Control Systems I
Lecture 7: Feedback and the Root Locus method

Readings:

Jacopo Tani

Institute for Dynamic Systems and Control
D-MAVT
ETH Zürich

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Date:
Thursday, 8. November, 2018
Time:
18:00 - 20:00
Location:
ML Building
Sonneggstrasse 3, 8092 Zurich
Opening:
ML E 12
Demos:
Various locations

Open Lab 2018
Institute for Dynamic Systems and Control
Prof. R. D’Andrea, Prof. E. Frazzoli, Prof. Ch. Onder, Prof. M. Zeilinger
## Tentative schedule

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Today’s learning objectives

- Poles and their effects on the response
- Zeros and their effects on the response
- Zeros and derivative action
- Effects of non-minimum-phase zeros
- Standard feedback control configuration and transfer function nomenclature
- Well-posedness
- Internal vs. I/O stability
- The root locus method
- Routh-Hurwitz condition and stability
Transfer Functions?

Real System:

Modeling:
\[ \dot{z} = f(z, w) \]
\[ \zeta = g(z, w) \]

Linearization/Normalization:
\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

Transfer Function (Today!)

Control System Design
From State Space to Transfer Function

- State-space model of LTI continuous time systems:
  \[
  \dot{x}(t) = Ax(t) + Bu(t) \\
  y(t) = Cx(t) + Du(t)
  \]

- Transfer function:
  \[
  G(s) = C(sI - A)^{-1}B + D
  \]

- State-space model of LTI discrete time systems:
  \[
  x[k + 1] = A_dx[k] + B_du[k] \\
  y[k] = C_dx[k] + D_du[k]
  \]

- In analogy to continuous time, one can obtain the transfer function for LTI discrete time systems:
  \[
  G(z) = C_d(zI - A_d)^{-1}B_d + D_d,
  \]

where \( z \in \mathbb{C} \) is the discrete time equivalent of \( s \).
In the general case (SISO system)

\[
G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d, \in \mathbb{C}.
\]
In the general case (SISO system)

\[ G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d, \in \mathbb{C}. \]

you can verify that the following is a minimal realization of \( G(s) \):

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & & & & \vdots \\
-a_0 & -a_1 & \ldots & \ldots & -a_{n-1}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
b_0 & b_1 & \ldots & b_{n-1}
\end{bmatrix}, \quad D = [d];
\]

This particular realization is called the controllable canonical form.
Different ways to write transfer functions

- **Partial fraction expansion**: useful to compute transient responses, and to assess how much different modes contribute to the response:

\[ G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \ldots + \frac{r_n}{s - p_n} + r_0, \]

where \( r_0, \ldots, r_n \) are called the “residues”.

- **Root-locus form**: This is useful to compute the value of \( G(s) \) “by hand”:

\[ G(s) = k_{rl} \frac{(s - z_1)(s - z_2) \ldots (s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_n)} \]

- **Bode form**: This is useful to use control design techniques like the Bode plot:

\[ G(s) = k_{Bode} \frac{(\frac{s}{-z_1} + 1)(\frac{s}{-z_2} + 1) \ldots (\frac{s}{-z_m} + 1)}{(\frac{s}{-p_1} + 1)(\frac{s}{-p_2} + 1) \ldots (\frac{s}{-p_n} + 1)} \]

- **Zeros**: \( z_1, \ldots, z_m \) are called the “zeros” of \( G(s) \), and are the roots of the numerator. \( G(z_i) = 0, i = 1, \ldots, m \).

- **Poles**: \( p_1, \ldots, p_n \) are called the “poles” of \( G(s) \), and are the roots of the denominator, which is the charachteristic polynomial of \( A \).

\[ \det(p_i I - A) = 0, i = 1, \ldots, n. \]
Impulse and Step responses

- Assume $D = 0$, $x(0) = 0$.
  - **Impulse response**: output when $u(t) = \delta(t)$:
    \[ y_{\text{imp}}(t) = \int_0^t Ce^{A(t-\tau)} B \delta(\tau) \, d\tau = Ce^{At} B. \]

  It is the same as the response to an initial condition $x(0) = B$.

