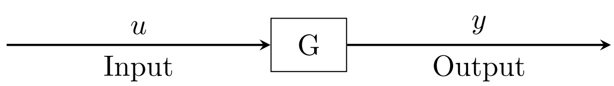
Lecture 2
Discrete-Time LTI Systems: Introduction

Outline

2.1 Classification of Systems

Recall that a DT system G is an operator that maps the input sequence u to the output sequence y, denoted $y = \mathrm{G}u$:



2.1.1 Memoryless

A system is called *memoryless* if the output at timestep n only depends on the input at the same timestep: $y[n] = f_n(u[n])$. For example, $y[n] = u^2[n]$ and $y[n] = a_n u[n]$ are memoryless, while $y[n] = u[n] - u[n-1]$ is not.

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2.1.2 Causal

A system is called *causal* or non-anticipative if at time n, the output $y[n]$ only depends on the present and past inputs $u[k]$, $k \leq n$. For example, $y[n] = u[n] - u[n-1]$ is causal, while $y[n] = u[n+1] - u[n]$, which depends on the future input $u[n+1]$, is not causal.

2.1.3 Linear

A system is called *linear* if $\mathrm{G}\{\alpha_1 u_1[n] + \alpha_2 u_2[n]\} = \alpha_1 \mathrm{G}\{u_1[n]\} + \alpha_2 \mathrm{G}\{u_2[n]\}$ holds for all input sequences $\{u_1[n]\}, \{u_2[n]\}$ as well as all constants α_1, α_2. This is also called the *superposition principle* and is a very useful property.

2.1.4 Time-invariant

Given the sequence u_1 and the shift $k \in \mathbb{Z}$, let $u_2[n] = u_1[n-k]$ for all n. The sequence u_2 is a *shifted* version of u_1. We denote this by $\{u_2[n]\} =: \{u_1[n-k]\}$.

Having defined a shifted sequence, we can now investigate the property of time-invariance. Let $\{u_2[n]\} = \{u_1[n-k]\}$, $y_1 = \mathrm{G}u_1$ and $y_2 = \mathrm{G}u_2$. If $\{y_2[n]\} = \{y_1[n-k]\}$, for all possible input sequences u_1, and for all time shifts k, then the system is called *time-invariant* or shift-invariant. A simple interpretation of time-invariance is that it does not matter when an input is applied: a delay in applying the input results in an equal delay in the output.

2.1.5 Stability of linear systems

To define the notion of stability, we need the concept of a bounded sequence. A sequence x is said to be *bounded* (by M) if there exists a finite value M such that $|x[n]| \leq M$ for all n.

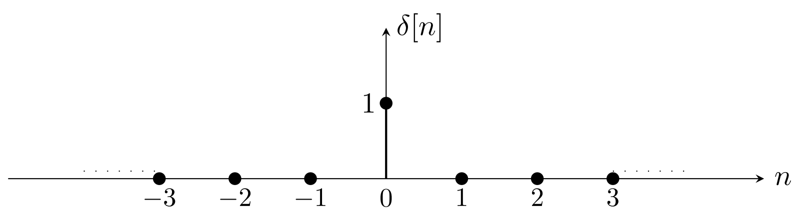
We can now define the notion of stability. A linear system is said to be *stable* if there exists a finite value M, such that for all input sequences u bounded by 1, the output sequence y is bounded by M. In general, this is referred to as *bounded input, bounded output (BIBO) stability* and can be generalized to non-linear systems.

2.2 Linear Time-Invariant (LTI) System Response to Inputs

2.2.1 Definitions of useful DT signals

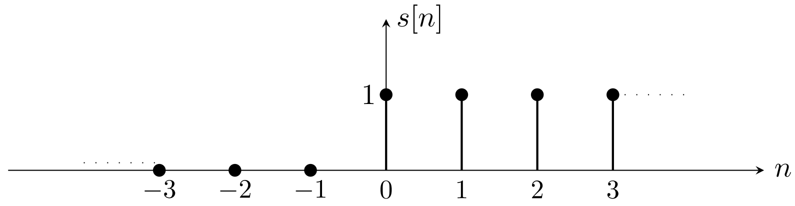
Unit impulse sequence $\{\delta[n]\}$ *with*

$$\delta[n] := \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



Unit step sequence $\{s[n]\}$ *with*

$$s[n] := \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Remark (CT vs. DT)
In CT, integrating the Dirac delta function $\delta(t)$ yields the Heaviside step function $s(t)$:

$$s(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau.$$

Likewise in DT, summing over the unit impulse sequence results in the unit step sequence $\{s[n]\}$ with

$$s[n] = \sum_{k=-\infty}^{n} \delta[k]. \tag{2.1}$$

In CT, differentiating the CT step function $s(t)$ [1] yields the Dirac delta function $\delta(t)$:

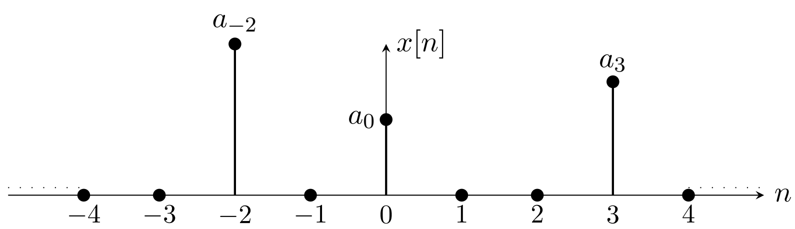
$$\frac{d}{dt} s(t) = \lim_{\varepsilon \to 0} \frac{s(t) - s(t-\varepsilon)}{\varepsilon} = \delta(t).$$

In DT, finite differences replace the process of differentiation: the unit impulse sequence $\{\delta[n]\}$ is given by the *backwards difference* of the DT step sequence $\{s[n]\}$:

$$\{s[n]\} - \{s[n-1]\} = \{\delta[n]\}. \tag{2.2}$$

2.2.2 Representing a sequence as a linear combination of impulses

We now show that DT signals can be expressed as a linear combination of time-shifted unit impulses. This form of expression is useful in DT and will allow us to calculate the response of LTI systems to arbitrary inputs. Consider the following example:



[1] In order to show this rigorously, the usage of distributional derivatives is required, which is beyond the scope of this class.

The above sequence can be represented as

$$x[n] = a_{-2}\delta[n+2] + a_0\delta[n] + a_3\delta[n-3], \text{ for all } n.$$

In particular, recalling that $\delta[n] = 0$ for $n \neq 0$ we have:

$$x[-2] = a_{-2}, \ x[0] = a_0, \ x[3] = a_3, \ x[n] = 0 \text{ otherwise.}$$

In general:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \text{ for all } n. \tag{2.3}$$

This is true also for entire sequences:

$$\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\}. \tag{2.4}$$

2.2.3 The system's impulse response and its response to arbitrary inputs

To derive the response of an LTI system G to an arbitrary input, we begin by defining the system's *impulse response* $\{h[n]\}$ as the output sequence given a unit impulse input sequence:

$$\{h[n]\} := \mathrm{G}\{\delta[n]\}.$$

Using Equation (2.3), we can write a sequence as a linear combination of time-shifted unit impulses. It follows that:

$$\begin{aligned} u[n] &= \sum_{k=-\infty}^{\infty} u[k]\delta[n-k] \\ &= \dots + u[-1]\delta[n+1] + u[0]\delta[n] + u[1]\delta[n-1] + \dots \text{ for all } n. \end{aligned}$$

Given that $\{y[n]\} = \mathrm{G}\{u[n]\}$, and by using the linearity (L) and time-invariance (TI) properties of G, we can write:

$$\{y[n]\} = \mathrm{G}\left(\sum_{k=-\infty}^{\infty} u[k]\{\delta[n-k]\}\right) \overset{\mathrm{L}}{=} \sum_{k=-\infty}^{\infty} u[k]\mathrm{G}\{\delta[n-k]\} \overset{\mathrm{TI}}{=} \sum_{k=-\infty}^{\infty} u[k]\{h[n-k]\}. \tag{2.5}$$

In Equation (2.4), we saw that a sequence can be represented by the summation of scaled and shifted unit impulses. Equation (2.5) demonstrates that the output of an LTI system can be represented by the summation of scaled and shifted versions of its impulse response.

