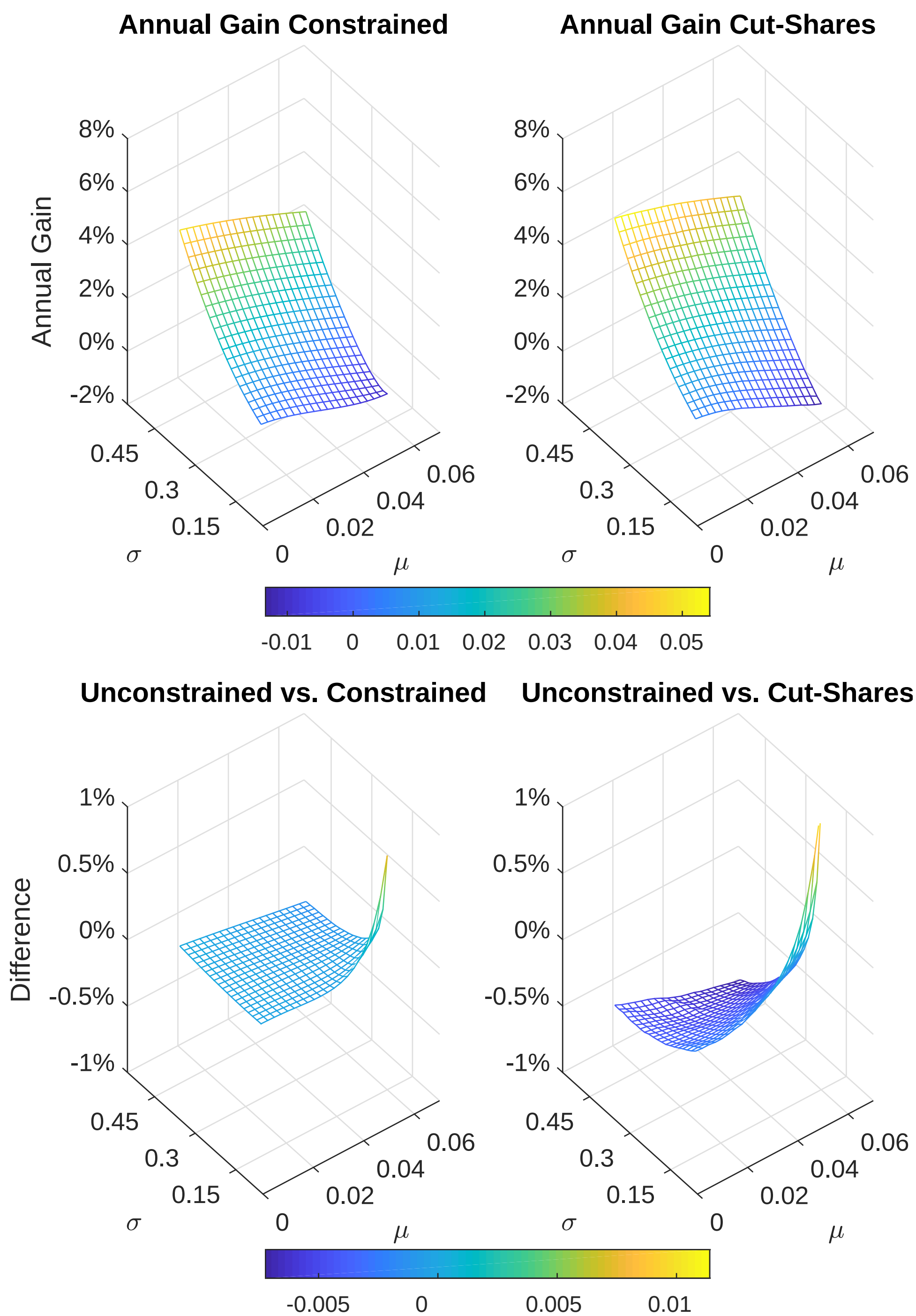


Figure 15.3: Annual gains are shown above, while the difference in performance can be seen below.

for this strategy than for the other two.

In order to measure the sensitivity on the drift and the volatility of the stock, we plot the performance for different parameters in Figure 15.3. With regard to Figure 15.2 we choose $\pi^b = 0.75$ in order to obtain significantly different equivalent target wealth for the different portfolio processes.

We observe that the annual gain is increasing in the volatility parameter, while decreasing in the drift parameter. For low volatilities, the benchmark strategy of investing 75% in the stock outperforms all other strategies by up to 1% annually. In the two bottom graphs in Figure 15.3 we compare the performance of the different portfolio strategies. We note that for low volatilities and high values for the drift parameters, the unconstrained portfolio process allows to enter a strong short position in the bank account, in order to invest heavily in the stock. Due to the low volatility, this only increases the risk slightly, while increasing the upside potential. The constrained portfolio process does not borrow as much money, in order to guarantee the non-negativity of the terminal wealth, while the strategy resulting from the cut-shares does not borrow at all, by definition. This explains why the unconstrained portfolio process outperforms the other two in those scenarios, while generally being outperformed everywhere else.

For this choice of parameters, the strategy resulting from cut-shares generally results in the best performance. We will see in the next section, if this stays true when we include inflation and contributions to the market. We already noted that in the case of contributions, the optimal strategies borrow against their future contributions from the beginning, which result in much larger short positions than in the market model of Part I.

15.3 Comparison in the Market Model of Part III

Similar to the last section, we start by calibrating the equivalent target wealth for different values π^b of the benchmark strategy. In order to obtain an equivalent terminal wealth for all the strategies, it is necessary to split the risky investment π^b between the stock and the inflation-linked bond. For simplicity, we split it equally.

We slighly alter the parameters of Example 9.1.10, i.e. we set the market parameters as $r_N = 4\%, \mu = 8\%$ and $\sigma_S = 0.2$. The investor starts with an initial wealth $x = 1000$ and invests for a time horizon of $T = 10$ years. In addition to the initial investment x, an additional $\delta = 10\%$ of the stochastic salary of the plan member is invested in a continuous manner. The parameters of the stochastic salary are given by $l = 20'000$, $\kappa = 0.015$, $\sigma_{LS} = 0.004$ and $\sigma_{LI} = 0.006$. Furthermore, the inflation parameters are given by $r_R = 3\%$, $\sigma_I = 0.08$, $\theta_I = 0.12$ and the volatility of the stock with respect to the inflation is given by $\sigma_{IS} = 0.08$.

$\boldsymbol{\pi^b = 0.2}$	**Unconstrained**	**Constrained**	**Cut-Shares**
Equivalent Wealth	39’200	39’200	63’800
Annual Gain	2.44%	2.43%	1.83%
Success Probability	95.5%	95.4%	91.9%
$\boldsymbol{\pi^b = 0.5}$			
Equivalent Wealth	39’400	39’500	67’700
Annual Gain	1.89%	1.88%	1.31%
Success Probability	95.5%	95.3%	91.5%
$\boldsymbol{\pi^b = 0.8}$			
Equivalent Wealth	40’500	40’600	88’300
Annual Gain	1.52%	1.51%	1.12%
Success Probability	94.9%	94.8%	89.4%

Table 15.2: The equivalent target wealth and the corresponding performance attributes for different values of π^b.

We see in Table 15.2 that the difference of equivalent target wealth between the unconstrained and the constrained portfolio process is much smaller than in the market model without contributions. This is no surprise, as due to the continuous contributions to the wealth process, the ruin probability of the unconstrained portfolio process is with maximally 0.05%

much smaller. Note that the equivalent target wealth for this set of parameters is slightly higher for the unconstrained strategy than for the constrained strategy. The order of difference is so small however, that this effect is more likely due to numerical inaccuracy than due to a significant difference in the underlying risk.

On the other hand, due to not borrowing against future contributions, the strategy resulting from cut-shares carries less risk. Thereby, the equivalent target wealth for the strategy resulting from cut-shares is much larger than for the other two strategies.

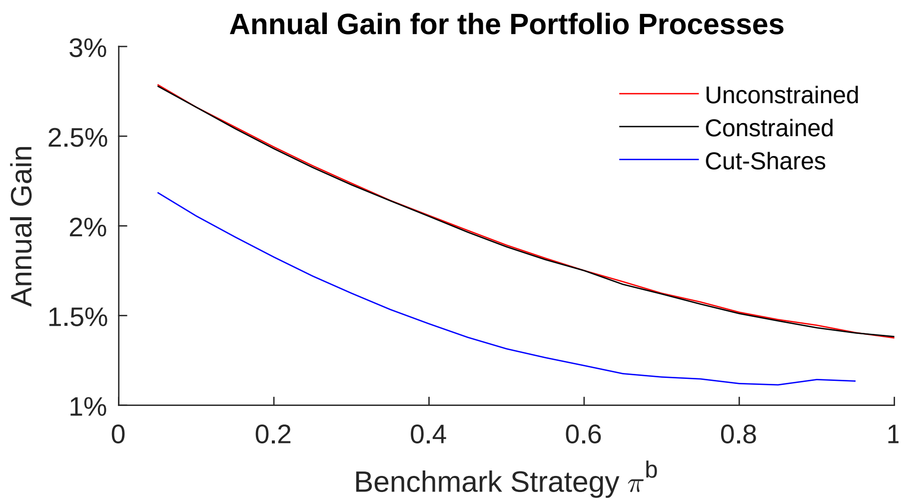


Figure 15.4: Annual gains for the different portfolio strategies for different values of π^b.

We see in Figure 15.4 that unlike in the previous section, the higher target wealth of the strategy resulting from cut-shares does not outweigh the inability to enter short position. For all values of π^b, the annual gain is lower than for the other two strategies. We see in Figures 15.5 and 15.6 that this is in part due to the specific choice of parameters. While the unconstrained and the constrained strategy lead to very similar annual gains for most parameters, the strategy resulting from cut-share may perform better for certain values of the inflation and salary parameters.

Note that the performance of the strategy resulting from cut shares drops of for low values of θ_I and high values of σ_I. For those parameters, the optimal strategy goes short in the inflation-linked bond, which is not possible under cut shares.

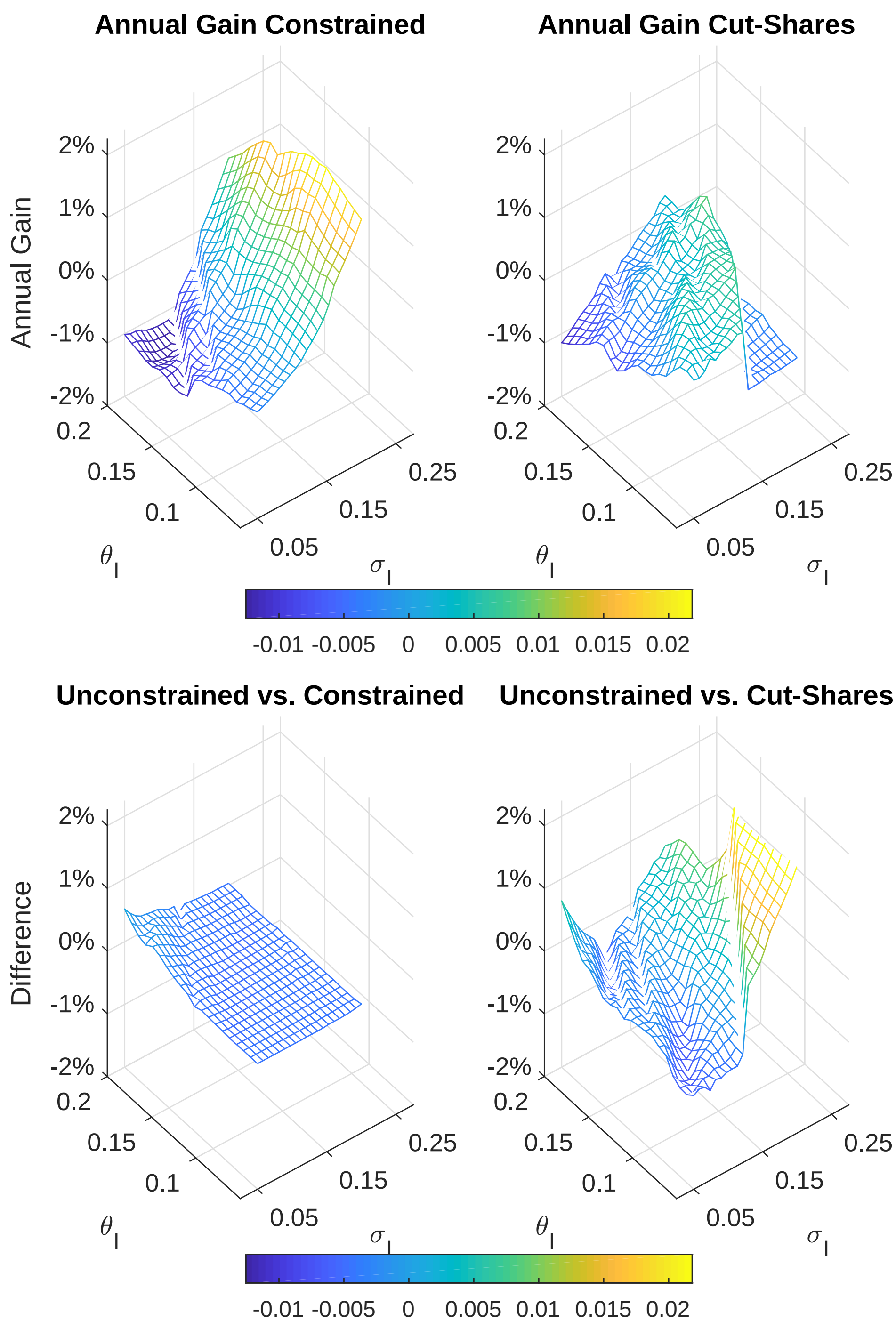


Figure 15.5: Annual gains and difference in performance for different parameters for the inflation volatility and market price of risk.

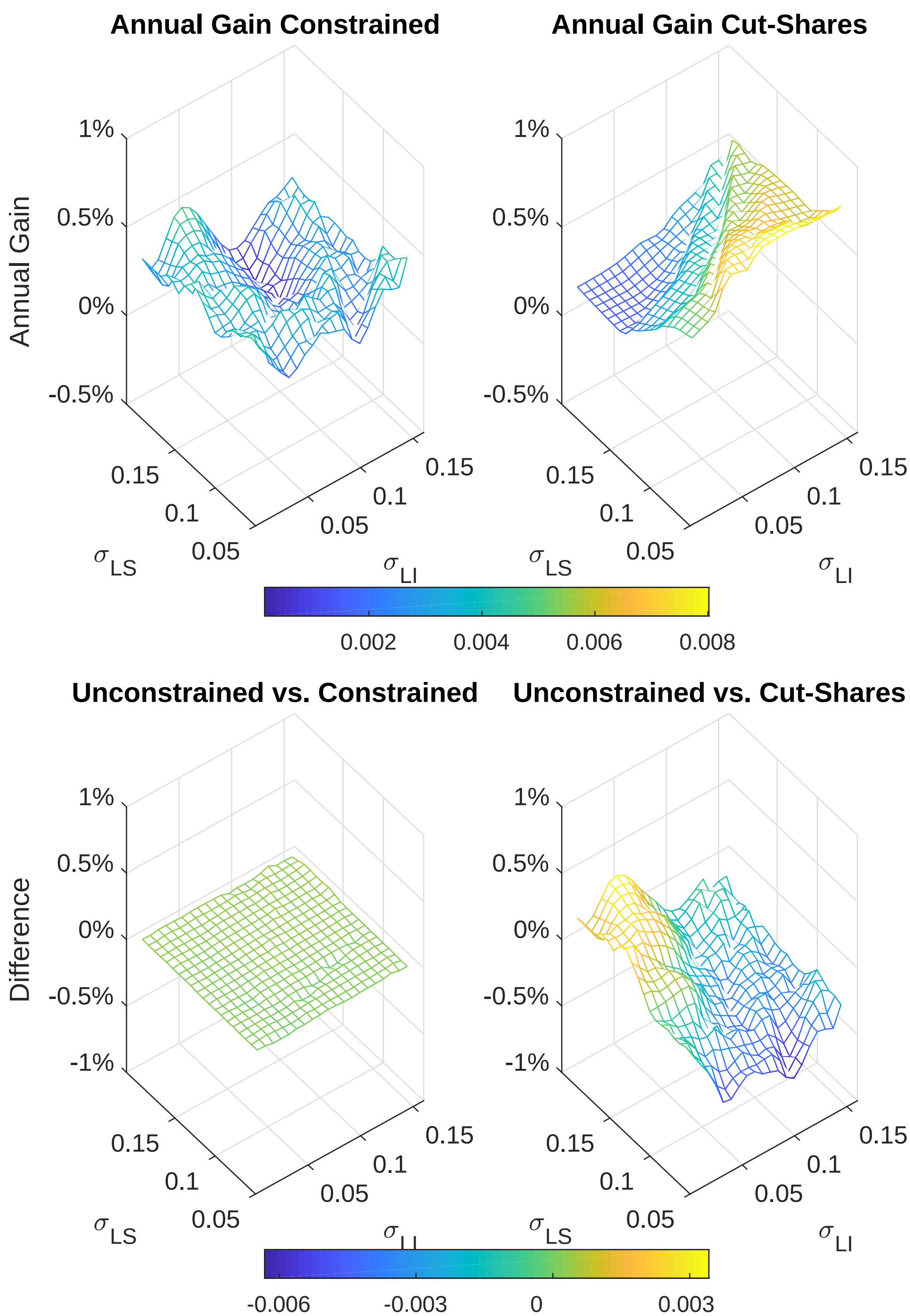


Figure 15.6: Annual gains and difference in performance for different parameters for the volatility of the salary process.

