Appendix D
Discrete Symmetries

Three discrete symmetries play an important role in modeling the fundamental interactions of nature: charge conjugation, parity, and time reversal. We here discuss these symmetries in the context of quantum electrodynamics. In the spirit of this book, our focus is on a concrete implementation of these symmetries given our choice of the Dirac matrices (3.5), (3.6) and the spinors (3.27)–(3.30). There is only a small number of fundamental interactions we are really interested in (the count may go down when unified theories are found or up when some phenomena are not properly described by existing theories; ideally it should end up at one). More general discussions of discrete symmetries in quantum field theory can, for example, be found in Sections 15.10 to 15.14 of [34] or Section 3.6 of [65].

D.1 Charge Conjugation

The basic idea of charge conjugation is to exchange the role of particles and antiparticles. It seems natural to assume that a systematic switch of matter and antimatter would not change the laws of nature.

The exchange of electrons and positrons can be described most conveniently in terms of the creation and annihilation operators introduced in Section 3.2.1. A self-adjoint, unitary, linear operator, $C = C^\dagger = C^{-1}$, is introduced by

$$C b^\sigma_{\boldsymbol{p}} C = -\mathrm{sgn}(\sigma)\, d^\sigma_{\boldsymbol{p}} \,, \qquad C d^\sigma_{\boldsymbol{p}} C = -\mathrm{sgn}(\sigma)\, b^\sigma_{\boldsymbol{p}} \,, \tag{D.1}$$

where the phase factors $-\mathrm{sgn}(\sigma)$ are a convenient choice for our spinor definitions (3.27)–(3.30). As the photon is its own antiparticle, we extend this definition by the transformation rule

$$C a^\alpha_{\boldsymbol{q}} C = -a^\alpha_{\boldsymbol{q}} \,, \tag{D.2}$$

where the phase factor -1 is introduced because we expect that the electric current density four-vector changes sign under charge conjugation and we want the interaction Hamiltonian (3.89) to be invariant. Note that, for any phase convention in the definitions (D.1) and (D.2), the free Hamiltonians (3.87) and (3.88) are invariant under charge conjugation. Finally, one should note that the (D.1) and (D.2) leave the canonical commutation and anticommutation relations for creation and annihilation operators invariant, which is a general requirement for all discrete symmetry operations. Whenever we introduce further particles into our basic Fock space, we must extend the above transformation rules for electrons, positrons, and photons.

The explicit construction of the linear operator C on Fock space is a straightforward exercise. We assume that the vacuum state possesses charge conjugation symmetry, or more precisely, $C\,|0\rangle = |0\rangle$. If $|\phi\rangle$ is any Fock space base vector obtained by acting with electron, positron and photon creation operators on the vacuum state $|0\rangle$ [see (1.11) and (1.17)], then $C\,|\phi\rangle$ is obtained by replacing all electron by positron creation operators, and vice versa; moreover, we must introduce a minus sign for every photon operator and for every electron/positron operator with positive spin. The extension from base vectors to arbitrary vectors is given by the linearity of C. The properties $C = C^\dagger = C^{-1}$ are obvious from this explicit construction. It often is more convenient to specify the rules (D.1), (D.2) than to describe the explicit construction of C, which directly reflects these rules.

The definition (D.1) implies the following transformation behavior for the spinor field (3.82),

$$C\psi_{\boldsymbol{x}}C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \bar{\psi}^T_{\boldsymbol{x}} \,, \tag{D.3}$$

where the matrix occurring in (D.3) is given by $-\mathrm{i}\gamma^2\gamma^0$. For the electric current density four-vector (3.91), the definition (D.1) implies the anticipated transformation behavior

$$CJ^\mu_{\boldsymbol{q}}C = -J^\mu_{\boldsymbol{q}} \,, \tag{D.4}$$

where the symmetry properties required for verifying this identity,

$$\mathrm{sgn}(\sigma\sigma')\; \bar{u}^\sigma_{\boldsymbol{p}}\,\gamma^\mu\, u^{\sigma'}_{\boldsymbol{p}'} = \bar{u}^{-\sigma'}_{\boldsymbol{p}'}\,\gamma^\mu\, u^{-\sigma}_{\boldsymbol{p}} \,, \tag{D.5}$$

and

$$\mathrm{sgn}(\sigma\sigma')\; \bar{v}^{-\sigma}_{-\boldsymbol{p}}\,\gamma^\mu\, u^{\sigma'}_{\boldsymbol{p}'} = -\bar{v}^{\sigma'}_{\boldsymbol{p}'}\,\gamma^\mu\, u^{-\sigma}_{-\boldsymbol{p}} \,, \tag{D.6}$$

follow directly from Table 3.1.

