Control Systems I
Lecture 12: Loop Shaping

Readings: Guzzella, Chapter 12
Åström and Murray, Chapter 11

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# Tentative schedule

<table>
<thead>
<tr>
<th>#</th>
<th>Date</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sept. 22</td>
<td>Introduction, Signals and Systems</td>
</tr>
<tr>
<td>2</td>
<td>Sept. 29</td>
<td>Modeling, Linearization</td>
</tr>
<tr>
<td>3</td>
<td>Oct. 6</td>
<td>Analysis 1: Time response, Stability</td>
</tr>
<tr>
<td>4</td>
<td>Oct. 13</td>
<td>Analysis 2: Diagonalization, Modal coordinates</td>
</tr>
<tr>
<td>5</td>
<td>Oct. 20</td>
<td>Transfer functions 1: Definition and properties</td>
</tr>
<tr>
<td>6</td>
<td>Oct. 27</td>
<td>Transfer functions 2: Poles and Zeros</td>
</tr>
<tr>
<td>7</td>
<td>Nov. 3</td>
<td>Analysis of feedback systems: internal stability, root locus</td>
</tr>
<tr>
<td>8</td>
<td>Nov. 10</td>
<td>Frequency response</td>
</tr>
<tr>
<td>9</td>
<td>Nov. 17</td>
<td>Analysis of feedback systems 2: the Nyquist condition</td>
</tr>
<tr>
<td>10</td>
<td>Nov. 24</td>
<td>Specifications for feedback systems</td>
</tr>
<tr>
<td>11</td>
<td>Dec. 1</td>
<td>PID Control</td>
</tr>
<tr>
<td>12</td>
<td>Dec. 8</td>
<td>Loop Shaping</td>
</tr>
<tr>
<td>13</td>
<td>Dec. 15</td>
<td>Implementation issues</td>
</tr>
<tr>
<td>14</td>
<td>Dec. 22</td>
<td>Robustness</td>
</tr>
</tbody>
</table>
Recap: Frequency-domain specifications

- The main point in using frequency domain specifications is to handle at the same time apparently conflicting requirements such as command tracking, and noise rejection.

- Typically commands and disturbances act at “low” frequency, e.g., no more than 10 Hz.

- Noise is typically a high-frequency phenomenon, e.g., more than 100 Hz.

- So we can reconcile both command tracking/disturbance rejection AND noise rejection by separating them frequency-wise!

  - Make \(|S(j\omega)| < < 1\) (hence \(|T(j\omega)| \approx 1\) at low frequencies.
    e.g., “ensure that commands are tracked with max 10% error up to a frequency of 10Hz.

  - Make \(|T(j\omega)| < < 1\) at high frequencies.
    e.g., “ensure that noise is reduced by a factor of 10 at the output at frequencies higher than 100 Hz.
Frequency-domain specifications on the Bode plot

- Frequency-domain specifications are usually expressed in terms of closed-loop frequency response.

- Can we write them in terms of the open loop frequency response? Indeed we can.

- Remember that for good command tracking / disturbance rejection, we want $|S(j\omega)| = |1 + L(j\omega)|^{-1}$ to be small at low frequencies, i.e., we want $|L(j\omega)|$ to be large at low frequencies.

- Typically this is written as $|S(j\omega)| \cdot |W_1(j\omega)| < 1$ for some function $|W_1(j\omega)|$ that is large at low frequency. This translates to $|1 + L(j\omega)| > |W_1(j\omega)|$, which is approximated as $|L(j\omega)| > |W_1(j\omega)|$.

- This can be seen as a “low frequency obstacle” on the magnitude Bode plot.
Frequency-domain specifications on the Bode plot

- For good noise rejection, we want $|T(j\omega)| = |L(j\omega)|/|1 + L(j\omega)|$ to be small at high frequencies.

- If $|T(j\omega)|$ is small, then $|L(j\omega)|$ has to be small, and $|T(j\omega)| \approx |L(j\omega)|$ (at high frequencies).

- Typically this is written as $|T(j\omega)| \cdot |W_2(j\omega)| < 1$ for some function $|W_2(j\omega)|$ that is large at high frequency. This translates to $|L(j\omega)| < |W_2(j\omega)|^{-1}$.

- This can be seen as a “high-frequency obstacle” on the Bode plot.
The bandwidth of the closed-loop system is defined as the maximum frequency $\omega$ for which $|T(j\omega)| > 1/\sqrt{2}$, i.e., the output can track the commands to within a factor of $\approx 0.7$.

Let $\omega_c$ be the crossover frequency, such that $|L(j\omega_c)| = 1$. If we assume that the phase margin is about 90°, then $L(j\omega_c) = -j$, and $T(j\omega_c) = 1/\sqrt{2}$.