Chapter 16

On the Importance of Hedging Inflation Risk

In Part II an inflation-linked bond was added as an additional market element and the optimization problem was expanded to include inflation risk. During Examples 6.2.6, 6.3.11 and 7.3.2 we found that for certain parameter values, the target-based optimal portfolio process lead to higher terminal wealth. Utilizing the performance methodology of Section 15.1 we are able to evaluate the impact of the inflation-linked bond more thoroughly.

16.1 Comparing the Terminal Wealth Distribution

We compare the unconstrained optimal portfolio process (9.22) of market model 8.3 to the optimal portfolio process (11.9), where inflation is not considered. The inflation-linked bond is used as another risky asset, but the portfolio problem to be optimized remains Problem 3.1.2. We calculate the equivalent target wealth for the two strategies using the benchmark strategy $\pi^b = 0.75$.

Example 16.1.1. Denote by C_I the equivalent target wealth for the portfolio process with inflation, and by C_N the equivalent target wealth without inflation. Using the same parameters as in Section 15.3, we obtain

$$C_I = 40'600,$$
$$C_N = 44'100.$$

As Problem 3.1.2 minimizes the difference to the nominal terminal wealth, which is usually higher than the real terminal wealth of Problem 9.1.5, it is unsurprising that the target wealth C_N is larger. For this set of parameters, the expected inflation over the duration of the investment period is $\mathbb{E}^{\mathbb{P}}[I(T)] = 1.22$, so the equivalent target wealth C_N is lower than $C_I \mathbb{E}^{\mathbb{P}}[I(T)]$, indicating higher underlying risk for the strategy not including inflation in the problem statement.

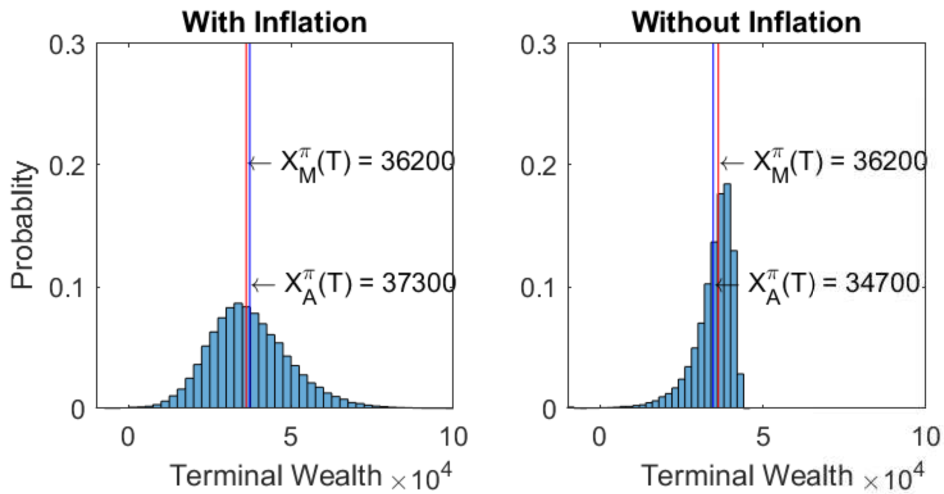


Figure 16.1: Histograms of the nominal terminal wealth distribution for the unconstrained portfolio problem with and without considering inflation.

In Figure 16.1 we plot the empirical terminal wealth for both strategies. Note that $X^{\pi}_{M}(T)$ denotes the median terminal wealth, whereas $X^{\pi}_{A}(T)$ denotes the mean terminal wealth. We observe that the empirical terminal distribution is much broader when inflation is considered in the problem statement. Furthermore, the probability of ending very close to the target in nominal terms is larger when inflation is not considered and we note that the median terminal wealth is very close for both strategies. The mean terminal wealth $X^{\pi}_{A}(T)$, on the other hand, differs quite substantially and indicates that the upside potential is much larger when inflation is considered.

We see in the analysis of Example 16.1.1 that the portfolio process including inflation in the problem statement leads to higher terminal wealth on average, even when the performance methodology of Section 15.1 is used. In order to understand the impact of the parameter values on the difference of the annual gain between the two strategies, we perform a sensitivity analysis on the inflation parameters σ_I and θ_I. We do not include the real interest

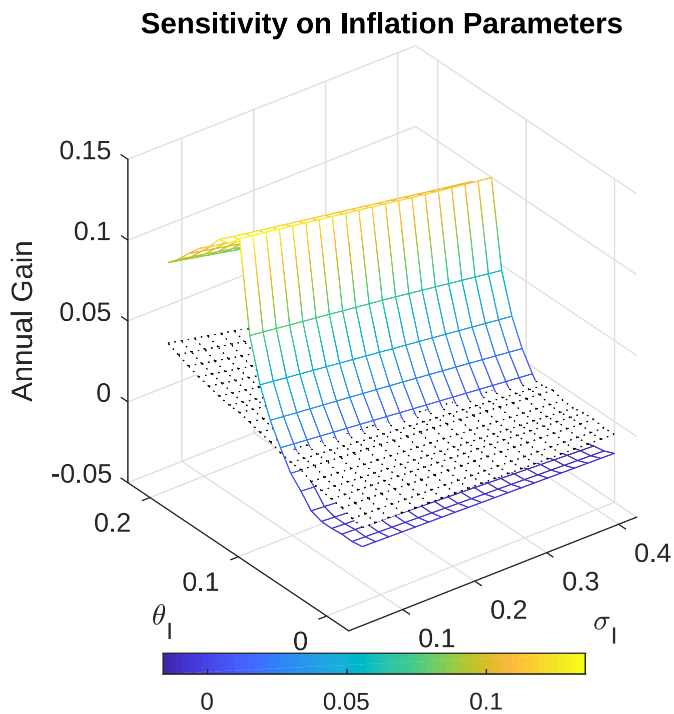


Figure 16.2: The sensitivity of the annual gain on the various inflation parameters. The mesh in black denotes the annual gain of the portfolio process not including inflation.

rate r_R in the analysis as we found that the annual gain of the portfolio process including inflation is fairly insensitive to r_R. For Figure 16.2 we calculate the equivalent target wealth for different parameter values of σ_I and θ_I and compare the performance of the two portfolio processes. We note that for most parameter values, including inflation to the analysis leads to a higher equivalent target wealth and increases the performance. The annual gain reacts the most to the market price of inflation risk θ_I and only sinks below the annual gain of the strategy not including the inflation-linked bond for small values of θ_I. Looking more closely at the form of the inflation index (5.1) we note that the parameter values for which the portfolio including inflation is outperformed correspond to periods of low inflation. Hence it is unsurprising that the resulting equivalent terminal wealth is lower and the performance is worse during those periods.

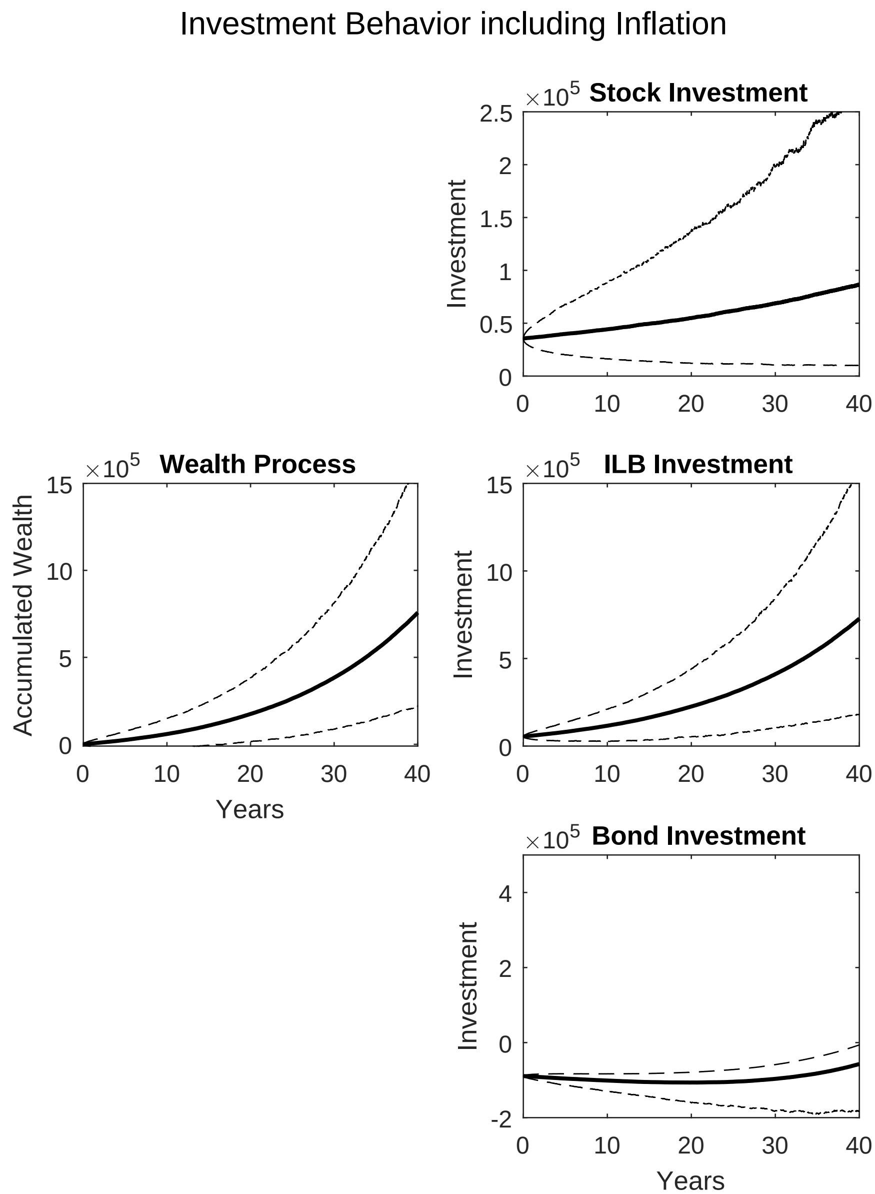


Figure 16.3: The investment behavior of the portfolio process including inflation in the problem statement. The straight lines denote the mean values, whereas dotted lines represent the 90%-confidence interval.

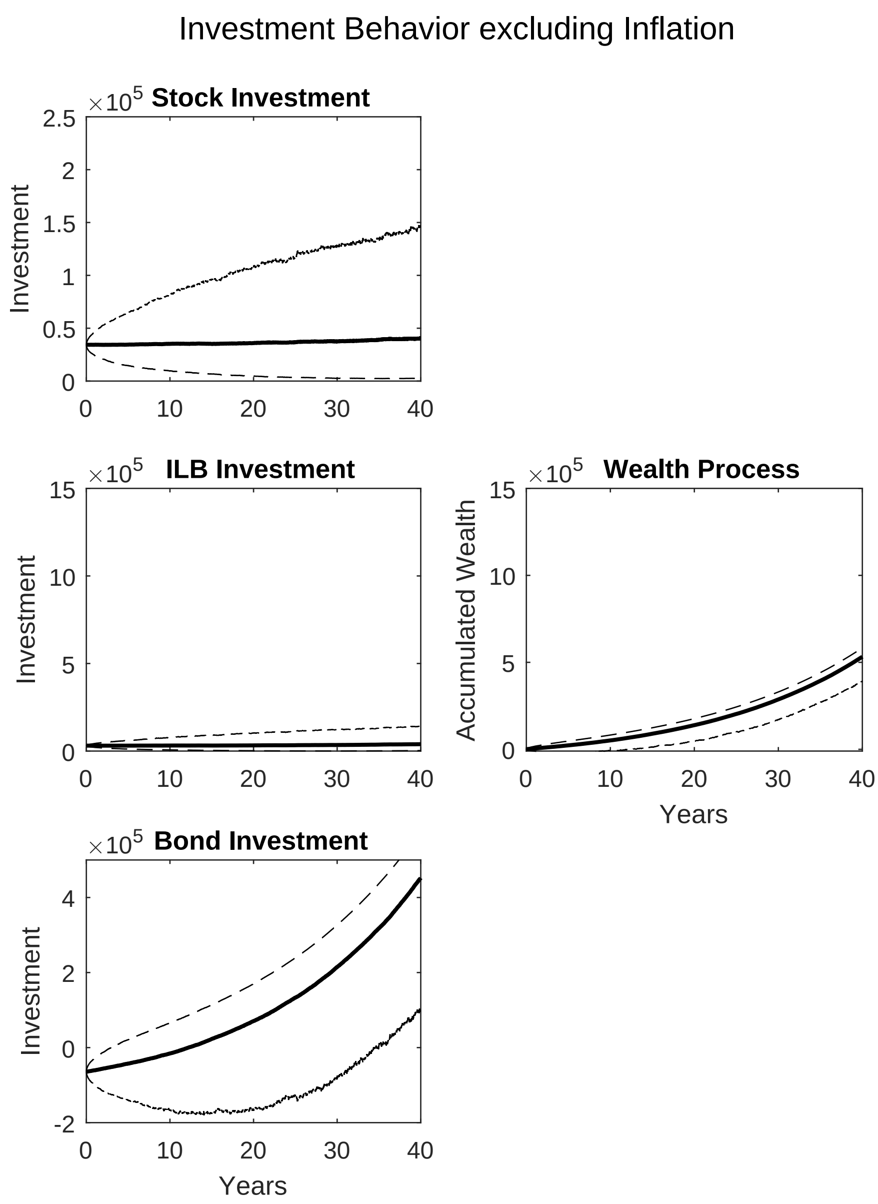


Figure 16.4: The investment behavior of the portfolio process not including inflation in the problem statement. The straight lines denote the mean values, whereas dotted lines represent the 90%-confidence interval.