With the additional transformation rules

$$CB_{\boldsymbol{q}}C = -B_{\boldsymbol{q}}\,, \qquad CD_{\boldsymbol{q}}C = -D_{\boldsymbol{q}}\,, \tag{D.7}$$

for the ghost photons, we can keep both the ghost contribution to the free Hamiltonian (3.111) and the BRST charge operators of the interacting theory in (3.125) and (3.126) invariant under charge conjugation. We have thus established the full charge conjugation symmetry of quantum electrodynamics in the BRST approach.

D.2 Parity

The unitary linear operator P describing the parity transformation, or point reflection at the origin, should reverse the momentum of every particle without flipping its spin,

$$Pb^{\sigma}_{\boldsymbol{p}}P = b^{\sigma}_{-\boldsymbol{p}}\,, \qquad Pd^{\sigma}_{\boldsymbol{p}}P = -d^{\sigma}_{-\boldsymbol{p}}\,, \tag{D.8}$$

where the relative sign between the two parts of (D.15) is very important for obtaining proper transformation formulas for the current density and the field operator. This transformation implies

$$PJ^{\mu}_{\boldsymbol{q}}P = -J_{-\boldsymbol{q}\,\mu}\,, \tag{D.9}$$

for the electric current density four-vector. Note that lowering the index μ implies that only the spatial components of the four-vector are reversed, whereas the temporal component (i.e., the charge density) remains unchanged under the parity transformation. Invariance of the interaction Hamiltonian (3.89) requires the same transformation behavior for the four-vector potential $A_{\boldsymbol{q}\,\mu}$ or, in view of the definitions (3.73) and (3.78), the somewhat more technical rules

$$Pa^0_{\boldsymbol{q}}P = a^0_{-\boldsymbol{q}}\,, \quad Pa^1_{\boldsymbol{q}}P = a^1_{-\boldsymbol{q}}\,, \quad Pa^2_{\boldsymbol{q}}P = -a^2_{-\boldsymbol{q}}\,, \quad Pa^3_{\boldsymbol{q}}P = a^3_{-\boldsymbol{q}}\,. \tag{D.10}$$

Invariance of the BRST charge operators in (3.125) and (3.126) under the parity transformation requires the transformation behavior

$$PB_{\boldsymbol{q}}P = B_{-\boldsymbol{q}}\,, \qquad PD_{\boldsymbol{q}}P = D_{-\boldsymbol{q}}\,, \tag{D.11}$$

for the ghost photons. We have now established the full symmetry of quantum electrodynamics under parity transformations in the BRST approach. For completeness, we also specify the transformation behavior of the spinor field (3.82),

$$P\psi_{\boldsymbol{x}}P = \gamma^0\psi_{-\boldsymbol{x}}\,, \tag{D.12}$$

which reflects the character of a simple point reflection at the origin most clearly.

D.3 Time Reversal

In our Hamiltonian setting, time reversal is more difficult to express than charge conjugation or parity. We take reversal of both the momentum and the spin of every particle as a hallmark of time reversal. If $|\phi\rangle$ is any Fock space base vector obtained by acting with electron and positron creation operators on the vacuum state, $T\,|\phi\rangle$ is obtained by reversing all momenta and all spins and by introducing a minus sign for every originally negative electron spin and positive positron spin. The extension of the operator T from base vectors to arbitrary vectors is now achieved by assuming antilinearity rather than linearity [see (1.3) vs. (1.4)], which is a general alternative option for expressing symmetries (according to Wigner's theorem (1931), any symmetry transformation is represented either by a linear and unitary or by an antilinear and antiunitary transformation of Hilbert space). It can be shown that a linear unitary operator cannot express time reversal (see p. 67 of [65]) whereas, for any antilinear T, linear H, and real t, the invariance condition $[T, H] = 0$ implies the operator identity

$$T\,\mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{-\mathrm{i}Ht}\,T\,, \tag{D.13}$$

which provides the best link to time reversal symmetry for Hamiltonian dynamics. If we define the adjoint $T^\dagger$ of any antilinear operator T by

$$s^{\mathrm{can}}(T\,|\phi\rangle\,, |\psi\rangle) = s^{\mathrm{can}}(|\phi\rangle\,, T^\dagger\,|\psi\rangle)^*, \tag{D.14}$$

for all vectors $|\phi\rangle$, $|\psi\rangle$, the above antilinear time reversal operator T has the properties $T = T^\dagger = T^{-1}$, just like the previously defined linear operators C and P.