- **Step response**: output when $u(t) = 1 = e^{0t}$:
  \[ y_{\text{step}}(t) = \int_0^t Ce^{A(t-\tau)} B \, d\tau = -CA^{-1}B + CA^{-1}e^{At} B \]

- The steady-state response for a step input is given by $y_{ss}(t) = G(0) = -CA^{-1}B$.

- For a first order system: $y_{\text{step}}(t) = y_{ss}(t)(1 - e^{at})$. I.e., the step response is the steady-state response minus the scaled impulse response.

The impulse response totally defines the response of a system (it is in fact the inverse Laplace transform of the transfer function)!
Higher-order system

- If we write the partial fraction expansion of \( G(s) \), assuming no repeated poles, we get
  \[
  G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \ldots + \frac{r_n}{s - p_n}.
  \]

- The response to an impulse will then be
  \[
  y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \ldots + r_n e^{p_n t}.
  \]

- The effect of the poles is then clear: each pole \( p_i \) generates a term of the form \( e^{p_i t} \) in the impulse response (and step response, etc.)

- As we know, these are simple exponentials if the pole \( p_i \) is real, and are sinusoids with exponentially-changing amplitude for complex-conjugate pole pairs.
Each pole $p_i = \sigma_i + j\omega_i$ with residue $r_i$ determines a term of the impulse response.

Each term's magnitude is bounded by $r_i e^{\sigma_i t}$ and oscillates at frequency $\omega_i$. 
Effects of zeros on the response

- Given $G(s)$ how can we compute the residues $r_i$? A convenient approach is the “cover-up” method.
  - For a non-repeated pole $p_i$ this takes the form:
    $$r_i = \lim_{s \to p_i} (s - p_i)G(s)$$
    which in practice means “remove the factor $(s - p_i)$ from the denominator and compute $G(p_i)$ only considering the other terms.”
  - This method works even for repeated poles $p_i$ with multiplicity $m_i$, the expression is somewhat more complex\(^1\).

- An alternative method is by “matching” (see example in the next slide).
- While the exponents in the terms of the response only depend on the poles $p_i$, the residues are affected by the zeros $z_i$.

\(^1\)Swarthmore College, Linear Physical Systems Analysis: tinyurl.com/brpwo2y.
Example

Consider

\[ G(s) = \frac{1}{(s + 1)(s + 1 + j)(s + 1 - j)} . \]

Using the cover-up method we get

\[ G(s) = \frac{1}{s + 1} + \frac{-1/2}{s + 1 + j} + \frac{-1/2}{s + 1 - j} . \]
Example - impulse response

Impulse Response

Time (seconds)

Amplitude

-0.2
0
0.2
0.4
0.6
0.8
1

p1
p2,p3
combined

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Example — adding a zero near a pole

Consider

\[ G(s) = \frac{s + 1 + \epsilon}{(s + 1)(s + 1 + j)(s + 1 - j)}. \]

Using the cover-up method we get

\[ G(s) \approx \frac{\epsilon}{s + 1} + \frac{1/2j}{s + 1 + j} + \frac{-1/2j}{s + 1 - j}. \]
A zero can reduce the residue (i.e., the effect) of a nearby pole.
Pole-zero cancellation

- What if a zero matches a pole exactly?

\[ G(s) = \frac{s + 1}{(s + 1)(s + 1 + j)(s + 1 - j)} = \frac{1}{(s + 1 + j)(s + 1 - j)}. \]

- One of the poles has been cancelled by the zero. Effectively its residue is zero, i.e.,

\[ G(s) = \frac{0}{s + 1} + \frac{1/2j}{s + 1 + j} + \frac{-1/2j}{s + 1 - j}. \]

- Recall from the modal (diagonal) form that the residue is also given by \( r_i = b_i c_i \); if the residue is zero, the \( i \)-th mode is either uncontrollable, unobservable, or both.

- This is ok if the \( i \)-th mode (i.e., \( p_i \)) is stable, but a big problem if it is unstable.

