Control Systems I
Lecture 13: Dealing with “nuisances”

Readings: notes

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Outline

1. Time Delays
2. Control of nonlinear systems
3. Integrator anti-windup
Introduction

Time delays are ubiquitous in control systems:

- Delays are incurred when the controller is implemented on a computer, which needs some time to compute the appropriate control input, given a certain error. More in general, the evaluation of sensory information aimed at deciding the best course of action, will require a finite computation time.

- In some systems, delays may also be part of the physical plant.
  
  - When taking a shower, the water temperature is felt on the body after it has traveled through the valve, pipe, and shower head.

  - On an airplane, the effect of lift variation at the main wing are felt on the tail when the vortices shed by the wing reach the tail plane.

  - A rather extreme example is remote tele-operation: communication with a deep-space spacecraft or planetary rover may require several minutes.

How to take into account the effects of time delays in control system design and analysis?
The transfer function of a time delay

- A time delay is an operator that transforms an input signal $t \mapsto u(t)$ into a delayed output signal $t \mapsto y(t)$, with $y(t) = u(t - T)$, where $T \geq 0$ is the amount of the delay.

- Clearly, this is a **linear operator**: the delayed version of a linear combination of signals is equal to the linear combination of the delayed signals.

- In order to compute the transfer function of this linear operator, consider an input of the form $u(t) = e^{st}$. The output will be

$$y(t) = e^{s(t-T)} = e^{-sT}u(t),$$

and hence the transfer function of a delay of $T$ seconds is $e^{-sT}$.

- Notice that this is **NOT** a rational transfer function!
The frequency response of a time delay

- In terms of frequency response:

\[ |e^{j\omega T}| = 1, \quad \angle (e^{-j\omega T}) = -\omega T. \]

- This is not unexpected, since the time-delayed version of a sinusoid of unit amplitude and zero phase \( u(t) = \sin(\omega t) \) is another sinusoid, \( y = \sin(\omega(t - T)) \), with unit magnitude and a phase delayed by \( \omega T \).
Polar and Bode plots of a time delay

\[ |e^{-j\omega}| \]

\[ \angle (e^{-j\omega}) \]

\[ \Re [e^{-j\omega}] \]

\[ \Im [e^{-j\omega}] \]
Effects of time delays on the loop transfer function

- Next, we are going to consider the effects that time delays have on closed-loop stability, and discuss methods to take such effects into account when designing feedback control systems.

- Let us consider a system with loop transfer function $L(s) = C(s)P(s)$, and include a time delay of $T$ seconds (this can be either in the controller, or in the plant). We would get a new loop transfer function

$$L'(s) = e^{-sT}L(s).$$

- The frequency response of the system with the time delay is obtained from the “ideal” frequency response with no time delay, by shifting the phase back by $\omega T$. In other words,

$$|L'(j\omega)| = |L(j\omega)|, \quad \angle L'(j\omega) = \angle L(j\omega) - \omega T, \quad \forall \omega > 0.$$
Example

- Consider a simple plant,
  \[ P(s) = \frac{1}{s + 1}, \]
  with a proportional controller \( C(s) = k \).

- Hence, the ideal loop transfer function is
  \[ L(s) = \frac{k}{s + 1}. \]

A quick check with the root locus or the Nyquist plot will show that the feedback system would be stable for all \( k > 0 \) (in fact, for all \( k > -1 \)).

- What is the effect of a time delay on the closed-loop stability of such a system?
Example

- The loop transfer function in the presence of a time delay $T$ is

$$L'(s) = e^{-sT} \frac{k}{s + 1}.$$ 

- The polar plot of $L'$ is obtained from the polar plot of $L$ by “smearing” it: in practice all points $L(j\omega)$ must be rotated clockwise about the origin by an angle $\omega T$. 

\[ \begin{array}{c}
\text{Re}[L(j\omega)] \\
\text{Im}[L(j\omega)] \\
|L(j\omega)| \\
\text{Arg}[L(j\omega)] \\
\end{array} \]
Example

- It is easy to recognize that while the polar plot of $L$ never crosses the negative real axis, the polar plot of $L'$ crosses it an infinite number of times.