After clarifying the detailed construction of the antilinear time reversal operator, we can now turn to the transformation behavior of the electron and positron annihilation operators under time reversal,

$$T b_{\boldsymbol{p}}^{\sigma} T = \mathrm{sgn}(\sigma)\, b_{-\boldsymbol{p}}^{-\sigma}\,, \qquad T d_{\boldsymbol{p}}^{\sigma} T = -\mathrm{sgn}(\sigma)\, d_{-\boldsymbol{p}}^{-\sigma}\,. \tag{D.15}$$

The occurrence of two antilinear factors of T on the left-hand side of these equations implies that linear operators are properly transformed into linear operators and that the adjoint of these equations can be formed in the usual way.

The definition (D.15) implies the following transformation behavior for the spinor field (3.82),

$$T\psi_{\boldsymbol{x}}T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \psi_{\boldsymbol{x}} \,, \tag{D.16}$$

where the matrix occurring in (D.16) is given by $\gamma^1\gamma^3$. By combing (D.5) with (3.44) and Table 3.1, we find the symmetry property

$$\mathrm{sgn}(\sigma\sigma')\;(\bar{u}^{\sigma}_{\boldsymbol{p}}\,\gamma^\mu\,u^{\sigma'}_{\boldsymbol{p}'})^* = -\bar{u}^{-\sigma}_{-\boldsymbol{p}}\,\gamma_\mu\,u^{-\sigma'}_{-\boldsymbol{p}'}\,, \tag{D.17}$$

and similarly

$$\mathrm{sgn}(\sigma\sigma')\;(\bar{v}^{-\sigma}_{-\boldsymbol{p}}\,\gamma^\mu\,u^{\sigma'}_{\boldsymbol{p}'})^* = -\bar{v}^{\sigma}_{\boldsymbol{p}}\,\gamma_\mu\,u^{-\sigma'}_{-\boldsymbol{p}'}\,. \tag{D.18}$$

These identities allow us to derive the following transformation behavior for the electric current density four-vector (3.91),

$$TJ^\mu_{\boldsymbol{q}}T = -J_{-\boldsymbol{q}\,\mu}\,. \tag{D.19}$$

In view of the antilinear character of T, this transformation behavior actually translates into

$$TJ^\mu_{\boldsymbol{x}}T = -J_{\boldsymbol{x}\,\mu}\,, \tag{D.20}$$

so that the flip of the sign of $\boldsymbol{q}$ now does not express a point reflection at the origin.

As the transformation law (D.19) for time reversal coincides with (D.9) for the parity transformation, we can immediately write down the transformation laws for photons and ghost particles,

$$Ta^0_{\boldsymbol{q}}T = a^0_{-\boldsymbol{q}}\,, \quad Ta^1_{\boldsymbol{q}}T = a^1_{-\boldsymbol{q}}\,, \quad Ta^2_{\boldsymbol{q}}T = -a^2_{-\boldsymbol{q}}\,, \quad Ta^3_{\boldsymbol{q}}T = a^3_{-\boldsymbol{q}}\,, \tag{D.21}$$

and

$$TB_{\boldsymbol{q}}T = B_{-\boldsymbol{q}}\,, \qquad TD_{\boldsymbol{q}}T = D_{-\boldsymbol{q}}\,, \tag{D.22}$$

which should be compared to (D.10) and (D.11). These equations complete our discussion of the time reversal symmetry of quantum electrodynamics in the BRST approach.

D.4 *CPT* Symmetry

From the transformations given in the previous sections, we realize the property

$$CPT\,X\,TPC = -X\,, \tag{D.23}$$

for all photon or ghost creation or annihilation operators X. The same transformation behavior is inherited by $X = A_{\boldsymbol{q}\,\mu}$. If a CPT invariant Hamiltonian contains a bilinear form in $A_{\boldsymbol{q}\,\mu}$ and $J^{\mu}_{\boldsymbol{q}}$, then also $X = J^{\mu}_{\boldsymbol{q}}$ must transform according to (D.23), as implied by (D.4), (D.9), and (D.19) for quantum electrodynamics. For the electron and positron annihilation operators, we have

$$CPT\, b^{\sigma}_{\boldsymbol{p}}\, TPC = d^{-\sigma}_{\boldsymbol{p}} \,, \qquad CPT\, d^{\sigma}_{\boldsymbol{p}}\, TPC = b^{-\sigma}_{\boldsymbol{p}} \,. \tag{D.24}$$

The transformation laws (D.23) and (D.24) are the basis for a very deep symmetry under CPT transformations, which is believed to hold for all fundamental interactions in nature. All the individual discrete symmetries are violated by at least one of the fundamental interactions. Most notable is the violation of CP symmetry, and hence of T symmetry, in the decay of neutral kaons. The CP violation is predominantly, or even exclusively, caused by the weak interaction.