- **Avoid unstable pole-zero cancellation!**
More effects of zeros...
Integrator and Differentiator

\[ u(t) \xrightarrow{\int} y(t) = \int_{-\infty}^{t} u(\tau) \, d\tau \]

- If the input is \( u(t) = e^{st} \), then the output will be \( y(t) = \frac{1}{s} e^{st} \).
- Hence, the transfer function of an integrator is \( G(s) = \frac{1}{s} \).

\[ u(t) \xrightarrow{\frac{d}{dt}} y(t) = \frac{du(t)}{dt} \]

- If the input is \( u(t) = e^{st} \), then the output will be \( y(t) = se^{st} \).
- Hence, the transfer function of an integrator is \( G(s) = s \).
Zeros as derivative action

- If we have a transfer function $G(s) = (s + z) \tilde{G}(s)$, we can decompose it into

\[ G(s) = z \tilde{G}(s) + s \tilde{G}(s). \]

- If the impulse response of $\tilde{G}(s)$ is given by $\tilde{y}(t)$, and the impulse response of $G(s)$ is $y(t)$, then remembering that $s$ is the transfer function of a differentiator, we can write

\[ y(t) = z \tilde{y}(t) + \dot{y}(t). \]

- In other words, the zero is effectively adding a derivative term to the output. This typically has an “anticipatory effect”.
With and without a zero / derivative
Non-minimum-phase zeros

- We know that poles with positive real part result in an unstable system. (The output diverges over time.)

- What happens when zeros have positive real part?

- The stability of the system is preserved (since the growth/decay of the terms in the response is not affected by the zeros — only the respective residues)

- However, a zero in the right half plane effectively means a “negative” derivative action. This is the opposite of anticipatory — indeed the output will tend to move in the “wrong” direction initially.

- These are called non-minimum phase zeros and are typically very bad news for control engineers, they make our work much harder.

- (Typically the presence of non-minimum-phase zeros depends on the choice of the output — to make your life easier, choose another output and/or move the sensors!)
Minimum-phase vs. non-minimum-phase zeros

Impulse Response

Time (seconds)

Amplitude

-0.2
-0.1
0
0.1
0.2
0.3
0.4

no zero

z = -1

z=+1
Towards feedback control

So far we have looked at how a given system, represented as a state-space model or as a transfer function, behaves given a certain input (and/or initial condition).

Typically the system behavior may not be satisfactory (e.g., because it is unstable, or too slow, or too fast, or it oscillates too much, etc.), and one may want to change it. This can only be done by feedback control!

The methods we will discuss next provide

- An analysis tool to understand how the closed-loop system (i.e., the system + feedback control) will behave for different choices of feedback control.

- A synthesis tool to design a good feedback control system.
Standard feedback configuration

- Transfer functions make it very easy to compose several blocks (controller, plant, etc.) Imagine doing the same with the state-space model!

- (Open-) **Loop** gain: \( L(s) = P(s)C(s) \)

- **Complementary sensitivity:** (Closed-loop) transfer function from \( r \) to \( y \)

\[
T(s) = \frac{L(s)}{1 + L(s)}
\]

- **Sensitivity:** (Closed-loop) transfer function from \( r \) to \( e \)

\[
S(s) = \frac{1}{1 + L(s)}
\]
Concern #1: Well-posedness

Assume both the plant and the controller are (static) gains.

Then the denominator of the closed-loop transfer functions would be $1 + D_2 D_1$. If $D_2 D_1 = -1$, the whole interconnections does not make sense — it is not well posed.
Concern #2: (Internal) Stability

- It may be tempting to just check that the interconnection is I/O stable, i.e., check that the poles of $T(s)$ have negative real part.

- However, consider what happens if $C(s) = \frac{1}{s-1}$ and $P(s) = \frac{s-1}{s+1}$:
  - The interconnection is I/O stable: $T(s) = \frac{1}{s+2}$.
  - The closed-loop transfer function from $r$ to $u$ is unstable: $S(s)C(s) = \frac{s+1}{(s-1)(s+2)}$.

- While the system seems to be stable, internally the controller is blowing up. The pole-zero cancellation made the unstable controller mode unobservable in the interconnection.

- Internal stability requires that all closed-loop transfer functions between any two signals must be stable.

- Never be tempted to cancel an unstable pole with a non-minimum-phase zero or viceversa.
How to determine closed-loop stability?

- Assuming the feedback interconnection is well-posed and internally stable, then all that remains to do is to design $C(s)$ in such a way that all the poles of $T(s)$ have negative real part.