- In particular, the “first” crossing occurs when

  \[ \angle \left( \frac{k}{j\omega + 1} \right) - \omega T = -180^\circ, \]

  i.e., when

  \[ \tan^{-1} \omega + \omega T = 180^\circ. \]

- The location of this point determines the gain margin, which will be finite for all $T > 0$ (it was infinite in the ideal case $T = 0$).
A similar analysis can be carried out on the Bode plot.

- The magnitude plot of $L'$ is exactly the same as the magnitude plot of $L$.
- The phase plot is obtained by adding the phase plot of $L$ to the phase plot of the time delay.

This lets you determine very easily the effect of a time delay on the phase margin. In fact, the following relationship holds:

$$\phi_{m,T} = \phi_0 - \omega_c T,$$

where $\phi_{m,T}$ and $\phi_{m,0}$ are the phase margins with and without the time delay, respectively, and $\omega_c$ is the crossover frequency (which does not depend on the time delay).

The above equation summarizes the main effect of a time delay, that is a reduction of the phase margin.

Moreover, the phase margin reduction increases as the crossover frequency increases.
A design procedure:

The previous considerations suggest a procedure to design a feedback control system in the presence of time delay, as follows:

1. Design a feedback control system *ignoring* the time delay.

2. Check the effective phase margin, using (12). If the effective phase margin is too small (or negative, indicating closed-loop instability), redesign the controller according to one or both of these criteria:
   - Increase the phase at crossover, e.g., using a phase lead controller.
   - Decrease the crossover frequency, e.g., reducing the gain, or possibly a phase lag controller (to maintain command-following performance).

3. Iterate until a satisfactory controller is found.
So far, we have not talked about the root locus method in conjunction with time delays.

The reason is simple: in order to apply the root locus method, the loop transfer function must be rational, i.e., the ratio of two polynomials in $s$. This is not the case when a time delay is present since the term $e^{-sT}$ is not a rational function.

In order to use the root locus method in this case, it is required to approximate the time delay with a rational transfer function.
A naïve approximation

- A first choice would be a Taylor series expansion, which will take the form

\[ e^{-sT} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{6}(sT)^3 \ldots \]

- Let's truncate this series and maintain only the terms up to the second order in \((sT)\), i.e., let us write

\[ e^{-sT} \approx 1 - sT + \frac{1}{2}(sT)^2. \]

- Notice that this is a non-proper transfer function with two non-minimum phase zeros. This would be a good approximation for \(|sT| \ll 1\).

- However, the magnitude of the frequency response diverges for \(\omega \to \infty\), while we know that the magnitude of \(e^{-j\omega T}\) is always equal to one.
A better approximation can be obtained by using what is called a *Padé approximant*.

What we will do is approximate the exponential representing the time delay with a ratio of two polynomials.

For simplicity, let us limit ourselves to the ratio of first-order polynomials in $s$, i.e., let us write

$$e^{-sT} \approx k \frac{s + p}{s + q}.$$
Computing the coefficients in the Padé approximation

- In other words, let us impose that

\[ k \frac{s + p}{s + q} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{6}(sT)^3 \ldots, \]

at least up to the terms that we can match using our three free parameters.

- We get

\[ ks + kp = s + q - s^2 T - q s T + \frac{1}{2} s^3 T^2 + \frac{1}{2} q(sT)^2 + \ldots. \]

- We can ensure that all the terms of order up to and including 2 match, by imposing the following choices:

  
  **Order 0**: \( kp = q \),
  
  **Order 1**: \( k = 1 - qT \),
  
  **Order 2**: \( 0 = -T + \frac{1}{2} q T^2 \).
Solving for $k$, $p$, and $q$, we get

\[
q = \frac{2}{T}, \\
k = -1, \\
p = -\frac{2}{T},
\]

and finally

\[
e^{-sT} \approx \frac{2/T - s}{2/T + s}.
\]

Such an approximation has the advantage that the magnitude of its frequency response is always equal to one.
Using the Padé approximation, we can represent a time delay on the root locus as a pole and zero, respectively at $\pm 2/T$. Notice the presence of a non-minimum-phase zero.