16.2 Comparing the Investment Behavior

In order to understand more thoroughly how adding inflation to the problem statement warps the investment behavior, we study the portfolio processes of the two strategies. We return to the parameters of Section 15.3, increasing the investment horizon to 40 years. We calculate the portfolio processes for $10'000$ paths and plot the resulting wealth and portfolio processes in Figures 16.3 and 16.4. The solid lines correspond to the average wealth and the average investment, while the dotted lines represent the extremes of an estimated range, including 90% of the paths. Clearly, the uncertainty of the investment increases as the investment horizon approaches, as the current wealth of the different paths deviate more heavily.

The main difference between the two strategies is the amount invested in the inflation-linked bond. The optimal portfolio process for Problem 9.1.5 invests much more heaviliy into the inflation-linked bond than the stock and finances this by borrowing from the bank account for the whole duration of the investment period. In contrast, the optimal portfolio process for Problem 3.1.2 with contributions invests almost nothing in the inflation-linked bond, but tends to put more money into the bank account, as maturity approaches.

Another difference of the two portfolio processes is the change in the investment in the risky assets. Including inflation, we find that the average amount invested in both the stock and the inflation-linked bond rises from the beginning. Therefore, contrary to the strategy without inflation, the riskiness increases throughout the investment period. This is also apparent, comparing the confidence intervals for the wealth processes.

To conclude, we note that the consideration of inflation is very important for target-based pension schemes. By altering the investment strategy considerably, it outperforms the strategy not considering inflation in most market situations. Due to low supply and market frictions, the current model may not portray the real market situation correctly. Nonetheless, pension fund managers should calculate the performance in real terms and add inflation-linked bonds to their portfolio.

Chapter 17

Comparing Different DC Plan Strategies

In order to evaluate the target-based strategy as a potential alternative to other DC plan strategies, we compare it to two popular investment plans. As a mathematical alternative, we compare the performance to that of the optimal portfolio process for a power utility maximizing investor, introduced in [Zhang et al., 2007]. The second strategy to which we compare the performance of the target-based strategy is the dynamic lifecycle strategy introduced in [Basu et al., 2011]. Being close to the most popular strategy in practice, while offering a dynamic upgrade, allows for comparing the more theoretical strategies to reality. The theory behind both alternative strategies is summarized in Appendix IV.D.

Consider the case of a pension plan member investing for the time horizon of $T = 40$ years with a starting salary of $\pounds 20'000$. The salary is assumed to follow the path of Figure 17.1, with empirical parameters $\kappa = 0.015$, $\sigma_{LS} = 0.004$ and $\sigma_{LI} = 0.006$. It follows the average salary process, increasing for the first 30 years, before beginning to stagnate and finally falling off before retirement. Throughout the investment period, contributions at a rate of $\delta = 10\%$ are credited to the pension plan monthly, with no contribution for the last month of the last year. With $x = \pounds 10'000$, the initial wealth is set higher than in previous discussions, as the optimal strategy for the power utility maximizing investor involves dividing by the current wealth, which lead to numerical inaccuracies whenever the current wealth is close to zero.

The inflation index is assumed to follow the path of Figure 17.2, with empirical parameters $r_N = 4\%$, $r_R = 3\%$, $\sigma_I = 0.08$ and $\theta_I = 0.12$.

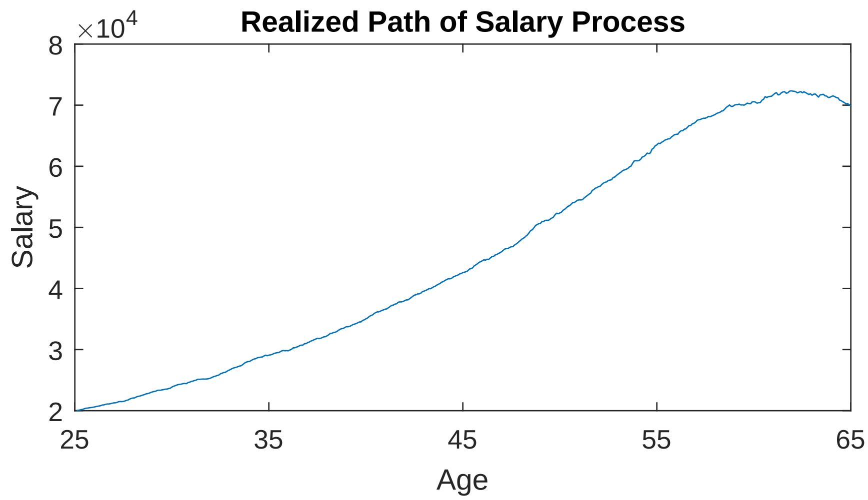


Figure 17.1: Realized path of the salary process for the hypothetical pension plan member.

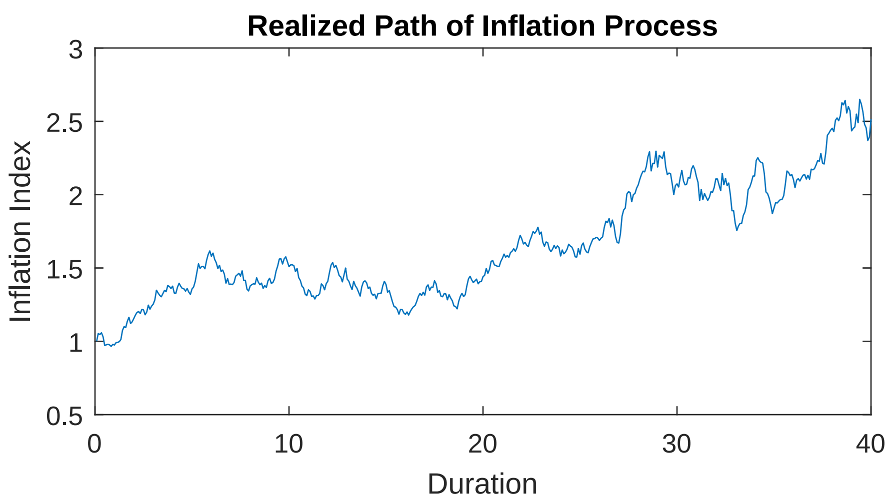


Figure 17.2: Realized path of the inflation index for the duration of the pension plan.

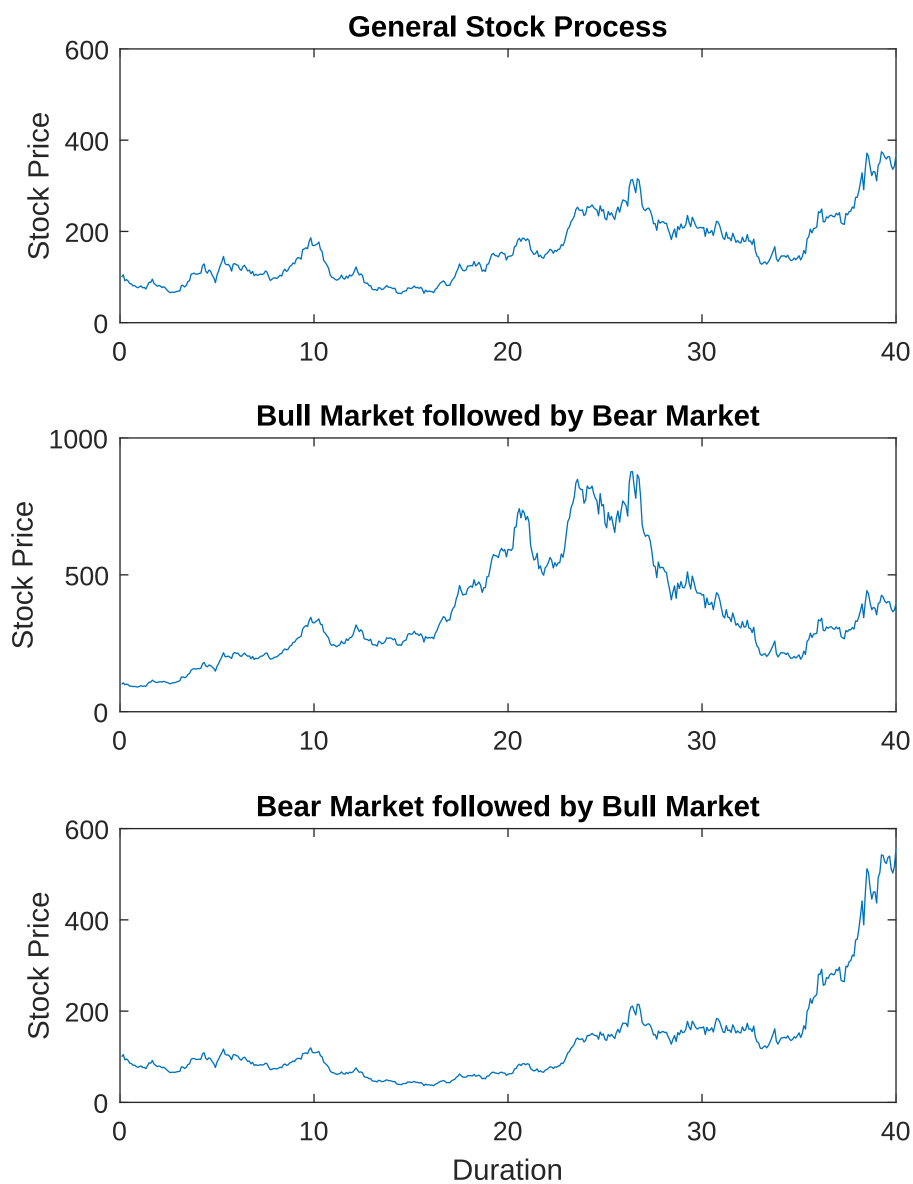


Figure 17.3: Three scenarios for the stock price process for the duration of the pension plan.

17.1 Comparing the Strategies with no Portfolio Constraints

The plan member can choose between four different investment strategies at the start of the pension plan accumulation phase. The unconstrained, target-based optimal portfolio strategy (P1), given by (9.22), the power utility maximizing portfolio strategy (P2), given by (18.6), the dynamic lifecycle strategy (P3), outlined in Appendix IV.D.2 and a deterministic lifecycle strategy (P4). The target wealth for strategy (P1) and the risk aversion parameter for strategy (P2) are chosen to obtain the same expected shortfall at the 95%-confidence interval as that of (P4), which switches from 50% stocks and 50% inflation-linked bonds to 100% investment in the bank account after the first 20 years. The target return r^* of strategy (P3) is set to equal the nominal rate of return.

	P1	**P2**	**P3**	**P4**
Parameter of Interest	C = 414'000	γ = -2.45	r* = 4.00%	\
Median Rate of Return	6.14%	5.38%	5.42%	5.17%
Success Probability	99.0%	97.6%	96.6%	97.7%

Table 17.1: Statistics of the different portfolio processes for the parameters of Chapter 17.

We observe in Table 17.1, that strategy (P1), the unconstrained portfolio process, performs best for the current set of parameters. The difference of 0.72% in the median rate of return results between the unconstrained portfolio process and the dynamic lifecycle strategy results in a difference of over £100′000 in the median terminal wealth. With the highest success probability and a ruin probability of only 0.6%, the unconstrained portfolio process clearly outperforms the other strategies for this choice of parameters.

The empirical terminal wealth distribution of the different portfolio processes can be seen in Figure 17.4. We note that even though strategy (P1) achieves the largest median terminal wealth, its terminal distribution is also the broadest, indicating more uncertainty. The other three strategies show a much narrower terminal wealth distribution, with more certainty about the outcome, but less upside potential.

In order to study the different investment character of the strategies, we analyze three specific stock market scenarios. The underlying parameters remain the same with $\mu = 8\%$, $\sigma_S = 0.2$ and $\sigma_{IS} = 0.08$, while the realized

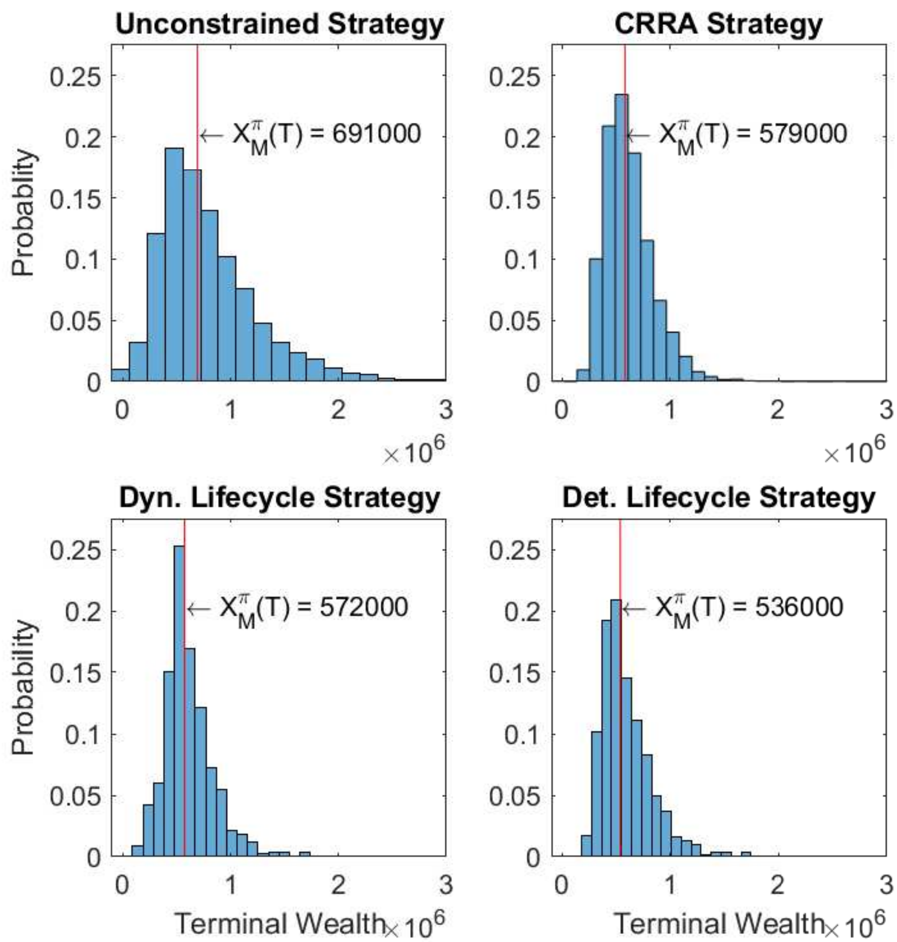


Figure 17.4: Terminal Distribution of the unconstrained portfolio process (9.22) and the power utility maximizing strategy (18.6). The target wealth C and the risk aversion parameter γ of Table 17.1 are used.

paths show the characteristics of a typical stock market, the characteristics of a bull market followed by a bear market, or the characteristics of a bear market followed by a bull market. We expect to observe the biggest advantage of the target-based optimal strategy in the third market situation, where high investment in the risky assets close to the retirement date should lead to a better performance.

General Stock Market

On the top of Figure 17.9 we see the stock price process on the left-hand side, while the different wealth processes are plotted on the right. Below, the investment behavior of the four strategies is shown.

For the stock price process in Figure 17.5 we note that the strategies have a very similar wealth process, with the strategy (P1) outperforming the others during the last five years. As both the unconstrained optimal portfolio and the dynamic lifecycle portfolio stay heavily invested in the risky assets, they are able to profit from the rise in the stock price during that period.

	P1	**P2**	**P3**	**P4**
Terminal Wealth	437'000	368'000	343'000	330'000
Rate of Return	4.6%	3.9%	3.6%	3.5%
Minimal Wealth	-5'050	6'260	9'920	9'920

Table 17.2: Properties of the different portfolio processes calculated for the salary process in Figure 17.1, the inflation process in Figure 17.2 and the stock price process in Figure 17.5.

In Table 17.2 we observe that although the strategy (P1) achieves the highest rate of return, it attains negative wealth during some period of the investment. As the deficit is fairly low compared to the contributions, it is remedied after a single period. Nonetheless, this may lead to problems in practice.

