## Tentative schedule

<table>
<thead>
<tr>
<th>#</th>
<th>Date</th>
<th>Topic</th>
<th>Chapter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sept. 23</td>
<td>Introduction, Signals and Systems</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Sept. 30</td>
<td>Modeling, Linearization</td>
<td>2, 3</td>
</tr>
<tr>
<td>3</td>
<td>Oct. 7</td>
<td>Analysis 1: Time response, Stability</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>Oct. 14</td>
<td>Analysis 2: Diagonalization, Modal Coordinates</td>
<td>5.1, 5.4-5.5</td>
</tr>
<tr>
<td>5</td>
<td>Oct. 21</td>
<td>Transfer functions 1: Definition and properties</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>Oct. 28</td>
<td>Transfer functions 2: Poles and Zeros</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>Nov. 4</td>
<td>Introduction to Feedback: internal stability, root locus</td>
<td>5.2-3, 9.1-3, 13.3</td>
</tr>
<tr>
<td>8</td>
<td>Nov. 11</td>
<td>Frequency response</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>Nov. 18</td>
<td>Analysis of feedback systems 2: the Nyquist condition</td>
<td>9.4-6</td>
</tr>
<tr>
<td>10</td>
<td>Nov. 25</td>
<td>Specifications for feedback systems</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>Dec. 2</td>
<td>Feedback control synthesis 1</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>Dec. 9</td>
<td>Feedback control synthesis 1, continued</td>
<td>11</td>
</tr>
<tr>
<td>13</td>
<td>Dec. 16</td>
<td>Feedback control synthesis 2</td>
<td>12</td>
</tr>
<tr>
<td>14</td>
<td>Dec. 23</td>
<td>Implementation issues</td>
<td>14</td>
</tr>
</tbody>
</table>
Examples

\[ r \rightarrow k \rightarrow \frac{s+1}{s^2+s+1} \rightarrow y \]

\[
\begin{align*}
\text{Re} & \quad \text{Im} \\
& \quad \times \\
& \quad \times \\
& \quad \times
\end{align*}
\]
Examples

$$\frac{1}{(s+1)(s^2+s+1)}$$
Stabilizing an inverted pendulum

The equations of motion for an inverted pendulum (see lecture 3) can be written as

\[ ml^2 \ddot{\theta} = mg l \theta + u; \]

The transfer function is given by

\[ G(s) = \frac{1}{s^2 - g/l}. \]

How does the pendulum’s behavior change if I attach a spring to the pendulum?
Proportional control of an inverted pendulum

\[ k \frac{1}{s^2 - g/l} \]
Proportional-derivative control of an inverted pendulum

\[ r \rightarrow k_p + k_d s \rightarrow \frac{1}{s^2 - g/l} \rightarrow y \]

\[ \text{Re} \quad \text{Im} \]
Root locus summary (for now)

- Great tool for back-of-the-envelope control design, quick check for closed-loop stability.

- Qualitative sketches are typically enough. There are many detailed rules for drawing the root locus in a very precise way: if you really need to do that, just use matlab or other methods.

- Closed-loop poles start from the open loop poles, and are “repelled” by them.

- Closed-loop poles are “attracted” by zeros (or go to infinity). Here you see an obvious explanation why non-minimum-phase zeros are in general to be avoided.

- Remember that the root locus must be symmetric wrt the real axis.
Classical methods for feedback control

- Remember: the main objective is to assess/design the properties of the closed-loop system by exploiting the knowledge of the open-loop system, and avoiding complex calculations. We have three main methods:

  - **Root Locus**
    - Quick assessment of control design feasibility. The insights are correct and clear.
    - Can only be used for finite-dimensional systems (e.g. systems with a finite number of poles/zeros)
    - Difficult to do sophisticated design.
    - Hard to represent uncertainty.

  - **Nyquist plot**
    - The most authoritative closed-loop stability test. It can always be used (finite or infinite-dimensional systems)
    - Easy to represent uncertainty.
    - Difficult to draw and to use for sophisticated design.

  - **Bode plots**
    - Potentially misleading results unless the system is open-loop stable and minimum-phase.
    - Easy to represent uncertainty.
    - Easy to draw, this is the tool of choice for sophisticated design.
Frequency response

- Remember the definition of the transfer function: the steady-state response of a linear, time-invariant systems to a complex exponential input of the form $u(t) = e^{st}$ is $y_{ss}(t) = G(s)e^{st}$.