This approximation method is useful since it allows us to use the root locus method for control design, and gain the insight provided by it.

In particular, notice the fact that we can immediately conclude that we cannot increase the gain arbitrarily, since eventually the closed-loop pole will converge to the non-minimum phase zero for large gains.
Example

- Consider again the example introduced previously.
- The root locus analysis will tell us that the closed-loop system will remain stable as long as $k < 3$.
- This is not correct, but only gives us a rough estimate of the gain margin. A better approximation may be achieved using a higher-order Padé approximation.
Summarizing, the Padé approximation allows the use of the root locus method to study the closed-loop stability of a system in the presence of a time delay.

However, the results may not be accurate, and it is recommended that the root-locus method be used only as a back-of-the-envelope tool providing some additional insight in the control design process.

Remember: the only tool that would always provide you with correct answers in all cases when a time delay is present (including, e.g., unstable open-loop poles and non-minimum phase zeros) is the Nyquist plot.
Outline

1. Time Delays

2. Control of nonlinear systems

3. Integrator anti-windup
Introduction to nonlinearities

- Most real-world systems are NOT linear.
  - Linear combinations of the inputs do not yield linear combinations of the outputs.
  - The principles of superposition of effects does not hold: the behavior will in general change depending on the initial conditions, as well as the amplitude and shape of the input signals!

Still there is a lot of insight that we can gain from the study of linear systems that transfers to nonlinear systems in many practical applications.

- In fact, most control systems in most applications (including flight control systems, spacecraft control systems) are primarily based on linear control theory!
Nonlinear systems

- A general model for (continuous-time) nonlinear system can be written as follows:

\[
\frac{d}{dt} x(t) = f(t, x(t), u(t)); \\
y(t) = h(t, x(t), u(t)).
\]

- Time-invariant systems do not depend explicitly on time:

\[
\frac{d}{dt} x(t) = f(x(t), u(t)); \\
y(t) = h(x(t), u(t)).
\]

- Clearly, non-linear systems include linear systems.
Equilibrium points

- Let us consider a nonlinear system
  \[
  \frac{d}{dt} x = f(x, u), \quad y = h(x, u).
  \]

- A state \( \bar{x} \) is an equilibrium point for the system if there exists a control \( \bar{u} \) such that
  \[
  f(\bar{x}, \bar{u}) = 0.
  \]

- In other words, if the system is at state \( \bar{x} \) at some time \( \bar{t} \), and the control input is set to \( \bar{u} \), then the system will remain at \( \bar{x} \) for all time \( t \geq \bar{t} \) (in principle).
  Also notice that \( y = h(\bar{x}, \bar{u}) = \text{const.} \) for all time \( t \geq \bar{t} \).
One can always introduce a change of coordinate such that the equilibrium point is at \((0, 0)\), e.g.,

\[
\xi = x - \bar{x}, \quad \nu = u - \bar{u}.
\]

So we can rewrite the system as

\[
\frac{d}{dt} \xi = \bar{f}(\xi, \nu), \quad y = \bar{h}(\xi, \nu),
\]

with \(\bar{f}(0, 0) = 0\), and \(\bar{h}(0, 0) = h(\bar{x}, \bar{u}) = \bar{y}\).

