Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
IDSC
Institut für Dynamische Systeme und Regelungstechnik

Dr. J. Tani, Prof. Dr. E. Frazzoli

151-0591-00 **Control Systems I** (HS 2018) **Solution**

Exercise Set 6 Transfer Functions 1: Definitions and Properties

leimarc@student.ethz.ch, 3rd November 2018

Exercise 1 (Control Block Calculations)

a) This is a relatively standard depiction of a feedback control system. The trick in finding the system transfer function Σ is generally to derive equations for all important signals separately and then combine them.

$$
\begin{aligned}
e &= r - \Sigma_3 y \\
y &= \Sigma_2 \Sigma_1 e
\end{aligned}
$$

Therefore $y = \Sigma_2\Sigma_1(r - \Sigma_3 y)$. This can be simplified to

$$
y = \frac{\Sigma_2\Sigma_1}{1+\Sigma_2\Sigma_1\Sigma_3}\cdot r \tag{1}
$$

b) In this diagram we are confronted with two loops of the type in **Case C**. Again we write equations for all important signals. Another approach could be to apply **Case C** two times.

$$
\begin{aligned}
e_1 &= u - y \\
e_2 &= e_1 - \Sigma_1 e_2 \\
y &= \Sigma_2\Sigma_1 e_2
\end{aligned}
$$

We can solve the second equation to be $e_2 = \frac{e_1}{1+\Sigma_1}$. Then we substitute:

$$
\begin{aligned}
y = \Sigma_2\Sigma_1 e_2 &= \frac{\Sigma_2\Sigma_1 e_1}{1+\Sigma_1} \\
&= \frac{\Sigma_2\Sigma_1(u-y)}{1+\Sigma_1} = \frac{\Sigma_2\Sigma_1 u}{1+\Sigma_1} - \frac{\Sigma_2\Sigma_1 y}{1+\Sigma_1} \\
y\left(1+\frac{\Sigma_2\Sigma_1}{1+\Sigma_1}\right) &= \frac{\Sigma_2\Sigma_1 u}{1+\Sigma_1}
\end{aligned}
$$

$$
y = \frac{\Sigma_2\Sigma_1}{1+\Sigma_1+\Sigma_2\Sigma_1}\cdot u \tag{2}
$$

c) This block diagram appears may appear more complex than it is:

$$
\begin{aligned}
y &= -\Sigma_1 u - \Sigma_2\Sigma_3 y \\
y(1+\Sigma_2\Sigma_3) &= -\Sigma_1 u
\end{aligned}
$$

$$
y = -\frac{\Sigma_1}{1+\Sigma_2\Sigma_3}\cdot u \tag{3}
$$

Exercise 2 (Definition and Derivation of the Transfer Function)

a) We apply Laplace with zero initial conditions:

$$\begin{aligned} \ddot{y}(t) + 5\dot{y}(t) + 10y(t) &= \dot{u}(t) + 10u(t) \quad /\mathcal{L} \\ s^2Y(s) - sy(0) - \dot{y}(0) + 5(sY(s) - y(0)) + 10Y(s) &= sU(s) - u(0) + 10U(s) \\ Y(s)(s^2 + 5s + 10) &= U(s)(s + 10) \end{aligned}$$

$$g(s) = \frac{Y(s)}{U(s)} = \frac{s + 10}{s^2 + 5s + 10} \tag{4}$$

b) System is given by:

$$\dot{x} = Ax + Bu \tag{5}$$
$$y = Cx + Du \tag{6}$$

now we can apply the Laplace transform to equations 5 and 6. We get:

$$\begin{aligned} sX(s) - x(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

As a next step, we have to remember that transfer functions are derived with zero initial conditions and solve for $g(s)$:

$$\begin{aligned} (sI - A)X(s) &= BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

$$\begin{aligned} X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$g(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \tag{7}$$

c)

$$g(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

With

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} s+2 & -3 \\ 0 & s+1 \end{bmatrix}$$

the inverse of this matrix can be described by

$$(sI - A)^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

and therefore

$$(sI - A)^{-1} = \frac{1}{(s+2)(s+1) - 0 \cdot (-3)} \cdot \begin{bmatrix} s+1 & 3 \\ 0 & s+2 \end{bmatrix}.$$

Therefore,

$$g(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \frac{1}{(s+2)(s+1) - 0 \cdot (-3)} \cdot \begin{bmatrix} s+1 & 3 \\ 0 & s+2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$
$$g(s) = \frac{s+1}{(s+2)(s+1)}$$