Note that due to the choice of parameters, strategy (P2) invests almost the same amount in the stock and the inflation-linked bond over the whole period. We also see that this strategy is much more similar to the lifecycle processes than the target-based strategy, as the investment in the risky assets follows a more constant path and increasing investment in the bank account starts after about half of the investment period.

The strategy (P3) is only able to reach its rate of return target for a short period after 25 years and otherwise stays invested fully in the risky assets. This leads to a more volatile wealth process than that of strategy (P4), but allows the investor to profit from increases in the stock price process during late stages of the investment period.

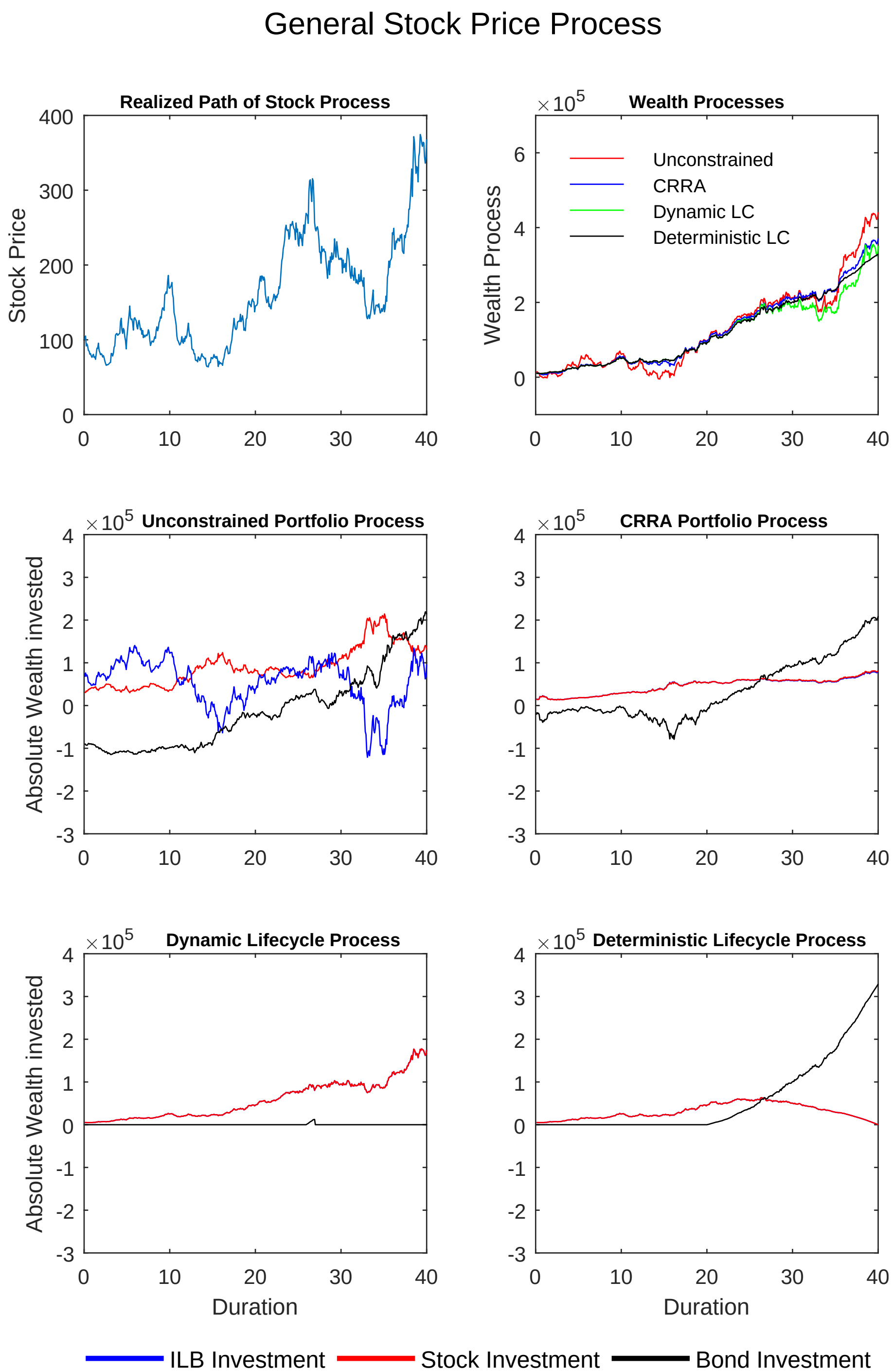


Figure 17.5: Performance of strategies with no portfolio constraints for a general stock price process.

Bull Market followed by Bear Market

The stock price process in Figure 17.5 increases quickly for the first 25 years, before falling for the following 10 years. This price process is most beneficial for strategies (P2) and (P4), which outperform the other strategies for the whole period. Both strategies initially profit from the good market situation and the rise of the stock prices during the bull market. In the second half of the investment horizon, they shift the wealth to the bank account, leading to smaller loss during the bear market.

	P1	**P2**	**P3**	**P4**
Terminal Wealth	243'000	342'000	266'000	314'000
Rate of Return	2.2%	3.6%	2.6%	3.3%
Minimal Wealth	3'550	8'940	10'000	10'000

Table 17.3: Properties of the different portfolio processes calculated for the salary process in Figure 17.1, the inflation process in Figure 17.2 and the stock price process in Figure 17.7.

Even though strategies (P2) and (P4) invest very similarly, they have some important differences, which lead to (P2) performing slightly better. During the first half of the investment period, strategy (P2) borrows some money in order to invest more heavily in the risky assets. Furthermore, it never invests fully in the bank account, allowing it to profit from stock price increases during late periods of the investment. We can see in Table 17.3 that those characteristics lead to the highest terminal wealth for strategy (P2).

On the other hand, strategies (P1) and (P3) are heavily invested in the risky assets for the whole duration, leading to a large loss of wealth during the bear market of the second half. Even though strategy (P3) achieved its target rate of return and started shifting the investment to the bank account, it reinvests everything as the stock prices fall, losing even more money in the process. However, we also note this also allows strategies (P1) and (P3) to profit from the stock price increase during the last five years.

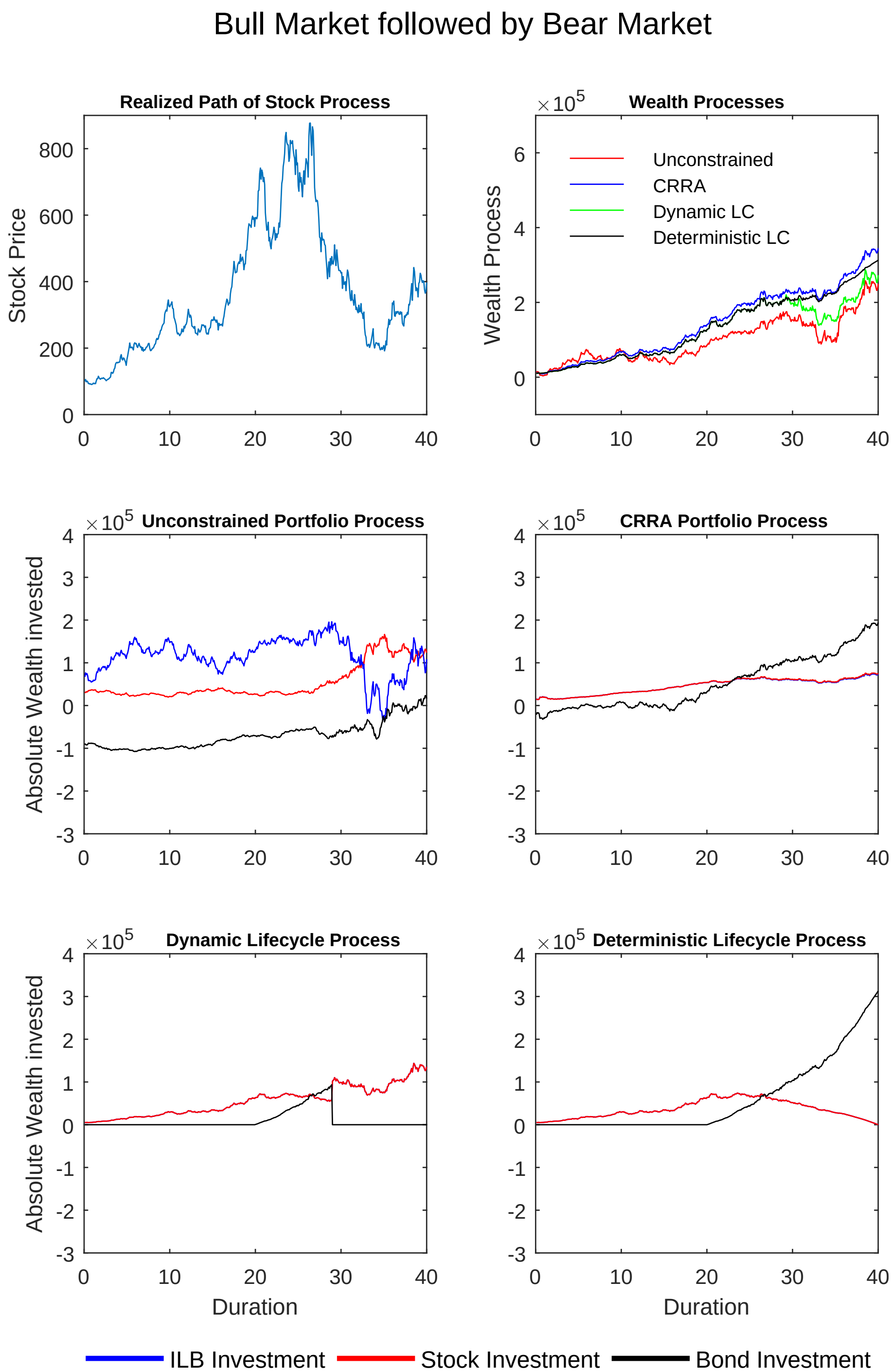


Figure 17.6: Performance of strategies with no portfolio constraints for a bull market followed by a bear market.

Bear Market followed by Bull Market

On contrary to the previous discussion, the stock price process in Figure 17.7 falls for the first 20 years, before recovering and rising quickly for the remaining period of the investment plan. As expected, the strategy (P1) outperforms the other strategies, due to staying invested in the risky assets for the whole duration. Strategy (P2) performs especially badly for this specific path of the stock price, and we note that it becomes very erratic during periods where the wealth process is close to zero. This is due to dividing by $X^*(t)$ in (18.6).

	P1	**P2**	**P3**	**P4**
Terminal Wealth	580'000	246'000	444'000	363'000
Rate of Return	5.7%	2.2%	4.7%	3.9%
Minimal Wealth	-55'400	-9	9'930	9'930

Table 17.4: Properties of the different portfolio processes calculated for the salary process in Figure 17.1, the inflation process in Figure 17.2 and the stock price process in Figure 17.7.

We note that strategy (P3) stays invested in the risky stock for much longer than strategy (P4), as the target rate of return is not reached until very late in the investment period. This in turn allows (P3) to profit from the bull market and lets the dynamic strategy attain a higher terminal wealth.

In Table 17.4 we see that the wealth process of strategy (P1) becomes heavily negative during the bear market in the first half of the investment period and we see in Figure 17.7 that the wealth process stays negative for almost 5 years. Even thoughit recovers during the second half, this basically means that the initial investment $x = 10'000$ and the contributions during the first 20 years of the pension plan were lost.

Even though the period of negative wealth for strategy (P1) won't be circumvented completely by using the constrained strategy or the strategy with a lower bound, the performance during that period would increase, as less money is invested in the risky assets for either strategy. As observed in the various examples so far, this would come at the price of a lower rate of return and may not be completely satisfying either. In order to prohibit negative wealth completely and we use strategies resulting from cut-shares during the next section.

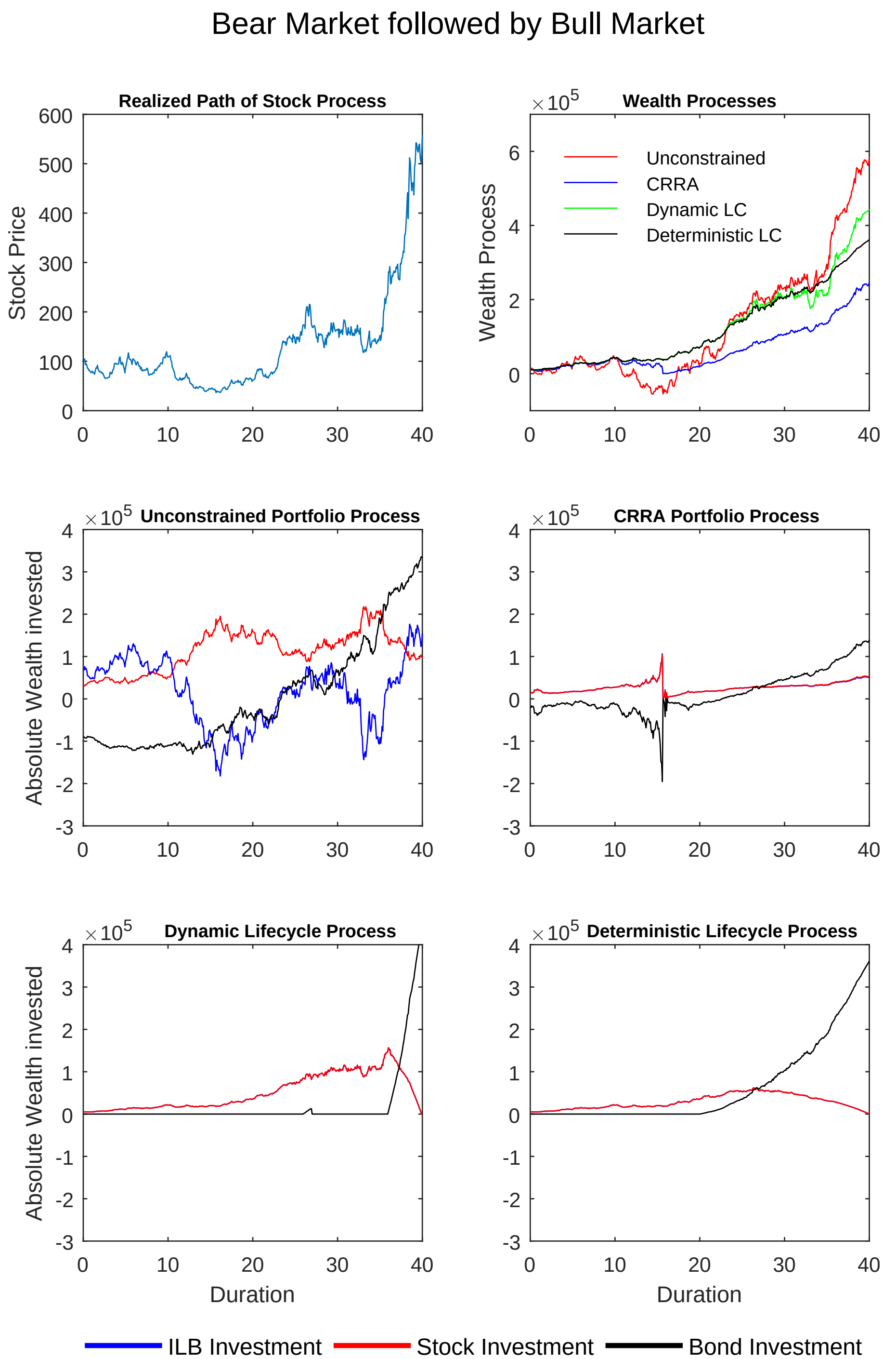


Figure 17.7: Performance of strategies with no portfolio constraints for a bear market followed by a bull market.

17.2 Comparing the Strategies under Cut-Shares

We have seen during the last section, that both the target-based optimal strategy and the power utility maximizing strategy may result in short positions. Not only do such position increase the underlying risk and may result in negative wealth, they are often regulated heavily by financial conduct authorities. Therefore, we assess the performance of the strategies under a no-shorting constraint.