  - In principle one could just pick a trial design for $C(s)$, go to a computer (python, matlab, ...), and check what the closed-loop response (e.g., $T(s)$) looks like.

- However, it is desired to find a systematic way to choose $C(s)$, while doing as little calculations as possible. The first automatic control engineers were working with paper, pencil, and possibly a slide-rule.

  - All of classical control can be summarized in: “exploit the knowledge of the loop gain $L(s)$ to figure out the properties of the closed-loop transfer functions $T(s)$ and $S(s)$ with the least effort possible.”
Classical methods for feedback control

- Remember: exploit $L(s)$ to find “good” $T(s)$, $S(s)$, with least possible effort.
  There are three main methods:

  - **Root Locus** (today)
    - Quick assessment of control design feasibility. The insights are correct and clear.
    - Can only be used for finite-dimensional systems (e.g. systems with a finite number of poles/zeros)
    - Difficult to do sophisticated design.
    - Hard to represent uncertainty.

  - **Bode plots** (Lec. 8)
    - Potentially misleading results unless the system is open-loop stable and minimum-phase.
    - Easy to represent uncertainty.
    - Easy to draw, this is the tool of choice for sophisticated design.

  - **Nyquist plot** (Lec. 9)
    - The most authoritative closed-loop stability test. It can always be used (finite or infinite-dimensional systems)
    - Easy to represent uncertainty.
    - Difficult to draw and to use for sophisticated design.
Evan’s Root Locus method

- Invented in the late ’40s by Walter R. Evans.
- Useful to study how the roots of a polynomial (i.e., the poles of a system) change as a function of a scalar parameter, e.g., the “gain.”

![Block Diagram](image)

- Let us write the loop gain in the “root locus form”

\[
kL(s) = k \frac{N(s)}{D(s)} = k \frac{(s - z_1)(s - z_2) \ldots (s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_n)}
\]

- The sensitivity function is

\[
S(s) = \frac{1}{1 + kL(s)} = \frac{D(s)}{D(s) + kN(s)}
\]

- The closed-loop poles are the solutions of:

\[
D(s) + kN(s) = 0.
\]
The root locus rules

- What can we say about the closed-loop poles?

1. Since the degree of $D(s) + kN(s)$ is the same as the degree of $D(s)$, the number of closed-loop poles is the same as the number of open-loop poles.

2. For $k \to 0$, $D(s) + kN(s) \approx D(s)$, and the closed-loop poles approach the open-loop poles.

3. For $k \to \infty$,
   - and the degree of $N(s)$ is the same as the degree of $D(s)$, then $\frac{1}{k}D(s) + N(s) \approx N(s)$, and the closed-loop poles approach the open-loop zeros.
   - If the degree of $N(s)$ is smaller, then the “excess” closed-loop poles “go to infinity” (we will look into this more).

4. The closed-loop poles need to be symmetric w.r.t. the real axis (i.e., either real, or complex-conjugate pairs), because $D(s) + kN(s)$ has real coefficients.
More rules: The angle and magnitude rules

- Let us rewrite the closed-loop characteristic equation as

\[
\frac{N(s)}{D(s)} = -\frac{1}{k}
\]

5 **The angle rule** — Take the argument on both sides:

\[
\angle(s - z_1) + \angle(s - z_2) + \ldots + \angle(s - z_m) - \angle(s - p_1) - \angle(s - p_2) - \ldots - \angle(s - p_n) = \begin{cases} 
180^\circ (\pm q \ 360^\circ) & \text{if } k > 0 \\
0^\circ (\pm q \ 360^\circ) & \text{if } k < 0 
\end{cases}
\]

6 **The magnitude rule** — Take the argument on both sides:

\[
\frac{|s - z_1| \cdot |s - z_2| \cdot \ldots \cdot |s - z_m|}{|s - p_1| \cdot |s - p_2| \cdot \ldots \cdot |s - p_n|} = \frac{1}{|k|}
\]
Graphical interpretation

- All points on the complex plane that could potentially be a closed-loop pole (i.e., the root locus) have to satisfy the angle condition—which is essentially THE rule for sketching the root locus.

\[ \angle s - p_1 - \angle s - p_2 - \angle s - z_1 = \pm \text{integer multiple of } 360° \]

“The sum of the angles (counted from the real axis) from each zero to s, minus the sum of the angles from each pole to s must be equal to 180° (for positive k) or 0° (for negative k), ± an integer multiple of 360°.”
All points on the real axis are on the root locus.