In other words, the (open-loop) crossover frequency is approximately equal to the bandwidth of the closed-loop system.
Bode-plot “obstacle course”

Load disturbance attenuation
Robustness
High frequency measurement noise

\[ \log |L(i\omega)| \]

\[ \log \omega \]

\[ W_1 \]

\[ W_2^{-1} \]

\[ \omega_{gc} \]
Loop Shaping

- How can we “steer” the open-loop frequency response through the Bode obstacle course?

- **Proportional (static) compensation**: Choose a proportional controller with transfer function $C(s) = k$.

- **Dynamic compensation**, i.e., by choosing a controller (compensator) with transfer function $C(s)$ so that $L(s) = P(s)C(s)$ satisfies the requirements.

- **Basic idea**: the basic approach to designing a dynamic compensator is to choose one or more of the following elements:

  \[
  \text{Gain : } k, \quad \text{Lead : } \frac{s/a + 1}{s/b + 1} \quad (\text{with } 0 < a < b), \quad \text{Lag : } \frac{s/a + 1}{s/b + 1} \quad (\text{with } 0 < b < a).
  \]
The effects of proportional control (i.e., a simple gain $k$) are to shift the magnitude plot of the loop transfer function up and down. The phase plot is not affected.

If the system is open-loop stable, we know that small enough gains ($k \to 0$) yield stable closed-loops.

However, we may not be able to meet the other constraints (crossover/bandwidth, or command tracking/disturbance rejection), without compromising stability.
Loop shaping: Lead compensator

\[ C_{\text{lead}} = \frac{s/a + 1}{s/b + 1} = \frac{b}{a} \frac{s + a}{s + b}, \quad 0 < a < b \]

- A lead compensator has the following main effects on the frequency response:
  - Increase the magnitude at high frequencies, by \( b/a \); magnitude at low frequencies is not affected.
  - Increase the slope of the magnitude at frequencies between \( a \) and \( b \) by 20 dB/decade.
  - Increase the phase around \( \sqrt{ab} \) (i.e., the mid point between \( a \) and \( b \) on the Bode plot, by up to 90 degrees.

- A lead compensator approximates \( PD \) control as \( a \to 0 \) (and \( k/a \) remains finite).
Lead compensator Bode plot

\[ C(s) = \frac{s/0.1+1}{s/10+1} \]
Using a lead compensator

- The typical use of a lead compensator is to increase the gain margin.
  - Pick $\sqrt{ab}$ at the desired crossover frequency.
  - Pick $b/a$ depending on the desired phase increase (the larger $b/a$ the larger the phase increase, but no more than 90 degrees).
  - Adjust $k$ to put the crossover at the desired frequency.

- Possible side effect: increase magnitude at high frequencies, and hence noise sensitivity
Lead compensation example

- **Plant**: $P(s) = \frac{1}{s^2 + 2s + 2}$
- Desired phase margin: at least 45 degrees.
- Desired bandwidth: 5 rad/s.

$\Rightarrow C(s) = 52 \frac{s + 2.5}{s + 10}$
Loop shaping: Lag compensator

\[ C_{\text{lag}} = \frac{s/a + 1}{s/b + 1} = \frac{b}{a} \frac{s + a}{s + b}, \quad 0 < b < a \]

- A lag compensator has the following main effects on the frequency response:
  - Decrease the magnitude at high frequencies, by \( b/a \); magnitude at low frequencies is not affected.
  - Decrease the slope of the magnitude at frequencies between \( a \) and \( b \), by 20 dB/decade.
  - Decrease the phase around \( \sqrt{ab} \) (i.e., the mid point between \( a \) and \( b \) on the Bode plot, by up to 90 degrees.

- A lag compensator approximates \( PI \) control as \( b \to 0 \) (and \( kb \) remains finite).
Lag compensator Bode plot

\[ C(s) = \frac{s/10+1}{s/0.1+1} \]
Using a lag compensator

- The typical use of a lag compensator is to **improve command tracking/disturbance rejection**
  - Pick $a/b$ as the desired increase in magnitude at low frequencies.
  - Pick $a$ so that it is sufficiently smaller than the crossover frequency, not to affect crossover frequency and phase margin.
  - Increase the gain $k$ by $a/b$

- Possible side effect: phase lag at low frequencies, potential reduction of phase margin.
Lag compensation example

- Plant: \( P(s) = \frac{52s + 2.5}{s + 10} \cdot \frac{1}{s^2 + 2s + 2} \)
- Previous specs + desired stead-state error to a unit step: less than 1%.

\[
\Rightarrow C(s) = \frac{s + 0.1}{s + 0.1/15.3}
\]
A general procedure for open-loop stable systems

Proceed from “the left,” i.e., from low frequencies to high frequencies.