The plan member can once again choose between four different investment strategies at the start of the pension plan accumulation phase. We replace the unconstrained, target-based optimal strategy (P1) with the target-based strategy resulting from cut-shares (P1') and replace the power utility maximizing portfolio strategy (P2) by cut-shares (P2') as well. Since both the dynamic lifecycle strategy (P3) and the deterministic lifecycle strategy (P4) do not include any short positions by definition, we include them to the comparison.

	P1	**P2**	**P3**	**P4**
Parameter of Interest	C = 971'000	γ = -2.24	r* = 4.0%	\
Median Rate of Return	5.72%	5.32%	5.42%	5.17%
Success Probability	99.1%	97.7%	96.6%	97.7%

Table 17.5: Statistics of the different portfolio processes for the parameters of Chapter 17.

Comparing Table 17.5 to Table 17.1 we see that the equivalent target wealth is higher for strategy (P1') and the factor of risk aversion is lower for (P2') than for their counterparts. This is in accordance to the discussion in Section 15, as the underlying risk is lower when shorting positions are impossible. We also note, that the resulting median rate of return is slighly lower for the strategies resulting from cut-shares.

The empirical terminal wealth distribution of the different portfolio processes can be seen in Figure 17.8. Note that compared to the distribution of the portfolio process without the no-shorting constraint in Figure 17.4, the distribution of the strategies resulting from cut-shares is much narrower. Therefore, we expect the performance of the different scenarios to be much more comparable than in the previous section. This comes at the price of a lower median terminal wealth, but nonetheless, the target-based strategy (P1') still outperforms the other strategies.

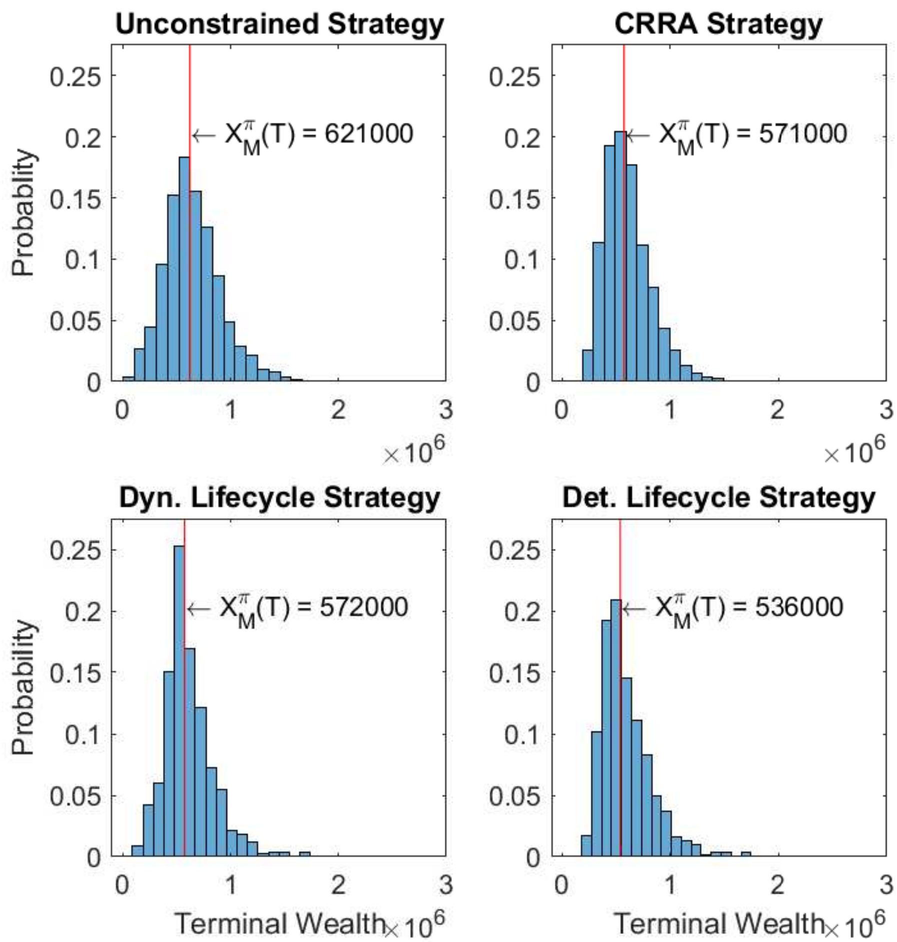


Figure 17.8: Terminal Distribution of the portfolio processes resulting from cut shares. The target wealth C and the risk aversion parameter γ of Table 17.5 are used.

For the specific scenarios of the stock price process, we expect to observe similar outcomes as in Section 17.1, while the terminal wealth of the strategies differ less between the three scenarios. The no shorting constraint restricts the amount that can be invested in the risky assets and therefore narrows the terminal wealth distribution, as seen in Figure 11.1.

General Stock Market

As expected, we observe that for the stock price process in Figure 17.9, the strategies are more similar than for the investment strategies without the no-shorting constraint. Once more, the strategies (P1') and (P3) profit from the stock price increase during the last five years, catching up to the better performance of the other strategies.

	P1'	**P2'**	**P3**	**P4**
Terminal Wealth	373'000	353'000	343'000	330'000
Rate of Return	4.0%	3.8%	3.6%	3.5%
Minimal Wealth	9'210	9'920	9'920	9'920

Table 17.6: Properties of the different portfolio processes calculated for the salary process in Figure 17.1, the inflation process in Figure 17.2 and the stock price process in Figure 17.9.

This can also be observed in Table 17.8, where all parameters shown are very similar for all strategies. We also note that contrary to the example for unconstrained portfolio processes, none of the strategies reach a negative wealth at any point of the investment period. Next to the regulatory necessity, this is one of the main advantages of the methodology of cut-shares.

Studying the investment behavior in Figure 17.9 more closely, we observe the similarity between strategy (P2') and the lifecycle strategy (P4). Both strategies start fully invested in the risky assets and start investing in the bank account only in the second half of the investment period. Note that contrary to strategy (P4), the allocation between the inflation-linked bond and the stock depends on the underlying parameters and is not set to 50%. Similarly, both the time after which money is invested in the bank account and the speed of the reduction in risky investment depends both on the parameters, as well as on the stock price process.

On the other hand, both the strategy (P1') and the strategy (P4) mainly invest in the risky assets for the whole duration of investment. This leads to a more volatile wealth process and the main difference between the strategies is the allocation between the inflation-linked bond and the stock, which is set externally for (P4).

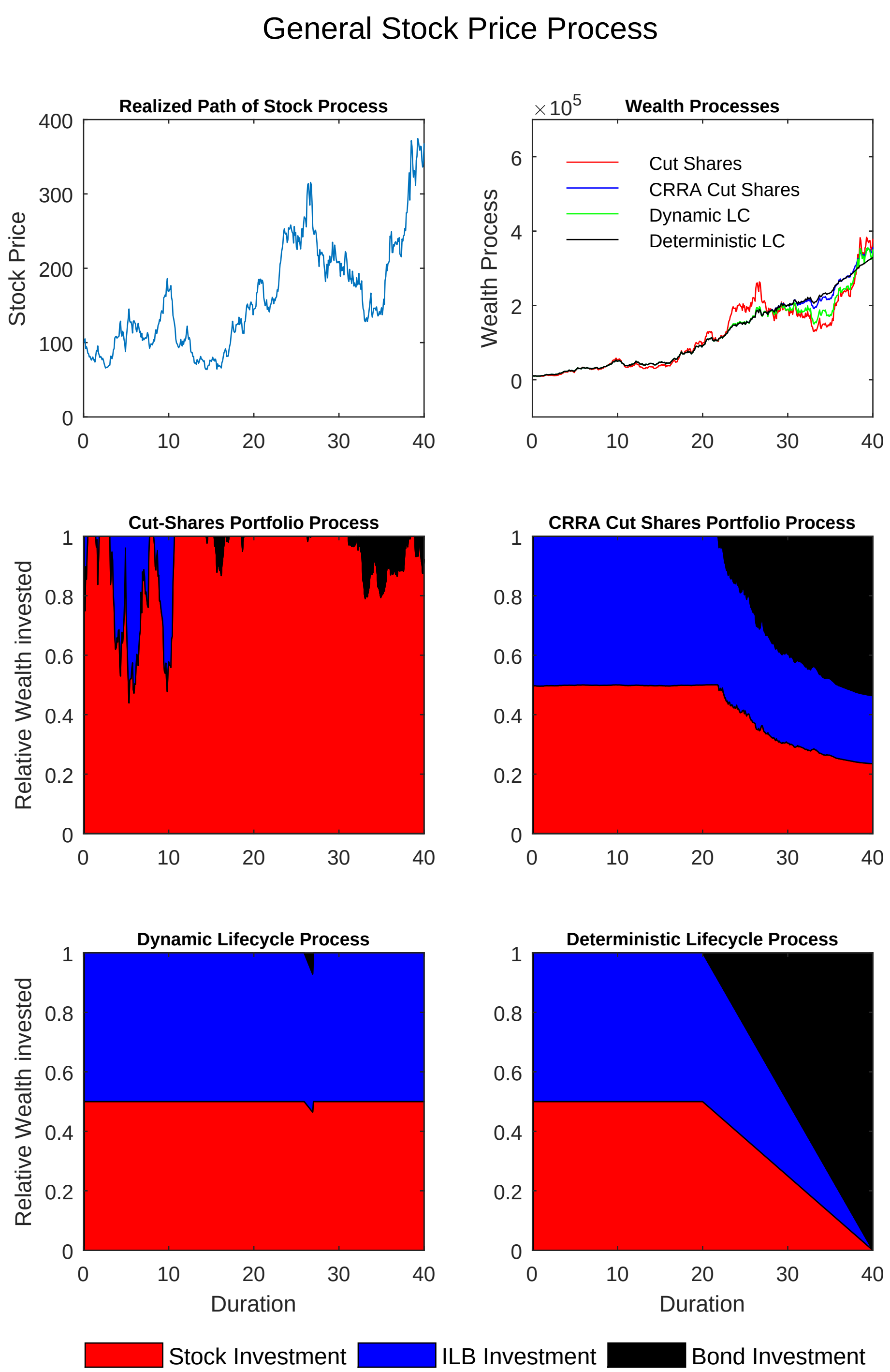


Figure 17.9: Performance of strategies with no shorting constraints for a general stock price process.

Bull Market followed by Bear Market

Unsurprisingly, both strategy (P1') and strategy (P2') perform worse than their respective counterparts in the previous section for the stock price process of a bull market followed by a bear market. Since they are not able to borrow against future contributions, less money can be invested during the period of the bull market, leading to a lower return. As more money is available during the second half, they still bear the brunt of the bear market, leading to a worse performance.

	P1	**P2**	**P3**	**P4**
Terminal Wealth	221'000	314'000	266'000	314'000
Rate of Return	1.7%	3.3%	2.6%	3.3%
Minimal Wealth	9'910	10'000	10'000	10'000

Table 17.7: Properties of the different portfolio processes calculated for the salary process in Figure 17.1, the inflation process in Figure 17.2 and the stock price process in Figure 17.10.

Nonetheless, we see in Table 17.8 that the ranking of the portfolio strategies remains the same, with the unconstrained strategy resulting from cut-shares being outperformed by all the other strategies. Analyzing Figure 17.10 we see why the target-based strategies are outperformed in this scenario.

During the first 20 years, the wealth processes of all four strategies follow a very similar path. On average, the investment is split equally between the stock and the inflation-linked bond, while almost nothing is invested in the bank account. After that period, strategies (P2'), (P3) and (P4) start investing more heavily in the bank account, while the stock price process enters a bear market. For strategies (P2') and (P4) this continues until the end of the investment period, while strategy (P3) reinvests in the risky assets, once the target rate of return is not attained anymore.

The target-based strategy does not start reinvesting in the bank account. On the contrary, the further away the current wealth is from the target wealth, the more money is invested in the stock, with the hope of recovering from the bear market. This increases the volatility of the wealth process and leads to larger loss for this specific scenario. The increase of the stock price during the last five years leads to some recovery of the wealth process of strategy (P1'), but not enough to overcome the bad performance during the bear market.

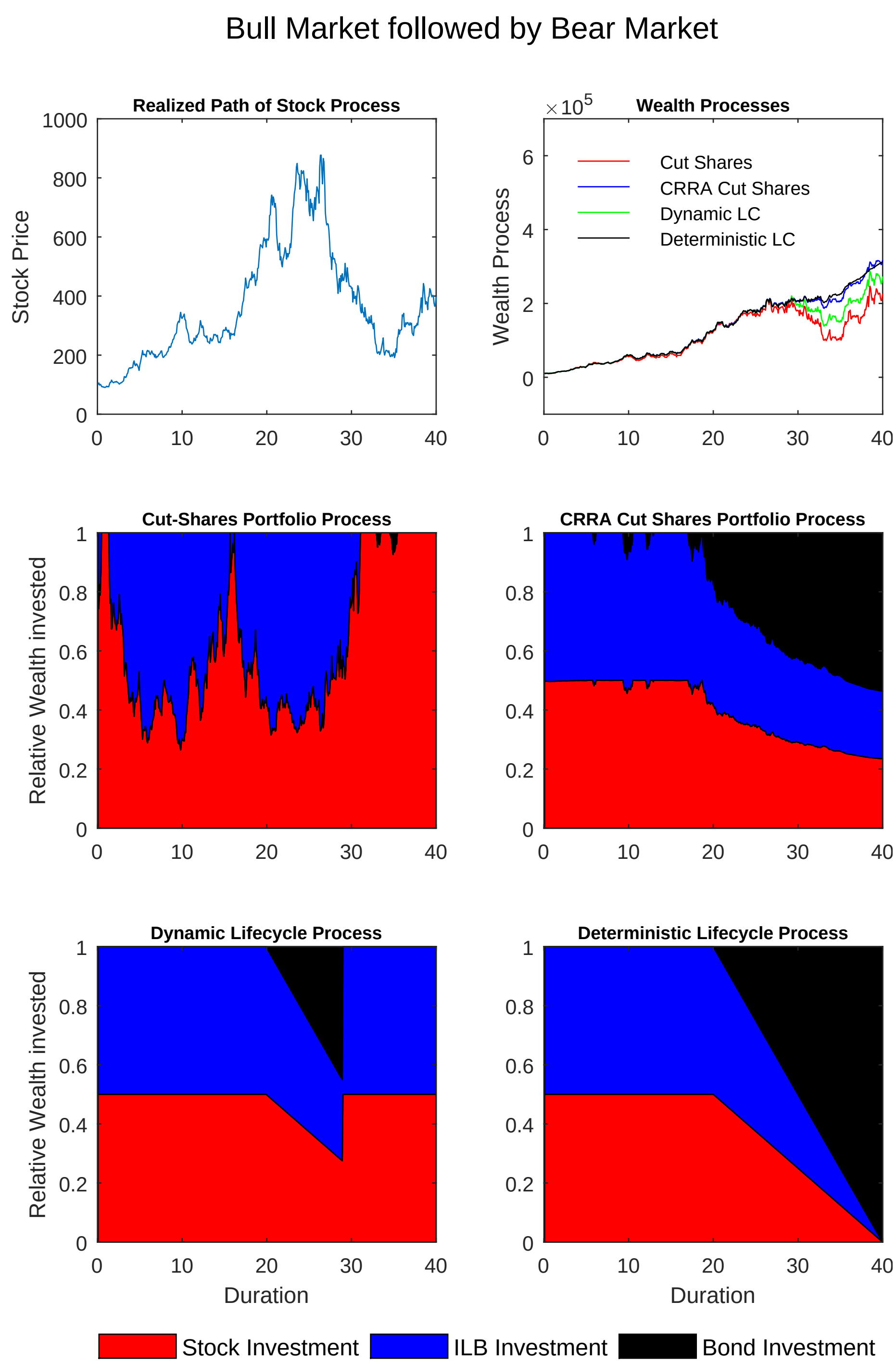


Figure 17.10: Performance of strategies with no shorting constraints for a bull market followed by a bear market.