- All points on the real axis to the left of an even number of poles/zeros (or none) are on the negative $k$ root locus.

- All points on the real axis to the right of an odd number of poles/zeros are on the positive $k$ root locus.

- When two branches come together on the real axis, there will be ‘breakaway” or “break-in” points.
Asymptotes

- So what happens when $k \to \infty$ and there are more open-loop poles than zeros? We can see, e.g., from the magnitude condition, that the “excess” closed-loop poles will have to go to “to infinity” ($s \to \infty$).
- Since this is the complex plane, we need to identify “in which direction” they go towards infinity. This is were we use the angle rule again.
- If we “zoom out” sufficiently far, the contributions from all the finite open-loop poles and zeros will all be approximately equal to $\angle s$, and the angle rule is approximated by $(m - n)\angle s = -(\angle - k \pm q360^\circ)$, $q \in \mathbb{N}$.
- In other words, as $k \to \infty$, the excess poles will go to infinity along asymptotes at angles of

$$\angle s = \frac{\angle - k \pm q360^\circ}{n - m}$$

- These asymptotes meet in a “center of mass” lying on the real axis at

$$s_{com} = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n - m}$$

(note that poles are “positive” unit masses, and zeros are “negative” unit masses in this analogy)

Examples
Examples

\[ r \xrightarrow{k} \frac{1}{(s-1)(s-2)} y \]

- Re
- Im

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Examples

\[ r \rightarrow k \rightarrow \frac{1}{s^2 + s + 1} \rightarrow y \]

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Examples

\[ r \rightarrow k \rightarrow \frac{1}{(s+1)(s^2+s+1)} \rightarrow y \]

\[ \text{Im} \]

\[ \text{Re} \]
Examples

\[ r \rightarrow k \rightarrow \frac{1}{(s+1)(s^2+s+1)} \rightarrow y \]

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Root locus summary (for now)

- Great tool for back-of-the-envelope control design, quick check for closed-loop stability.

- Qualitative sketches are typically enough. There are many detailed rules for drawing the root locus in a very precise way: if you really need to do that, just use a computer or other methods.

- Closed-loop poles start from the open loop poles, and are “repelled” by them.

- Closed-loop poles are “attracted” by zeros (or go to infinity). Here you see an obvious explanation why non-minimum-phase zeros are in general to be avoided.

- Remember that the root locus must be symmetric w.r.t. the real axis.

- If denominator factorization is non trivial, how to find immaginary axis crossings ($k$ that makes the closed loop system unstable)?
Stability and Routh-Hurwitz Condition

- We have seen how to determine the stability from eigenvalues of the matrix $A$ or the poles of the transfer function.

- Can the stability of a system be checked without having to determine the poles?

- Let $G = \frac{p(s)}{q(s)}$, where $q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$.

- A necessary condition for stability of linear systems $\longrightarrow$ All coefficients of $q(s)$ must have the same sign and non-zero if all of its roots are in the left-hand plane.

- A necessary and sufficient stability condition for linear systems $\longrightarrow$ Routh-Hurwitz condition
Routh-Hurwitz Condition: A first look

- Consider the characteristic polynomial

\[ q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0. \]
Consider the characteristic polynomial

\[ q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0. \]
Routh-Hurwitz Condition: A first look

Consider the characteristic polynomial

\[ q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0 \]

\[ \begin{array}{c|cccc}
  s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
  s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
  s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
  s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots \\
  s^0 & h_{n-1} & & & \\
\end{array} \]

\[ b_{n-1} = -\frac{1}{a_{n-1}} \left| \begin{array}{cc}
  a_n & a_{n-2} \\
  a_{n-1} & a_{n-3} \\
\end{array} \right|, \]
Routh-Hurwitz Condition: A first look