1. Figure out how many integrators are needed in $C(s)$. This depends on the order of the ramp that must be tracked with zero steady-state error.

2. Fix the gain in such a way that the low-frequency asymptote clears the command-tracking/disturbance specification (the low-frequency Bode obstacle).

3. Add terms of the form $(\tau s + 1)$ at the numerator or denominator in such a way that the Bode magnitude plot intersects the 0dB line with a slope of about 20 dB/s (this would give you a 90-degree phase margin. Poles steer “down”, zeros steer “up.” Note that normalizing the zero-order term to 1 makes it so that these do not affect the Bode plot on the left of the pole/zero.

4. Past the crossover, add poles as needed to clear the high-frequency Bode obstacle (noise reduction/uncertainty).
A general design example

Consider the system with transfer function

\[ P(s) = 0.01 \frac{(s + 10)^2}{s(s^2 + 2s + 2)}; \]

Design a feedback control system such that

- The closed-loop is stable;
- The steady-state error to a unit step is zero.
- Commands up to 0.05 rad/s are followed with at most 1% error.
- The phase margin is no less than 40 degrees;
- Noise at frequencies higher than 100 rad/s is attenuated by at least 40 dB.
A general design example
Why does PID work so well?

- The form of an “implementable” PID is the following:

\[
PID(s) = k \cdot \frac{(s/z_1 + 1)(s/z_2 + 1)}{s(s/p + 1)},
\]

where \( p >> 1 \) refers to an additional “fast” pole introduced to make \( PID(s) \) a proper transfer function.

- This can also be interpreted as follows:

\[
PID(s) = k \cdot \frac{s/z_1 + 1}{s + 0} \cdot \frac{s/z_2 + 1}{s/p + 1},
\]

i.e., a PID corresponds to an “extreme” lead-lag compensator, with one pole at \( s = 0 \), and one pole at \( -s = p >> 1 \)!
Loop shaping for non-minimum-phase/unstable systems

- Can we extend the techniques learned so far to systems with poles/zeros in the right half plane? We can — but please always double check with Nyquist and/or the root locus.

- That said... factor the plant transfer function as follows:

\[ P(s) = P_{mp}(s)D(s), \]

where

- \( P_{mp}(s) \) is obtained by replacing all poles/zeros of \( P(s) \) in the right half plane with their mirror image wrt the imaginary axis.

- \( D(s) \) contains all the poles/zeros of \( P(s) \) in the right half plane, times the inverse all the mirror images introduced in \( P_{mp}(s) \). Note that \(|D(j\omega)| = 1\) for all \( \omega \), i.e., \( D(s) \) is an all-pass filter. Choose the sign of \( D(s) \) so that the phase of \( D(j\omega) \) is negative.

- Example: if \( P(s) = \frac{s-z}{s-p} \), then \( P_{mp}(s) = \frac{s+z}{s+p} \), and \( D(s) = \frac{z-s}{s+z} \cdot \frac{s+p}{s-p} \).
Pretend that you have to stabilize a “nice” plant $P_{mp}(s)$, but a little “Bode demon” placed $D(s)$ inside your control system.

$D(s)$ does not alter the magnitude plot of $L(s)$, but messes up its phase plot—by introducing a possibly massive phase lag, and hence reducing the phase margin.
Loop shaping for non-minimum-phase system

\[ P(s) = \frac{1 - s}{s^2 + 2s + 2} \]

We have that \( D(s) = \frac{-s-1}{s+1} \). Note that \( \angle D(0) = 0 \), \( \angle D(j) = -90^\circ \), and \( \lim_{\omega \to \infty} \angle D(j\omega) = -180^\circ \).

- NMP zeros limit the crossover frequency and the feedback gain.
Loop shaping for open-loop unstable system

\[ P(s) = \frac{1}{(s + 2)(s - 1)} \]

- We have that \( D(s) = \frac{s+1}{s-1} \). Note that \( \angle D(0) = -180 \), \( \angle D(j) = -90^\circ \), and \( \lim_{\omega \to \infty} \angle D(j\omega) = 0^\circ \).

- Unstable open-loop poles force the crossover frequency to be large.
In this lecture, we covered the following material

- Specifications on the Bode plot;
- Loop-shaping concept for control design
- Use standard elements such as
  - Gain: increase crossover frequency, reduce steady-state error, etc.
  - Lead: increase the phase margin at crossover frequency.
  - Lag: decrease the steady-state error.
- ... or use poles/zeros to “steer” the Bode plot through the obstacle course.
- Non-minimum-phase zeros limit the crossover frequency (closed-loop bandwidth)
- Open loop unstable poles require the crossover frequency to be higher.