- In particular, if we choose $s = j\omega$ (i.e., the real part of the input is $\text{Re}[e^{j\omega t}] = \cos(\omega t)$), and we assume that the system is stable, then the steady-state output will be given by

$$y_{ss}(t) = G(j\omega)e^{j\omega t} = |G(j\omega)|e^{j\omega t + \angle G(j\omega)}.$$  

The real part of the input is now

$$\text{Re}[y_{ss}(t)] = |G(j\omega)| \cos(\omega t + \angle G(j\omega)).$$
In other words, the steady-state response to a sinusoidal input of frequency $\omega$ is a sinusoidal output of the same frequency such that:

1. the **amplitude** of the output is $|G(j\omega)|$ times the amplitude of the input;
2. the **phase** of the output lags the phase of the input by $\angle G(j\omega)$.
How can we display/visualize the frequency response?
The frequency response $G(j\omega) \in \mathbb{C}$ is a complex function of a single real argument $\omega \in \mathbb{R}$.

We basically have two options to plot the frequency response:

1. A parametric curve showing $G(j\omega)$ in the complex plane, in which $\omega$ is implicit. This leads to the polar plot and eventually to the Nyquist plot.

2. Two separate plots for, e.g., real and imaginary part of $G(j\omega)$ or — better — the magnitude and phase of $G(j\omega)$ as a function of $\omega$. The latter choice leads to the Bode plot.
The Bode Plot

- The Bode plot is actually composed of two plots: the magnitude and the phase plots.

- On the horizontal axis of both plots, we report the frequency $\omega$ on a logarithmic scale (base 10).

- On the vertical axis we report:
  1. The logarithm of $G(j\omega)$ (in base 10), or, equivalently, in dB (deciBels). Note that we use the convention that

     $$|G(j\omega)|[dB] = 20 \log_{10} |G(j\omega)|.$$ 

     Note that “one decade” = 20 dB.
  2. The phase $\angle G(j\omega)$. Usually given in degrees (ok to use radians though).

- Note: Since magnitudes multiply (i.e., their logs add) and phases add, these choices of vertical coordinates makes it possible to just add Bode plots of serial connections.

- Also, inverting the transfer function is equivalent to reflection about the horizontal axis, in both Bode plots.
Bode plots from data

- Assuming that we have a stable plant, we can choose several values of $\omega$, and for each measure amplitude and phase of the steady-state response to a sinusoidal input $\sin(\omega t)$.

- No need for an analytical models—but one can derive an analytical model from the experimental frequency response.

- One can also maintain error bounds for the uncertainty in the frequency response.
If $G(s) = k > 0$, then clearly

$$|G(j\omega)| = k, \quad \angle G(j\omega) = 0^\circ.$$
The frequency response of $G(s) = 1/s$ is simply $G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega}$, hence

$$|G(j\omega)| = \frac{1}{\omega}, \quad \angle G(j\omega) = -90^\circ.$$
Consider \( G(s) = \frac{1}{\tau s + 1} \), with \( \tau = -1/p > 0 \).

It is useful to construct an approximation of the Bode plots, for \( \omega \to 0^+ \), and for \( \omega \to +\infty \). We have:

1. For \( \omega \to 0^+ \), \( G(j\omega) \approx 1 \), i.e.,
   \[
   |G(j\omega)| \approx 1 = 0\, dB, \quad \angle G(j\omega) \approx 0.
   \]

2. For \( \omega \to +\infty \), \( G(j\omega) \approx \frac{1}{j\tau \omega} \), and
   \[
   |G(j\omega)| \approx \frac{1}{\tau \omega}, \quad \angle G(j\omega) \approx -90^\circ.
   \]

3. For \( \omega = 1/\tau \), \( G(j\omega) = \frac{1}{j + 1} \), yielding
   \[
   |G(j\omega)| = \frac{1}{\sqrt{2}} \approx -3\, dB, \quad \angle G(j\omega) = -45^\circ
   \]
Asymptotic Bode plots — single real, stable pole

\[ G(s) = \frac{1}{s+1} \]
Asymptotic Bode plots — complex-conjugate, stable poles

- Consider \( G(s) = \frac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1} \)
  (i.e., the poles are at \( p = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \)).