- Consider the characteristic polynomial

\[ q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0 \]

\[
\begin{array}{c|cccc}
  s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
  s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
  s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
  s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  s^0 & h_{n-1} & \\
\end{array}
\]

\[ b_{n-3} = -\frac{1}{a_{n-1}} \left| \begin{array}{cc}
  a_n & a_{n-4} \\
  a_{n-1} & a_{n-5} \\
\end{array} \right|, \]
Routh-Hurwitz Condition: A first look

Consider the characteristic polynomial

\[ q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0 \]

\[
\begin{array}{c|cccc}
  s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
  s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
  s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
  s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  s^0 & h_{n-1} & & & \\
\end{array}
\]

\[ c_{n-1} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}, \]
Routh-Hurwitz Condition: A first look

- Consider the characteristic polynomial

\[ q(s) = a_ns^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1s + a_0 \]

\[ \begin{array}{c|cccc}
  s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
  s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
  s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
  s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  s^0 & h_{n-1} & & & \\
\end{array} \]

\[ c_{n-3} = -\frac{1}{b_{n-1}} \begin{vmatrix}
  a_{n-1} & a_{n-5} \\
  b_{n-1} & b_{n-5}
\end{vmatrix}, \]
The Routh-Hurwitz criterion states that the number of roots of $q(s)$ with positive real parts is equal to the number of sign changes in the first column of the Routh table.

Example:

$$q(s) = s^3 + 10s^2 + 31s + 1030$$

<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
<th>31</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>10</td>
<td>1</td>
<td>1030</td>
</tr>
<tr>
<td>$s^1$</td>
<td>-72</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>103</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
The Routh-Hurwitz criterion states that the number of roots of $q(s)$ with positive real parts is equal to the number of sign changes in the first column of the Routh table.

Example:

$$q(s) = s^3 + 10s^2 + 31s + 1030$$

$$\begin{array}{c|ccc}
  s^3 & 1 > 0 & 31 & 0 \\
  s^2 & 1 > 0 & 103 & 0 \\
  s^1 & -72 < 0 & 0 & 0 \\
  s^0 & 103 > 0 & 0 & 0 \\
\end{array}$$
Special case: zero in the first column

- Replace the zero with $\epsilon$ (we can assume $\epsilon > 0$ or $\epsilon < 0$).

- Example:

\[
q(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3
\]

<table>
<thead>
<tr>
<th>$s^5$</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^4$</td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>$s^3$</td>
<td>$\emptyset$</td>
<td>$\epsilon$</td>
<td>7</td>
</tr>
<tr>
<td>$s^2$</td>
<td>$6\epsilon - 7$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Special case: zero in the first column

- Replace the zero with \( \epsilon \) (we can assume \( \epsilon > 0 \) or \( \epsilon < 0 \)).

- Example:

\[
q(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3
\]

| \( s^5 \) | 1 | \( \epsilon \) |
| \( s^4 \) | 2 | 3 |
| \( s^3 \) | 6 | 7 |
| \( s^2 \) | 6\( \epsilon \) - 7 | 2 |
| \( s^1 \) | \( 42\epsilon \) - 49 - 6\( \epsilon^2 \) | 0 |
| \( s^0 \) | 3 | 0 |
Special case: zero in the first column

- Replace the zero with $\epsilon$ (we can assume $\epsilon > 0$ or $\epsilon < 0$).

- Example:

$$q(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3$$

<table>
<thead>
<tr>
<th>$s^5$</th>
<th>$1 &gt; 0$</th>
<th>3</th>
<th>5</th>
</tr>
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<td>$s^4$</td>
<td>$2 &gt; 0$</td>
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</tr>
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<td>$s^3$</td>
<td>$\epsilon &gt; 0$</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>$s^2$</td>
<td>$\frac{6\epsilon - 7}{\epsilon} &lt; 0$</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14} &gt; 0$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>$3 &gt; 0$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Form an auxiliary polynomial $a(s)$ using the entries of row above row of zeros as coefficient, then differentiate with respect to $s$, finally use coefficients to replace the rows of zeros and continue the RH procedure.

Example:

$$q(s) = s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$$
Today’s learning objectives

- Poles and their effects on the response
- Zeros and their effects on the response
- Zeros and derivative action
- Effects of non-minimum-phase zeros
- Standard feedback control configuration and transfer function nomenclature
- Well-posedness
- Internal vs. I/O stability
- The root locus method