So, if \(\bar{f}\) and \(\bar{h}\) are sufficiently “smooth” (continuous and differentiable) at the equilibrium point, we can write

\[
\frac{d}{dt} \xi = \bar{f}(\xi, \nu) \approx \bar{f}(0, 0) + \left. \frac{\partial \bar{f}(\xi, \nu)}{\partial \xi} \right|_{(0,0)} \xi + \left. \frac{\partial \bar{f}(\xi, \nu)}{\partial \nu} \right|_{(0,0)} \nu,
\]

\[A \quad \text{and} \quad B\]
After similar calculations for the output, we get

\[
\frac{d}{dt} \xi \approx A \xi + B \nu, \quad y - \bar{y} \approx C \xi + D \nu.
\]

where

\[
A = \left. \frac{\partial \bar{f}(\xi, \nu)}{\partial \xi} \right|_{(0,0)}, \quad B = \left. \frac{\partial \bar{f}(\xi, \nu)}{\partial \nu} \right|_{(0,0)},
\]

\[
C = \left. \frac{\partial \bar{h}(\xi, \nu)}{\partial \xi} \right|_{(0,0)}, \quad D = \left. \frac{\partial \bar{h}(\xi, \nu)}{\partial \nu} \right|_{(0,0)}.
\]

The matrices \(A, B, C, D\) are the Jacobian of \((\bar{f}, \bar{h})\) with respect to \(\xi, \nu\).
Validity of the Jacobian linearization

- The approximation is only valid when $\xi, \nu$ are very small.
- Let us assume that we have a linear feedback controller $\nu = K\xi$; in such a case, the nonlinear system only depends on $\xi$:

$$\frac{d}{dt}\xi = \bar{f}(\xi, K\xi) = \bar{f}_{cl}(\xi).$$

- The same holds for the linearized version:

$$\frac{d}{dt}\xi = (A - BK)\xi.$$

- There is an important theorem (Hartman-Großman) that says that if the linearized system is closed-loop BIBO stable, then the nonlinear system is also stable, for $(\xi, \nu)$ in a neighborhood of $(0, 0)$.

- Note: we don’t know how large this neighborhood is, just that it exists.
Example: Car cruise control

Equation of motion in the direction parallel to the road surface:

\[ m \frac{d v}{dt} = F_{\text{eng}} + F_{\text{aero}} + F_{\text{frict}} + F_g. \]

where

\[ F_{\text{aero}} = -\frac{1}{2} \rho C_d A v \cdot |v|, \]

\[ F_g = -mg \sin(\theta), \]

\[ F_{\text{frict}} = -mg C_r \cos(\theta) \text{ sgn}(v). \]

---

1 The example is taken from Åström and Murray: Feedback Systems, 2008
Engine model

- Engine torque (at full throttle): \( T_\omega = T_m \left(1 - \beta \left(\frac{\omega}{\omega_m} - 1\right)^2\right) \), where \( \omega = \frac{n}{r} v =: \alpha_n v \) (\( n \) is the gear ratio, and \( r \) the wheel radius).
- The engine driving force can hence be written as

\[
F_{\text{eng}} = \alpha_n T(\alpha_n v)u, \quad 0 \leq u \leq 1.
\]
Jacobian Linearization

- Any (feasible) speed corresponds to an equilibrium point.
- Choose a reference speed $\bar{v} > 0$, and solve for $dv/dt = 0$ with respect to $u$, assuming a horizontal road ($\theta = 0$).
- We get

$$0 = \alpha_n T(\alpha_n \bar{v})\bar{u} - \frac{1}{2} \rho C_d A \bar{v}^2 - mg C_r$$

i.e.,

$$\bar{u} = \frac{\frac{1}{2} \rho C_d A \bar{v}^2 + mg C_r}{\alpha_n T(\alpha_n \bar{v})}.$$

- Linearized system ($\xi = v - \bar{v}$, $\upsilon = u - \bar{u}$):

$$m \frac{d}{dt} \xi = \left( \alpha_n \frac{\partial T(\alpha_n v)}{\partial v} \Bigg|_{\bar{v}} \bar{u} - \rho C_d A \bar{v} \right) \xi + \alpha_n T(\alpha_n \bar{v}) \upsilon.$$
Example: numerical values