At this point, we can see that we have the term $s + 1$ both in the numerator and the denominator of the transfer function. With the cancellation of these factors, we lose information about our system, namely that there is a pole and a zero at $s = -1$. This phenomenon is called pole-zero cancellation. The transfer function can be written as

$$g(s) = \frac{1}{(s+2)} \tag{8}$$

Exercise 3 (Realisation of Transfer Functions)

a) The transfer function can be rewritten as

$$g(s) = \frac{s^3 + 3 \cdot s^2 + s + 5}{s^3 + 8 \cdot s^2 + 15s + 0}$$

This transfer function has a relative degree of 0. As stated in the task description (eq. 10), we want to have a part of the transfer function to be strictly proper and add a second term D.

Given the general form

$$g(s) = \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + a_2 s^s + a_1 s + a_0}$$

we can simply rewrite the transfer function as

$$g(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^s + a_1 s + a_0} + D$$

with

$$\begin{aligned} D &= \beta_0 \\ b_0 &= \beta_0 - D a_0 \\ b_1 &= \beta_1 - D a_1 \\ b_2 &= \beta_2 - D a_2 \end{aligned}$$

This formula could be expanded to a higher degree.

The solution then is

$$g(s) = \frac{-5 \cdot s^2 - 14 \cdot s + 5}{s^3 + 8 \cdot s^2 + 15 \cdot s + 0} + 1 \tag{9}$$

b) With the given definitions, we can easily determine the controllable canonical form to be

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -15 & -8 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

$$C = \begin{bmatrix} 5 & -14 & -5 \end{bmatrix} \; D = 1 \tag{11}$$

c) No, we can derive the controllable canonical form for all systems, even if they were not controllable or observable in the first place. As you saw in exercise 2, observability and controllability are no conditions to derive a transfer function from a given state space description. As the same holds for the derivation of the controllable canonical form, it follows that neither controllability or observability are necessary to derive the minimal realisation. In other words, we only realise the controllable and observable subspace of a given system when deriving the controllable canonical form.

This means that the non-controllable or non-observable parts of a system get lost during the transformation. The loss of information occurs in the frequency domain if we have a pole-zero-cancellation (see exercise 2.c).

Exercise 4 (Poles and Zeroes of a Transfer Functions)

a) We can determine the zeroes of this transfer function by solving the equation

$$Z(s) = s^2 + 5s + 3 = 0$$

It follows

$$z_{1,2} = \frac{-5 + -\sqrt{13}}{2}$$

Equivalently, we can determine the poles of this transfer function by solving

$$P(s) = s(s+3)(s+5) = 0$$

It follows

$$p_{1,2,3} = 0, -3, -5$$

b) The given transfer function does not have any zeroes. The poles can be determined by solving the equation

$$(s+5)^3(s+2)(s-1) = 0$$

In is easy to see that the poles are

$$\begin{aligned} p_{1,2,3} &= -5 \\ p_4 &= -2 \\ p_5 &= -1 \end{aligned}$$

Note that this system actually has three poles at $s = -5$. The concept of multiple poles is important for future applications, such as the root locus method or the frequency response (exercise sets 8-9).

c) With

$$s^2 + 6s + 9 = (s+3)^2$$

We can observe a pole-zero cancellation, as the factors $(s+3)$ can be cancelled from the transfer function. Therefore, $g(s)$ simplifies to

$$g(s) = \frac{1}{(s+3)(s-2)}$$

and we can see that the transfer function has no zeroes and two poles at $p_1 = -3$ and $p_2 = 2$.

d) The given transfer function describes a 2nd order system. It has no zeroes and the poles can be calculated by solving the equation

$$s^2 + 2\sigma s + (\sigma^2 + \omega^2) = 0$$

It follows that

$$s_{1,2} = -\sigma \pm i \cdot \omega$$

Exercise 5 (Causality of Transfer Functions)

a) With the given transfer function, we can see that $degree(Y) = 3 > degree(U) = 2$. Therefore, the system is strictly proper.

b) With the given transfer function, we can see that $degree(Y) = 0 < degree(U) = 5$. Therefore, the system is improper. It is important to note that such a system is not realisable.

c) With the given transfer function, the factor $(s + 3)$ is part of the numerator and the denominator polynomial. As we will see in further lectures, it is unproblematic to cancel these factors as the pole and zero described are negative. In this case we have a strictly proper system with $degree(Y) = 2 < degree(U) = 0$.