Bear Market followed by Bull Market

The optimal market situation for the target-based strategy resulting from cut-shares can be observed in Figure 17.11. During the analysis in Section 17.1 we already noted the advantage of the target-based optimal strategy (P1) over the other portfolio strategies. This carries over to the strategy (P1'), as the portfolio stays heavily invested in the stock for the whole duration of the investment.

	P1	**P2**	**P3**	**P4**
Terminal Wealth	704'000	439'000	444'000	363'000
Rate of Return	6.4%	4.6%	4.6%	3.9%
Minimal Wealth	9'220	9'930	9'930	9'930

Table 17.8: Properties of the different portfolio processes calculated for the salary process in Figure 17.1, the inflation process in Figure 17.2 and the stock price process in Figure 17.11.

Comparing Table 17.8 to Table 17.2, we see that the strategies resulting from cut-shares outperform their unconstrained counterparts. Due to the no-shorting constraint, they are not able to borrow against the future contributions at the start and hence lose less money during the bear market of the first 20 years. This is most apparent while comparing the minimal wealth attained, which is negative for both (P1) and (P2), while being close to the initial wealth for (P1') and (P2'). As more money is available for the second half, during the bull market, both strategies can profit by investing in the risky assets.

In Figure 17.11 another advantage of the target-based approach becomes apparent. While the allocation to the stock and the inflation-linked process is very similar for the three market scenarios for the other portfolio processes, it varies more heavily for strategies (P1) and (P1'). This is already apparent from the form of the optimal portfolio process (11.5). The investment in the inflation-linked bond depends in a different way on the current optimal wealth than the investment in the stock, which allows to vary the allocation depending on the wealth attained. For the CRRA strategy (18.6), we see that the inflation-linked bond is treated the same as an additional stock and hence does not allow for the same degree of variation in allocation.

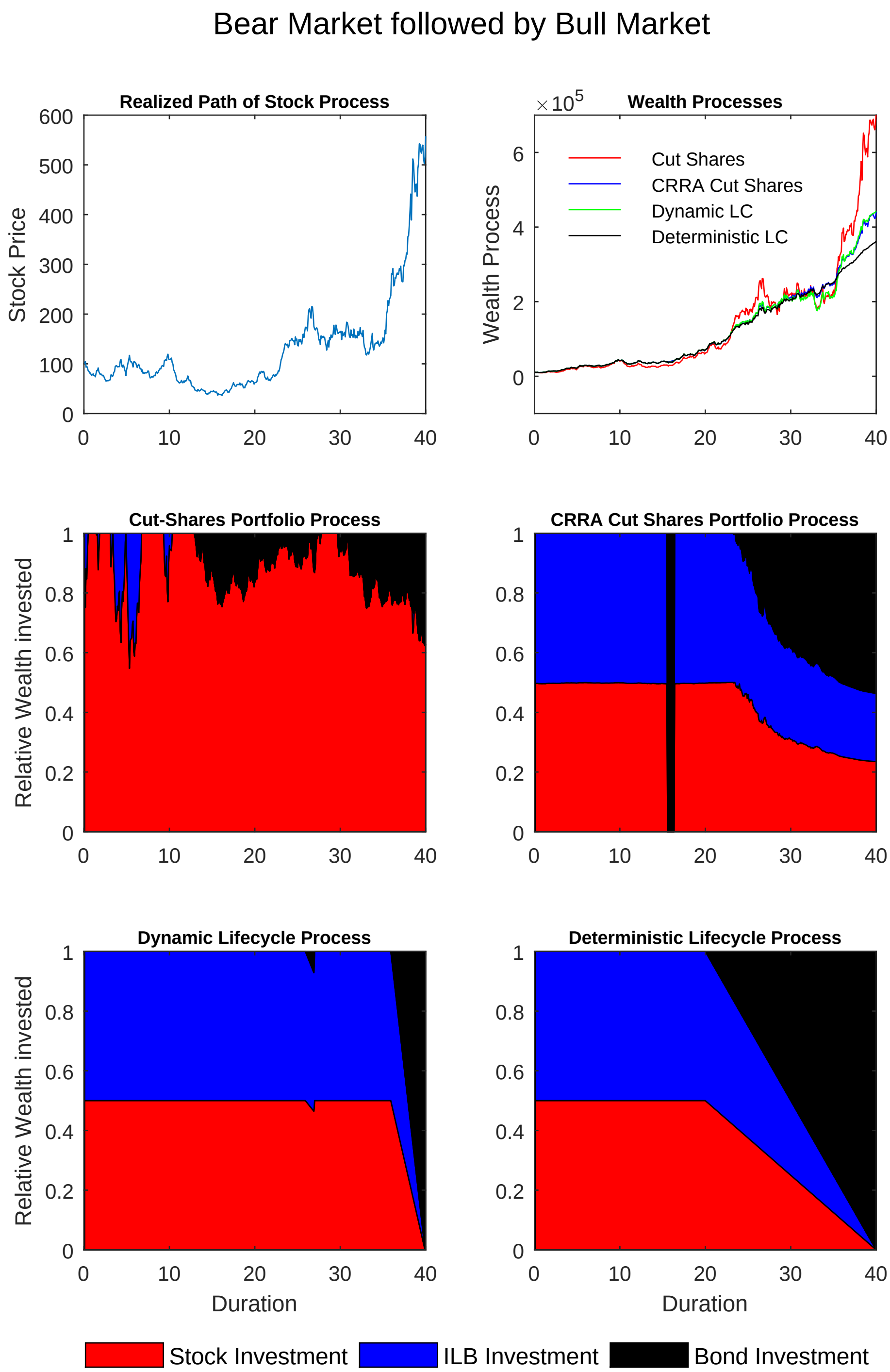


Figure 17.11: Performance of strategies with no shorting constraints for a bear market followed by a bull market.

Chapter 18

Conclusion and Further Work

The goal of this thesis was to analyze the DC pension plan strategy resulting from a target-based approach. We believe that the resulting investment plan is a viable alternative to other strategies used in practice. With life-cycle strategies being simple to explain but lacking a proper mathematical framework and other utility maximizing strategies working with abstract risk aversion coefficients, the target-based strategy fills a niche in combining the best of both worlds.

We have shown that the target wealth can fulfill two roles. It can either be chosen by a plan member, taking a figurative step back towards DB schemes, where the risk is still borne by the pension plan member, but a certain direction is provided. On the other hand, the target wealth can also be chosen by the performance methodology of Section 15.1 in order to limit downside risk and thereby identifying the risk profile of the pension plan member. Both roles are more suitable for the average pension fund member, as it is easier to select a wealth target rather than an abstract index.

In addition to the psychological advantages, the numerical analysis of Chapter 17 has shown that the target-based optimal portfolio performs very well in many situation. While it misses a certain time component, after which the asset allocation is shifted to a less risky portfolio, it is able to react to changes in the market quickly and thoroughly. As the amount invested in the different underlyings starts varying more heavily close to maturity, it seems reasonable to reassess the portfolio strategy more frequently towards the end of the pension plan.

Unlike for other utility maximizing portfolio strategies, the inflation-linked bond is not simply seen as an additional stock, but used as an important tool to hedge inflation risk. During the analysis in Chapter 16 we validated that the stronger the inflation present in the market, the more important it is to consider real wealth instead of nominal wealth as the target. Even though more money is invested in the risky asset, the underlying risk can be reduced by diversification benefits to a degree that the performance of the strategy including the inflation-linked bond is simply better than that of the strategy without.

We see two extensions of the quadratic utility maximizing portfolio as the most promising approaches for future research on the target-based optimal portfolio problem. Firstly, non-negativity constraints on the terminal wealth are shown to be of less importance in the market model including contributions. A general lower bound may improve the performance somewhat, however we believe that constraints on the portfolio processes offer more potential. We have seen that by enforcing the no-shorting constraint on the portfolio process ex-post, the performance improves for certain scenarios. As this is not a mathematically optimal portfolio, it only gives an upper bound on the optimal portfolio process with those constraints.

Secondly, in order to utilize the target-based optimal portfolio process in practice, the parameter values need to be estimated in some better way. Optimally, one would find a solution for the optimization problem using predictable processes instead of deterministic ones. Since very little theoretic advance has been made in this regard, it may be more profitable to include more stochastic factors to the market. This can be done via the regime-switching model touched upon in Section 14.3.1 or by modeling some of the parameters as stochastic processes directly. This leads to a similar procedure as in Part II, where the stochastic inflation index was added to the market.

Concluding, we remark that the target-based approach for DC pension schemes is an important alternative to consider, for both practitioners and researchers. Even though some points may still be improved upon, the numerical analysis in Part IV leads us to believe that the target-based approach outperforms the lifecycle strategy as well as other popular, utility maximizing, investment plans. Together with the advantage of setting a target wealth instead of some abstract risk factor, this gives us great hope for the use of the quadratic risk minimization approach in practice.

Appendix

In order to estimate parameters and to compare the quality of the different models, we heavily rely on statistical tools. In this section, we summarize the techniques used and provide insight into different models used for parameter estimation. Most proofs will not be given in this thesis, but a reference is always provided.

IV.A Terminology and the Maximum Likelihood Estimator

We start with a short introduction to the terminology and notation used in this section, mainly based on [Van de Geer, 2017] and [Wüthrich, 2017]. Suppose we are given a set of data points $(X, y) \in \mathbb{R}^p \times \mathbb{R}$ and try to find the characteristic of the relation between X and y, given by

$$y = f(X) + \varepsilon,$$

where $f : \mathbb{R}^p \to \mathbb{R}$ is a deterministic function which is to be determined and ε captures noise terms, model errors and other inabilities of the model to capture the underlying relation between X and y. The variables $(X(1), \ldots, X(p))$ are called *predictor variables*, whereas y is the *response variable*.

As the real function f is very hard to find, we estimate f by some function $\hat{f}$ and predict the response by $\hat{y} = \hat{f}(X)$. In order to measure the quality of the chosen function $\hat{f}$, we utilize a measure for the goodness-of-fit, the mean squared error of prediction.

Definition IV.A.1. Suppose we are given a set of data points $(X_i, y_i) \in \mathbb{R}^p \times \mathbb{R}$, for $i = 1, \ldots, n$. The *mean squared error of prediction* is given by

$$MSE = \frac{1}{n} \sum_{i=1}^{n} \big(y_i - \hat{f}(X_i)\big)^2.$$

Note that minimizing the mean squared error of prediction over all possible functions $\hat{f}$ is very hard, as the set of possible functions is very large. Moreover, no knowledge on the response variable y is used. In most applications, $\hat{f}$ is chosen to be of a certain form, which implies certain properties of the predicted response $\hat{y}$. These properties may then be used to find specific parameters for $\hat{f}$.

Assume that the $y_1, \ldots, y_n$ are independent and identically distributed from some distribution P. Further model assumptions then concern the modelling of the distribution P, which is typically indexed by a parameter ϑ in some parameter space Θ. We write $P = P_\vartheta, \vartheta \in \Theta$. Suppose that the densities p_ϑ exist.

Definition IV.A.2. The *likelihood function* $\mathcal{L}(\vartheta) : \Theta \to \mathbb{R}$ is given by

$$\mathcal{L}(\vartheta) = \prod_{i=1}^{n} p_\vartheta(y_i).$$

The *maximum likelihood estimator*, henceforth MLE, is given by

$$\hat{\vartheta} = \arg\max_{\vartheta \in \Theta} \mathcal{L}(\vartheta),$$

if it exists.

Remark. Alternatively, we may maximize the log-likelihood

$$\hat{\vartheta} = \arg\max_{\vartheta \in \Theta} \mathcal{L}(\vartheta) = \arg\max_{\vartheta \in \Theta} \sum_{i=1}^{n} \log P_\vartheta(y_i),$$

as the logarithm is an increasing transformation.

Note that the MLE does not always exist and other methods may be used to estimate parameters. However, for the applications in this thesis, the MLE is sufficient. For properties of the MLE we refer to [Lehmann, 2003].

IV.A.1 Model Selection

In the previous section we presented the MLE as a way to estimate parameters of some models such that they lead to a good fit to the data at hand. However, the MLE does not tell us if the form of the model function $\hat{f}$ is suitable for the underlying data. Graphical tools can be used to determine the usefulness of certain models, but more methodological tools offer more insight.

If the particular continuous distribution function for the model can be found, different tests, like the Kolmogorov-Smirnov test, or the Anderson-Darling test may be used to compare it to the empirical distribution function. As for the models applied here, the distribution function cannot be found easily, we present some information criteria which can be used to compare different models instead, and refer to [Wüthrich, 2017] for the tests mentioned above instead. Note that these information criteria will not give a test for one model, but instead provide a tool to compare two different models.

Definition IV.A.3. The *Akaike information criterion*, henceforth AIC, is defined by

$$AIC = -2\log\mathcal{L}(\vartheta) + 2d,$$

whereas the *Bayesian information criterion*, henceforth BIC, is defined by

$$BIC = -2\log\mathcal{L}(\vartheta) + \log(n)d.$$

Here, $\mathcal{L}$ denotes the likelihood function of the density chosen for the model and d denotes the number of parameters to be estimated.

The preferred model, from a selection of models, is then the one that has minimum AIC or BIC of the group. Both information criteria penalise the model on the number of parameters that need to be estimated, which decreases the likeliness of overfitting. We do not discuss the justification of these criteria here, but just mention two results.

- The AIC is an unbiased estimate of a distance between the fitted and the true model.
- Compared to the BIC, the AIC favors complex models and does not provide a consistent estimate of the true order.

See e.g. [Shumway and Stoffer, 2011][Chapter 2] for a more thorough discussion on information criteria.

IV.B The Autoregressive Moving-Average Model

Having introduced tools to estimate parameters and compare different models in the previous section, we study a specific model, where the predictor variables are previous values of the response variable and its innovations. Namely, we view the stochastic process $X(t)$ as a process experiencing some form of autocorrelation, i.e. depending on its path and we assume that its volatility is constant in time. The family of autoregressive, moving-average models, henceforth ARMA, allows to make certain statements and predictions for such processes. For the proofs of the theorems in this section, we refer to [Shumway and Stoffer, 2011][Chapter 2].

ARMA models make the strong assumption that the distribution of the underlying time series is unchanged for arbitrary shifts in time. Most financial time series have non-constant volatility in time, which cannot be captured in the family of ARMA models. The next section introduces a model extension that is able to portray the conditional heteroskedasticity of certain time series.

Definition IV.B.1. A stochastic process $\big(X(t)\big)_{t\in\mathbb{N}}$ is *stationary* if the distribution of $\big(X(t_1), \dots X(t_k)\big)$ is identical to the distribution of $\big(X(t_1 + h), \dots X(t_k + h)\big)$ for all $k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{N}$ and $h \in \mathbb{Z}$.

Recall the notion of a random walk, where each term $X(t)$ is dependent solely upon the previous term $X(t-1)$ and some random innovation. The family of ARMA models is an extension of this notion, where the dependence may reach further in the past and furthermore includes a dependence on the innovations as well.

Definition IV.B.2. A stochastic process $\big(X(t)\big)_{t\in\mathbb{N}}$ is called ARMA(p,q) if it is stationary and

$$\begin{aligned} X(t) = &\phi(1)X(t-1) + \dots + \phi(p)X(t-p) \\ &+ W(t) + \theta(1)W(t-1) + \dots + \theta(q)W(t-q), \end{aligned} \tag{18.1}$$

where $W(t)$ is independent of all $X(s)$ for $s < t$ and $\phi(p) \neq 0 \neq \theta(q)$. The variable $W(t)$ is called the *innovation* at time t. We assume that $W(t) \sim \mathcal{N}(0, \sigma_W^2)$ for all t.