- It is useful to construct an approximation of the Bode plots, for \( \omega \to 0^+ \), and for \( \omega \to +\infty \). We have:
  1. For \( \omega \to 0^+ \), \( G(j\omega) \approx 1 \), i.e.,
     \[
     |G(j\omega)| \approx 1 = 0 \text{ dB}, \quad \angle G(j\omega) \approx 0.
     \]
  2. For \( \omega \to +\infty \), \( G(j\omega) \approx \frac{\omega_n^2}{-\omega^2} \), and
     \[
     |G(j\omega)| \approx \frac{\omega_n^2}{\omega^2}, \quad \angle G(j\omega) \approx -180^\circ.
     \]
  3. For \( \omega = \omega_n \), \( G(j\omega) = \frac{1}{2\zeta j} \), yielding
     \[
     |G(j\omega)| = \frac{1}{2\zeta}, \quad \angle G(j\omega) = -90^\circ.
     \]
Asymptotic Bode plots — complex-conjugate, stable poles

\[ G(s) = \frac{1}{s^2 + 0.5s + 1}, \quad G(s) = \frac{1}{s^2 + 2s + 1} \]
Asymptotic Bode plots — single real, minimum-phase zero

Consider $G(s) = \tau s + 1$, with $\tau = -1/z > 0$.

It is useful to construct an approximation of the Bode plots, for $\omega \to 0^+$, and for $\omega \to +\infty$. We have:

1. For $\omega \to 0^+$, $G(j\omega) \approx 1$, i.e.,
   
   $|G(j\omega)| \approx 1$, $\angle G(j\omega) \approx 0$.

2. For $\omega \to +\infty$, $G(j\omega) \approx j\tau \omega$, and
   
   $|G(j\omega)| \approx \tau \omega$, $\angle G(j\omega) \approx +90^\circ$.

3. For $\omega = 1/\tau$, $G(j\omega) = j + 1$, yielding
   
   $|G(j\omega)| = \sqrt{2}$, $\angle G(j\omega) = +45^\circ$. 
The Bode plots for $1/G(s)$ can be obtained by “flipping” the Bode plots for $G(s)$ about the horizontal (frequency) axis. Can you draw the Bode plots for a differentiator, for complex-conjugate minimum-phase zeros, etc.?
Asymptotic Bode plots — single real, non-minimum-phase zero

- Consider $G(s) = -\tau s + 1$, with $\tau = 1/z > 0$.

- It is useful to construct an approximation of the Bode plots, for $\omega \to 0^+$, and for $\omega \to +\infty$. We have:
  1. For $\omega \to 0^+$, $G(j\omega) \approx 1$, i.e.,
     $$|G(j\omega)| \approx 1 = 0 \text{ dB}, \quad \angle G(j\omega) \approx 0.$$
  2. For $\omega \to +\infty$, $G(j\omega) \approx -j\tau\omega$, and
     $$|G(j\omega)| \approx \tau \omega, \quad \angle G(j\omega) \approx -90^\circ.$$
  3. For $\omega = 1/\tau$, $G(j\omega) = -j + 1$, yielding
     $$|G(j\omega)| = \sqrt{2}, \quad \angle G(j\omega) = -45^\circ.$$
Asymptotic Bode plots — single real, non-minimum-phase zero

The phase plot for a non-minimum-phase zero is the same as that for a stable pole.
Putting it all together: Bode plots for complicated transfer functions
Example

- Sketch the Bode plots of

\[ G(s) = \frac{1}{2} \frac{(s + 2)(s + 10)}{(s^2 + s + 1)(s + 5)} \]

- First thing: write the transfer function in the "Bode" form:

\[ G(s) = 2 \frac{(s/2 + 1)(s/10 + 1)}{(s^2 + s + 1)(s/5 + 1)} \]

- Second: draw the Bode plot for each factor in the transfer function.

- Third: add all of the above together to get the final Bode plot.
Example

\[ G(s) = 2 \frac{(s/2 + 1)(s/10 + 1)}{(s^2 + s + 1)(s/5 + 1)} \]
Example

\[ G(s) = 2 \frac{(s/2 + 1)(s/10 + 1)}{(s^2 + s + 1)(s/5 + 1)} \]
Bode’s Law

- In the Bode plot, the magnitude slope and the phase are not independent.

- In particular, if the slope of the Bode magnitude plot is $\kappa$ dB/decade over a range of more than $\approx 1$ decade, the phase in that range will be approximately $\kappa \cdot 90^\circ$. 