- Let us use the following numerical values (all units in SI):
  
  \[ T_m = 190, \beta = 0.4, \omega_m = 420, \alpha_5 = 10, C_r = 0.01, \]
  
  \[ m = 1500, g = 9.81, \rho = 1.2, C_d A = 0.79. \]

- For \( \tilde{v} = 25 \) (90 km/h, or 55 mph), we get \( \tilde{u} = 0.2497. \)

- The linearization yields:

  \[ A = -0.0161, \quad B = 1.1837, \quad C = 1, \quad = 0, \]
  
  i.e.,

  \[ G(s) = \frac{1.1837}{s + 0.0161}. \]
Cruise control design

- A proportional controller would stabilize the closed-loop system.

- Assume we want to maintain the commanded speed (cruise control): we need to add an integrator.

- A PI controller will work, e.g.,

\[ C(s) = 1.5 \frac{s + 1}{s} \]
Check with BOTH linear AND non-linear simulation

- Transfer Fcn1: 1.5s + 1.5
- Transfer Fcn2: 1.5s + 1.5
- Gain3: B / (s - A)
- Gain: Faero
- Integrator
- Saturation
- Product
- Add
- Scope
Response to a 4 degree (7%) slope
Summary

- (Jacobian) linearization:
  - Find the desired equilibrium condition (state and control).
  - Linearize the non-linear model around the equilibrium.

- Control design:
  - Design a linear compensator for the linear model.
  - If the linear system is closed-loop stable, so will be the nonlinear system—in a neighborhood of the equilibrium.
  - Check in a (nonlinear) simulation the robustness of your design with respect to “typical” deviations.
Outline

1. Time Delays
2. Control of nonlinear systems
3. Integrator anti-windup
Effects of the saturation

- What if the slope is a little steeper (say 4.5 degrees)?

- What is wrong?
Integrator wind-up

- Once the input saturates, the integral of the error keeps increasing.

- When the error decreases, the large integral prevents the controller from resuming “normal operations” quickly (the integral error must decrease first!)

- **Idea**: once the input saturates, stop integrating the error (can’t do much about it anyway!)
Implementing Anti-windup logic

One option is the following logic for the integral gain:

\[
K'_I = \begin{cases} 
K_I & \text{if the input does not saturate;} \\
0 & \text{if the input saturates}
\end{cases}
\]

Another option is the following:

- Compare the actual input and the commanded input.
- If they are the same, the saturation is not in effect.
- Otherwise, reduce the integral error by a constant times the difference.

With this choice, under saturation, the integral error behaves like a simple lag, and converges to zero. If there is no saturation, the anti-windup scheme has no effect.
Anti-windup scheme

\[ a_5 T(a_5 v) \]

\[ f(u) \]

\[ \frac{1.5}{s} \]

\[ \frac{1.5}{s} + 1.5 \]

\[ B \]

\[ \frac{B}{s - A} \]

\[ f(u) \]

\[ \frac{1}{s} \]

\[ v \]

\[ \text{Saturation} \]

\[ \text{Product} \]

\[ \text{Gain} \]

\[ \text{Integrator} \]

\[ \text{Step} \]

\[ F \]

\[ F_g \]

\[ F_{\text{frict}} \]

\[ \text{Scope} \]

\[ v \bar{b} \]

\[ \text{ Constant } \]

\[ \text{Constant1} \]

\[ \text{Constant2} \]

\[ \text{Constant3} \]

\[ \text{Constant4} \]

\[ \text{Constant5} \]
Response to a 4.5 degree (8%) slope
Anti-wind up summary

- Anti-wind up schemes guarantee the **stability of the compensator** when the (original) feedback loop is effectively opened by the saturation.

- Prevent divergence of the integral error when the control cannot keep up with the reference.

- Maintain the integral errors “small.”