Remark. If $p = 0$, $\big(X(t)\big)_{t\in\mathbb{N}}$ is called a moving average process of order q, denoted by MA(q). Similarly, if $q = 0$, $\big(X(t)\big)_{t\in\mathbb{N}}$ is called an autoregressive process of order p, denoted by AR(p).

For an ARMA(p,q) model, denote by $\Phi(z) = 1 - \phi(1)z - \dots - \phi(p)z^p$ and by $\Theta(z) = 1 + \theta(1)z + \dots + \theta(q)z^q$.

Proposition IV.B.3. *Assume that Φ and Θ have no common zeroes. If all zeros of Φ and Θ are outside of the unit circle, then the stochastic process given by* (18.1) *is stationary.*

Therefore, it is clear that a stochastic process is non-stationary if it contains a unit root. In that case, differencing or applying non-linear transformations may make the model stationary.

IV.B.1 Testing for Stationarity

In order to fit an ARMA model to the historical data of interest rates, the stationarity of the process needs to be checked and the unknown parameters need to be estimated.

We test the stochastic process for stationarity by testing for a unit root. By Proposition IV.B.3, the innovation at time t can be written as

$$W(t) = \sum_{i=0}^{\infty} \rho(i) X(t-i), \tag{18.2}$$

for some coefficients ρ, see e.g. [Shumway and Stoffer, 2011][Property 3.2]. Therefore, $X(t)$ can be written as a function of $X(t-1)$ and a series of differenced lag terms, i.e.

$$\begin{aligned} X(t) &= \big(\rho(1) + \rho(2) + \dots\big) X(t-1) - \big(\rho(2) + \rho(3) + \dots\big) \Delta X(t-1) \\ &\quad - \big(\rho(3) + \rho(4) + \dots\big) \Delta X(t-2) - \dots + W(t) \\ &= \tilde{\rho}(1) X(t-1) - \tilde{\rho}(2) \Delta X(t-1) - \tilde{\rho}(3) \Delta X(t-2) - \dots + W(t), \end{aligned} \tag{18.3}$$

where $\tilde{\rho}(i) = \rho(i) + \rho(i+1) + \dots$ denotes the cumulative sum of all the following parameters and Δ denotes the *difference operator* $(\Delta X)(t) = X(t) - X(t-1)$. Introducing the *backshift operator*, defined by

$$(BX)(t) = X(t-1),$$

we can rewrite (18.2) as $\big(1 - \rho(1)B - \rho(2)B^2 - \dots\big) X(t) = W(t)$. Therefore, the existence of a unit root is equicalent to $\tilde{\rho}(1) = 1$ in (18.3). This is the so called *augmented Dickey Fuller test*, the t-test for the null hypothesis

$$H_0 : \tilde{\rho}(1) = 1,$$

based on the regression (18.3).

IV.B.2 Autocorrelation and Partial Autocorrelation

In order to determine the size of the model, i.e. the parameters p and q, graphical tools can be used to determine the range of those parameters, before applying the information criteria of Definition IV.A.3. These graphical tools are based on additional information on the distribution of $X(t)$.

Definition IV.B.4. The *auto-covariance function* of a stochastic process $\big(X(t)\big)_{t\in\mathbb{N}}$ is defined by

$$\begin{aligned}\gamma(s,t) &= Cov(X(s),X(t)) \\ &= \mathbb{E}[(X(s)-\mu(s))(X(t)-\mu(t))], \quad \text{for all } s,t\in\mathbb{N},\end{aligned}$$

where $\mu(t)$ denotes the mean of $X(t)$. Similarly, the *auto-correlation function* is defined as

$$\begin{aligned}\rho(s,t) &= Corr(X(s),X(t)) \\ &= \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}, \quad \text{for all } s,t\in\mathbb{N}.\end{aligned}$$

Example IV.B.5. Assume $X(t)$ is an MA(q) process, i.e.

$$X(t) = \sum_{i=0}^{q} \theta(i)W(t-i),$$

where $\theta(0)=1$, and assume that the $W(t)$ are independent and identically distributed with $W(t)\sim\mathcal{N}(0,\sigma_W^2)$. Note that by the stationarity of $X(t)$, the covariance is invariant under time-shifts and we write γ without its second argument, i.e. $\gamma(h)=\gamma(h,0)$ for all $h\in\mathbb{N}$. Moreover,

$$\begin{aligned}\gamma(h) &= Cov\big(\sum_{i=0}^{q}\theta(i)W(t+h-i),\sum_{j=0}^{q}\theta(j)W(t-j)\big) \\ &= \begin{cases}\sigma_W^2\sum_{i=0}^{q-h}\theta(j)\theta(j+h) & 0\le|h|\le q, \\ 0 & \text{else.}\end{cases}\end{aligned}$$

The auto-correlation function is then

$$\rho(h) = \begin{cases} 1 & h=0 \\ \frac{\sum_{i=0}^{q-h}\theta(i)\theta(i+h)}{1+\theta^2(1)+\ldots+\theta^2(q)} & 1\le|h|\le q, \\ 0 & \text{else.}\end{cases}$$

Hence it is clear, that for an MA(q) process, the autocorrelation function is zero after a lag of q. This can be seen in Figure 18.1, where a realization of an MA(1) process is shown.

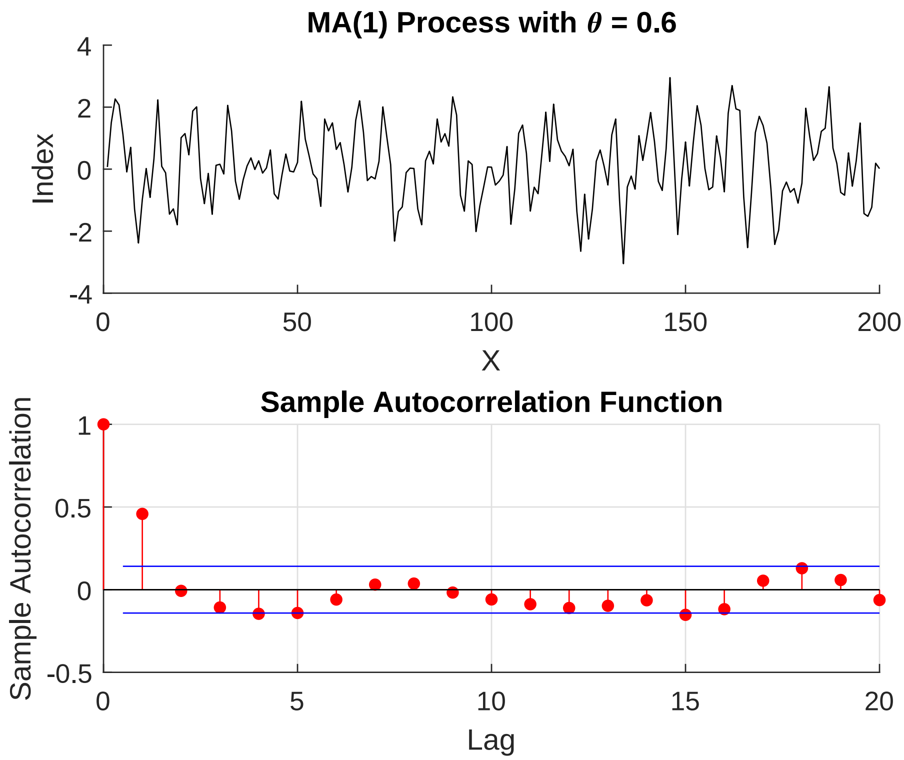


Figure 18.1: An example of an MA(1) process with θ=0.6. The autocorrelation function decreases below the confidence bounds after one lag, indicating the structure of the model.

For a general ARMA(p, q) process, we can write

$$\begin{aligned}\gamma(h) &= \sum_{i=1}^{p} \phi(i) Cov(X(h-i), X(0)) + \sum_{j=0}^{q} \theta(j) Cov(W(h-j), X(0)) \\ &= \sum_{i=1}^{p} \phi(i)\gamma(h-i) + \sum_{j=h}^{q} \theta(j) Cov(W(h-j), X(0)),\end{aligned}$$

where we have used the property that $W(t)$ is independent with $X(0)$ for $t > 0$. In particular, for $h \geq \max(p, q+1)$, the autocovariance function decays exponentially fast. Notice that the general pattern of the autocovariance function is not different from that of an AR(p) process. Hence, we cannot determine the presence of a moving average part simply on the autocorrelation function alone. Therefore, we pursue a function that will behave in similar ways to the autocorrelation function for MA models, but for AR models instead.

Consider an AR(1) model and write $X(t) = \phi X(t-1) + W(t)$. Then

$$\gamma(2) = Cov(\phi^2 X(t-1) + \phi W(t-1) + W(t), X(t-2) = \phi^2 \gamma(0).$$

The correlation between $X(t)$ and $X(t-2)$ is not zero, as the dependence is carried on through $X(t-1)$. The idea is now to remove the effect of $X(t-1)$ and consider the correlation between $X(t) - \phi X(t-1)$ and $X(t-2) - \phi X(t-1)$, which is clearly zero. For general stochastic processes $X(t)$, the best linear prediction of $X(t)$, based on $(X(r), \ldots, X(s))$, for $r \leq s$ is the linear combination

$$\hat{X}^{r:s}(t) = \sum_{j=0}^{s-r} \beta(k) X(s-k),$$

which minimizes the mean square error of prediction.

Definition IV.B.6. The *partial autocorrelation function* of a stationary stochastic process $X(t)$ is defined by

$$\tau(h) = Corr(X(0) - \hat{X}^{1:h-1}(0), X(h) - \hat{X}^{1:h-1}(h)).$$

Example IV.B.7. Assume $X(t)$ is an AR(p) process, i.e.

$$X(t) = \sum_{i=0}^{p} \phi(i) X(t-i) + W(t).$$

When $h > p$, it can be shown that the best linear prediction of $X(t+h)$ on $(X(t+1), \ldots, X(t+h-1))$ is

$$\hat{X}(t+h) = \sum_{i=1}^{p} \phi(i) X(t+h-i),$$

and hence

$$\begin{aligned}\tau(h) &= Corr(X(t+h) - \hat{X}(t+h), X(t) - \hat{X}(t)) \\ &= Corr(W(t+h), X(t) - \hat{X}(t)) = 0.\end{aligned}$$

Hence it is clear, that for an AR(p) process, the partial auto-correlation function is zero after a lag of p.

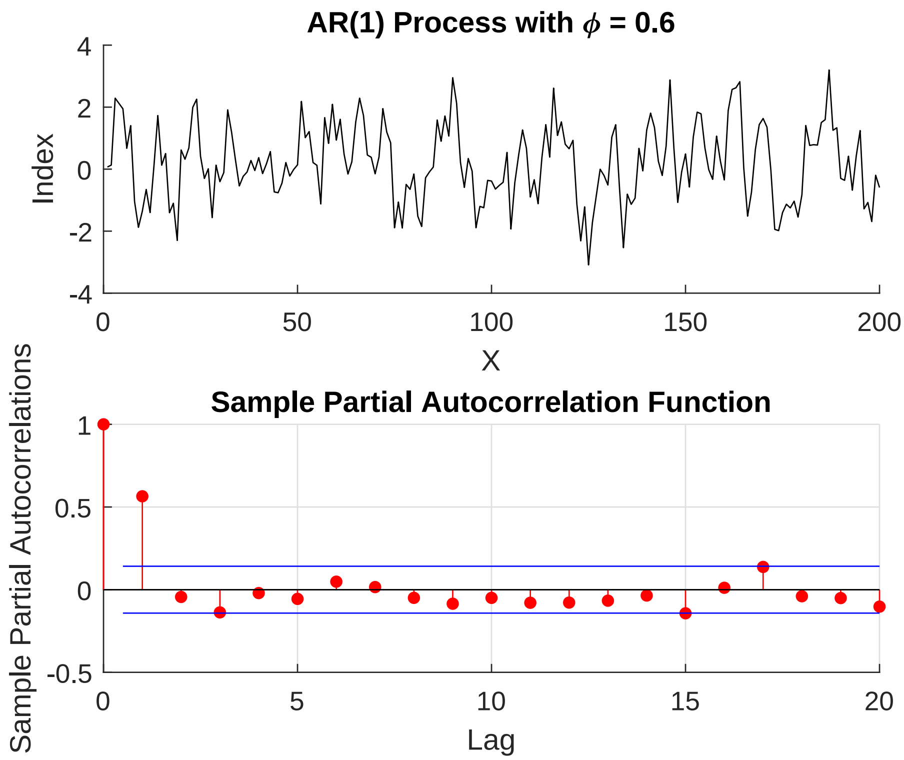


Figure 18.2: An example of an AR(1) process with ϕ=0.6. The partial autocorrelation function decreases below the confidence bounds after one lag.

Similar to the discussion above, one can show that the partial autocorrelation function of an $\mathrm{MA}(q)$ process never cuts off, as in the case of an $\mathrm{AR}(p)$ process. We summarize the findings in Table 18.1

Example IV.B.8. In Figures 18.2 and 18.1 we see a realization of an AR(1) and of an MA(1) process repectively. In both these models, it is quite clear from the autocorrelation function and the partial autocorrelation function, which model structure should be chosen. Now assume that $X(t)$ follows an $\mathrm{ARMA}(1,1)$ model and is given by

$$X(t) = \phi X(t-1) + W(t) + \theta W(t-1).$$

In Figure 18.3 we see a realization of such a process with $\phi = 0.8$ and $\theta = -0.6$ and the corresponding correlation graphics. Neither the autocorrelation function nor the partial autocorrelation function cut off after a

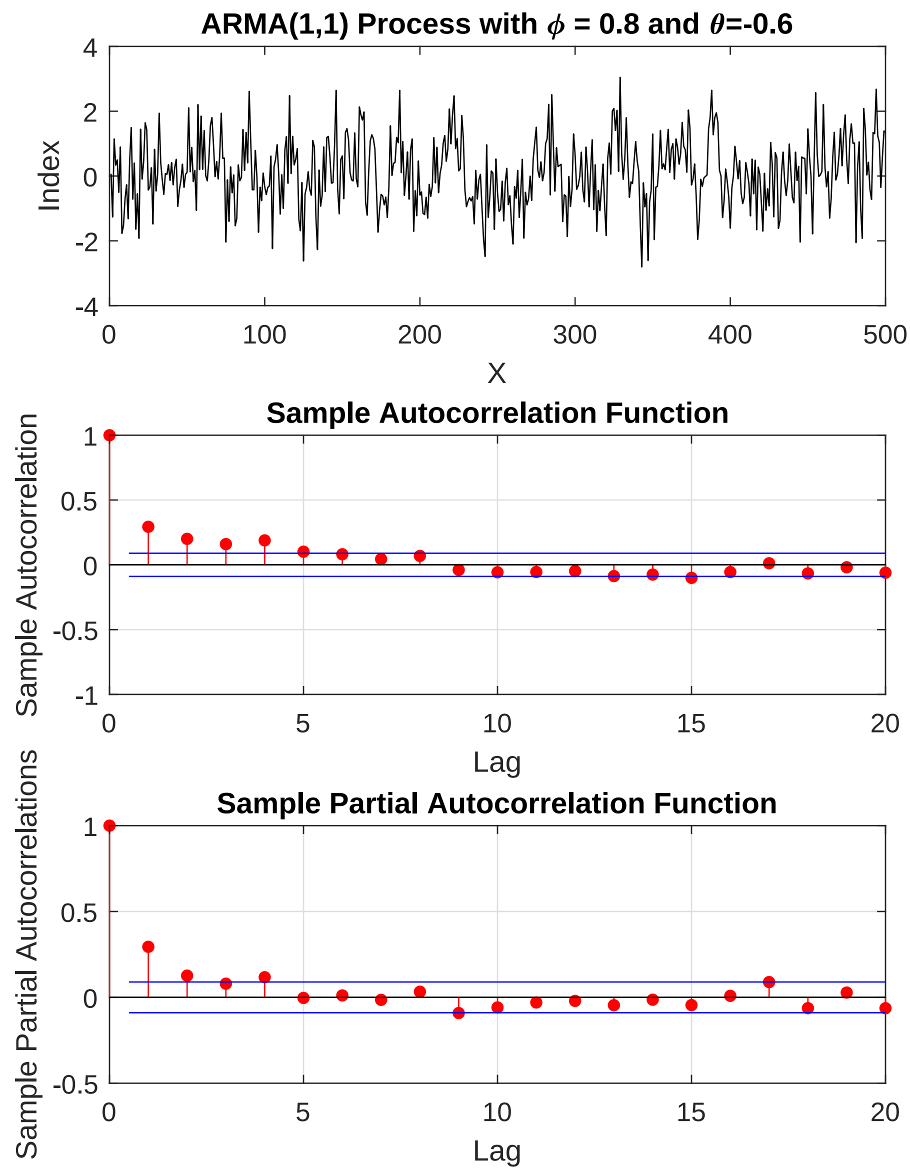


Figure 18.3: An example of an ARMA$(1, 1)$ process with ϕ=0.8 and $\theta = -0.6$. Both the autocorrelation function and the partial autocorrelation function decreases below the confidence bounds exponentially.

	AR(p)	**MA**(q)	**ARMA**(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

Table 18.1: Behavior of the ACF and the PACF of ARMA Models.

certain lag, hence we can guess the ARMA structure of the model. In order to find the size of the parameters p and q from the simulation alone, either the AIC or the BIC could be used.

IV.B.3 Using the ARMA Model on Financial Data

The ARMA model structure offers certain advantages in modelling financial parameters which show some degree of autocorrelation. It offers an accessible way to forecast future values of stationary time series based on its path and in turn offers us an easy way to predict certain parameters of the financial markets (2.2), (5.1) and (8.3) deterministically.

However, the ARMA model is not applicable to non-stationary time series. There are certain ways to make a time series stationary, e.g. detrending, differencing and non-linear transformations, but if there is evidence of conditional heteroscedasticity in the data, the requirements to apply the ARMA model are not met. We call a series of random variables heteroskedastic if there are certain subsets of variables, within the larger set, that have a different variance from the remaining variables. This is a clear violation of the stationarity of the underlying data, which would require the variance to be independent of time. This effect can frequently be seen in the variance of financial stocks, where an increase in variance is correlated to a further increase in variance.

One possible way find the presence of conditional heteroskedasticity in data is the Engle test for residual heteroscedasticity. We will not provide the theory behind this test, and instead refer to [Engle, 1982]. However, we note that a weakness of the Engle test is, that it assumes the heteroskedasticity to be a linear function of the underlying variables. Failing to find evidence of heteroskedasticity with the Engle test does not rule out a nonlinear relationship. The next section outlines one possible model which is able to model conditional heteroskedasticity.

IV.C The GARCH Model

The generalized autoregressive conditional heteroskedasticity model, henceforth GARCH, builds on the ARMA model of the previous section in the sence that it utilises an ARMA process for the variance itself. Introduced in [Engle, 1982] it eases the restriction of constant volatility in the Black-Scholes model is a discrete setting.

Definition IV.C.1. A stochastic process $\big(\epsilon(t)\big)_{t\in\mathbb{N}}$ is called GARCH(p,q) if

$$\epsilon(t) = \sigma(t)z(t),$$

where $z(t)$ conditionally follows some distribution D, i.e. $z(t)|\mathcal{F}_{t-1} \sim D(0,1)$ and $\sigma^2(t)$ is given by

$$\sigma^2(t) = \alpha(0) + \sum_{i=1}^{q} \alpha(i)\epsilon^2(t-i) + \sum_{j=1}^{p} \beta(j)\sigma^2(t-j),$$

and $\alpha(q) \neq 0 \neq \beta(p)$.

For most applications we will assume that the $z(t)$ are standard normal distributed.Note that in order for the conditional variance $\sigma^2(t)$ to remain positive, we assume that all the coefficients are positive. Positive coefficients are sufficient but not necessary conditions for the positivity of conditional variance. For more general conditions, we refer to [Nelson and Cao, 1992].

We can utilize most of the theory on ARMA processes for GARCH processes as well. This follows from the fact that a GARCH model can be expressed as an ARMA model of squared residuals. Consider for example a GARCH$(1,1)$ model

$$\sigma^2(t) = \alpha(0) + \alpha(1)\epsilon^2(t-1) + \beta(1)\sigma^2(t-1).$$

Since $\mathbb{E}^{\mathbb{P}}\big[\epsilon^2(t)|\mathcal{F}_{t-1}\big] = \sigma^2(t)$, this can be rewritten as

$$\epsilon^2(t) = \alpha(0) + \big(\alpha(1) + \beta(1)\big)\epsilon^2(t-1) + u(t) - \beta(1)u(t-1), \qquad (18.4)$$

which is an ARMA$(1,1)$ model with $u(t) = \epsilon^2(t) - \mathbb{E}^{\mathbb{P}}\big[\epsilon^2(t)|\mathcal{F}_{t-1}\big]$.

IV.C.1 Testing for GARCH Effects

By (18.4) we can test for heteroskedasticity using the autocorrelation function and the partial autocorrelation function, introduced in Section IV.B.2, of the squared returns. The significance of these autocorrelations can then be quantified by the Ljung-Box test, introduced in [Ljung and Box, 1978]. Let $\hat{\rho}(j)$ denote the j-lag sample autocorrelation of the squared returns. Then the Ljung-Box statistic is defined as

$$\mathrm{LJ}(p) = n(n+2) \sum_{j=1}^{p} \frac{\hat{\rho}^2(j)}{n-j}.$$

Under the null hypothesis of independently distributed data, one can show that $\mathrm{LJ}(p)$ follows a chi-squared distribution with p degrees of freedom.

After evidence for heteroskedasticity has been found in the model, a combination of the autocorrelation function and the partial autocorrelation function of the squared returns can be used together with either the AIC or the BIC to determine the size of the GARCH model.

Example IV.C.2. In Figure 18.4 we see a realization of a GARCH$(1,1)$ process with $\alpha(0) = 0.01, \alpha(1) = 0.45$ and $\beta(1) = 0.5$. As the autocorrelation function does not show any correlation, we can guess that the underlying process is not of the autoregressive form. As the autocorrelation function of the squared process does show significant correlation, however, the underlying GARCH structure becomes apparent.

IV.C.2 The ARMA/GARCH Model

By Definition IV.C.1, it is apparent, that a GARCH process has constant conditional mean, but nonconstant conditional variance. On the other hand, ARMA processes are just the opposite. If both the conditional mean, as well as the conditional variance depend on the past, those two models can be combined.

Definition IV.C.3. For $\epsilon(t) = \sigma(t)z(t)$, a stochastic process $\big(X(t)\big)_{t\in\mathbb{N}}$ is called ARMA(p_A, q_A)/GARCH(p_G, q_G) process, if

$$X(t) = \sum_{i=1}^{p_A} \phi(i) X(t-i) + \sum_{j=0}^{q_A} \theta(j)\epsilon(t-j),$$

where the $z(t)$ are independent of all $X(s)$ for $s < t$, identically standard normal distributed and $\phi(p_A) \neq 0 \neq \theta(q_A)$. Here, $\sigma(t)$ satisfies the GARCH equation

$$\sigma^2(t) = \alpha(0) + \sum_{i=1}^{q_G} \alpha(i)\epsilon^2(t-i) + \sum_{j=1}^{p_G} \beta(j)\sigma^2(t-j),$$

for $\alpha(q_G) \neq 0 \neq \beta(p_G)$.

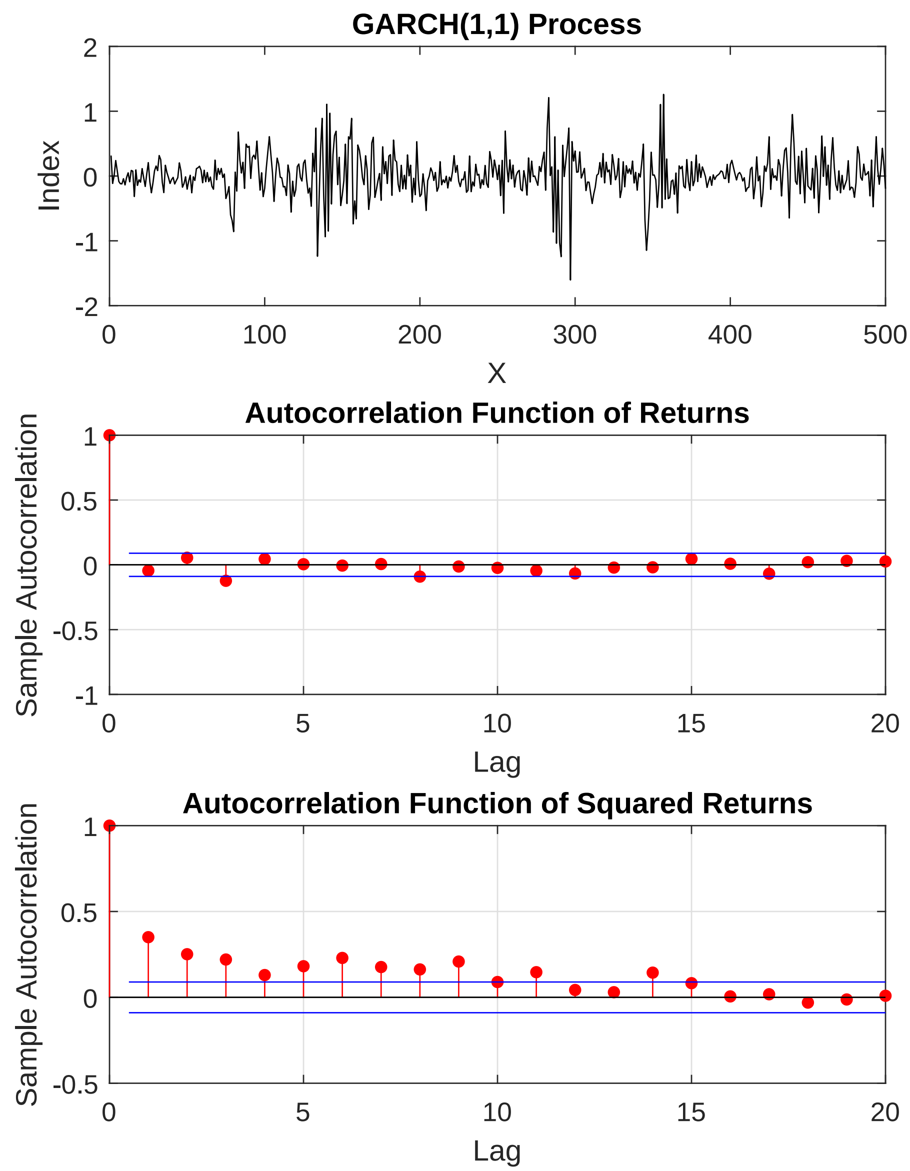


Figure 18.4: An example of a GARCH$(1,1)$ model with $\alpha(0) = 0.01, \alpha(1) = 0.45$ and $\beta(1) = 0.5$. Note that even though the process itself is uncorrelated, the squared process is not.

Note that since the $\epsilon(t)$ are independent, $X(t)$ still shows the auto-correlation function of an ARMA process, while $\epsilon^2(t)$ shows the auto-correlation function of a GARCH process.

IV.D Alternative Strategies

IV.D.1 The CRRA Utility Function

One of the more popular utlity function used in practice to determine investment strategies is the power utility function, given by

$$U(x) = \frac{x^\gamma}{\gamma}, \tag{18.5}$$

for $\gamma \in (-\infty, 1) \setminus \{0\}$, which is used as a factor of risk aversion. Define the relative risk aversion by

$$R(x) = -\frac{xU''(x)}{U'(x)}.$$

Then the power utility function has constant relative risk aversion, called CRRA, given by $R(x) = 1 - \gamma$.

Assume the market model (8.3) with a single risky stock and constant parameters and recall the notion of the wealth process (8.5), satisfying

$$dX^\pi(t) = \Big(r_N(t)X^\pi(t) + \pi(t)'\sigma(t)\theta(t) + \delta L(t)\Big)dt + \pi(t)'\sigma(t)dW(t).$$

Similar to Problem 3.2.3, we are interested in maximizing

$$J(x, \pi) = \mathbb{E}[U(X^\pi(T))],$$

over the class

$$\mathcal{A}_2(x) = \Big\{\pi \in \Pi \mid \mathbb{E}[U^-(X^\pi(T))] < \infty\Big\},$$

but instead of using the quadratic utility function, we use the power utility function (18.5). Denote the optimal portfolio process for a power utility maximizing investor by $\pi^*(t)$. Using the Lagrangian expression (3.9) with the admissibility constraint (9.4), we write

$$\mathcal{L} = \mathbb{E}^{\mathbb{P}}\big[U(X^\pi(T))\big] + y\big(x + d - \mathbb{E}^{\mathbb{P}}\big[U(X^\pi(T))\xi(T)\big]\big),$$

which reaches its extremum for $U(x) = \frac{x^\gamma}{\gamma - 1}$ at

$$X^{\pi^*}(T) = \big(y\xi(T)\big)^{1-\gamma}.$$

By setting

$$y = \Big(\frac{x + d}{\mathbb{E}^{\mathbb{P}}\big[\xi(T)^{\frac{\gamma}{\gamma-1}}\big]}\Big)^{\gamma - 1},$$

the admissibility constrained (9.4) is satisfied.

Proposition IV.D.1. *The optimal portfolio process for a power utility maximizing investor is given by*

$$\pi^*(t) = \left(\sigma'\right)^{-1}\left(\frac{\frac{1}{1-\gamma}\left(X^{\pi^*}(t) + D(t)\right)\theta - D(t)\sigma_L}{X^{\pi^*}(t)}\right), \tag{18.6}$$

where the optimal wealth process is given by

$$X^{\pi^*}(t) = (x+d)e^{\left(r_N + \frac{1}{2}\|\theta\|^2 t\left(1-(\frac{\gamma}{\gamma-1})^2\right) + \left(1-\frac{\gamma}{\gamma-1}\right)\theta' W(t)\right.} - D(t).$$

Proof. The proof follows along the same lines as the proofs of Proposition 3.3.1 and Theorem 3.4.3. We refer to [Zhang et al., 2007] for the complete proof. □

IV.D.2 The Dynamic Lifecycle Strategy

The dynamic lifecycle strategy proposed in [Basu et al., 2011] is used to compare the portfolio processes to an alternative more closely related to the investment strategies used in practice. Contrary to the deterministic lifecycle strategy, the plan member sets an annual target for the internal rate of return and the investment is adjusted depending if that target has been reached. We adjust the procedure taken in [Basu et al., 2011], such that the resulting strategy differs more heavily from the strategy investing 100% in the risky assets.

The strategy is divided into two periods. During the first period, set to one half of the investment horizon, 100% of the wealth is invested in the risky assets. During the second period, the investment behavior depends on the rate of return achieved. The investment in the risky assets is reduced linearly to zero if the target return set at the beginning is reached. However, failing to achieve the target at any point during the second half results in full investment in the risky assets once again. This procedure is repeated annually, so the portfolio may start reducing the investment multiple times.

For simplicity, we assume that the investment in risky assets is split equally between the stock and the inflation-linked bond. Finally, since the strategy carries more risk than the deterministic lifecycle strategy by definition, we do not calculate the target rate of return by setting the expected shortfall equal to that of the deterministic lifecycle strategy. This would simply result in a target rate of return of 0%. The target is set to the nominal interest rate instead.

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