2.2.4 Convolution

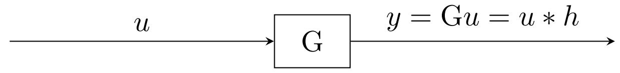
The *convolution* between two sequences x and h is denoted as $x * h$ and is defined as

$$x * h = \{x[n]\} * \{h[n]\} := \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\}.$$

Comparing this definition to Equation (2.5), we see that the the output of an LTI system G is the convolution between its impulse response h and its input u:

$$y = u * h. \tag{2.6}$$

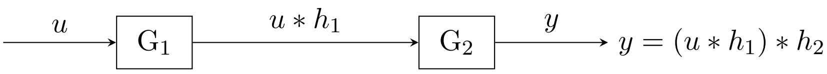
This can be graphically represented as:



The convolution operation is:

- Commutative: $x * h = h * x$
- Associative: $(x * h_1) * h_2 = x * (h_1 * h_2)$
- Distributive: $x * (h_1 + h_2) = x * h_1 + x * h_2$

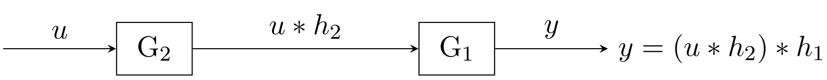
Example (Cascaded systems)
Let us look at a useful example of the convolution properties in action. Consider systems G_1 and G_2 with impulse responses h_1 and h_2 respectively. We cascade these systems as shown in the figure below:

(1) 

By Equation (2.6), we can write the output of the cascade to input u as $y = \mathrm{G}_2(\mathrm{G}_1 u) = (u * h_1) * h_2$. Using the associative property, we can rewrite this as $y = u * (h_1 * h_2)$. Defining the equivalent system $\mathrm{G} = \mathrm{G}_2\mathrm{G}_1$ to have impulse response $(h_1 * h_2)$, we can redraw the cascade as:

(2) 

Rewriting the output again, using the commutative and associative property, we arrive at the equivalent expression $y = u * (h_1 * h_2) = u * (h_2 * h_1) = (u * h_2) * h_1$. We can again redraw the cascade as

(3) 

and observe that the order in which LTI systems are cascaded does not matter because of the commutative and associative properties of convolution. Furthermore, the impulse response of the single equivalent system is the convolution of the individual impulse responses.

2.2.5 Step response

The *step response* $\{r[n]\}$ of a system is defined as its output to a unit step $\{s[n]\}$ input. We therefore have

$$\{r[n]\} := \{s[n]\} * \{h[n]\} = \{h[n]\} * \{s[n]\} = \sum_{k=-\infty}^{\infty} h[k]\{s[n-k]\} = \{\sum_{k=-\infty}^{n} h[k]\}. \quad (2.7)$$

In Equation (2.1), we saw that $\{s[n]\}$ can be obtained from $\{\delta[n]\}$ via a summation. Similarly, Equation (2.7) shows that $\{r[n]\}$ can be obtained from $\{h[n]\}$ via a summation.

In Equation (2.2), we saw that $\{\delta[n]\}$ is the backwards difference of $\{s[n]\}$. Similarly, $\{h[n]\}$ is the backwards difference of $\{r[n]\}$:

$$r[n] - r[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] = h[n], \text{ for all } n.$$

2.2.6 System properties from impulse response

Causality

Recall that

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k], \text{ for all } n. \quad (2.8)$$

If causality is to hold for all possible input sequences, then all terms $h[n-k]$ with $k > n$ must be zero. Therefore we have

$$\text{System is causal} \iff h[n] = 0 \text{ for } n < 0.$$

We call a sequence x *causal* if $x[n] = 0$ for $n < 0$. We typically work with causal signals and systems because physical systems are causal, and because we can assume, without loss of generality, that experiments start at time zero. For a causal system with causal input, Equation (2.8) becomes:

$$y[n] = \sum_{k=0}^{n} u[k]h[n-k] = \sum_{k=0}^{n} h[k]u[n-k], \text{ for all } n.$$

Note that if a system is causal and its input sequence is causal, the output sequence will also be causal.

Stability

An LTI system is stable iff

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty. \quad (2.9)$$

Proof
We now prove one direction of the above statement: if an LTI system's impulse response satisfies Equation (2.9), the system is stable.

Let $M = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$ and let u be any input sequence bounded by 1. It follows that, for all n:

$$\begin{aligned}
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} u[k]h[n-k] \right| = \left| \sum_{k=-\infty}^{\infty} h[k]u[n-k] \right| && \text{(Equation (2.5))} \\
&\leq \sum_{k=-\infty}^{\infty} |h[k]u[n-k]| = \sum_{k=-\infty}^{\infty} |h[k]|\,|u[n-k]| && \text{(Triangle Inequality)} \\
&\leq \sum_{k=-\infty}^{\infty} |h[k]|\,1 = M < \infty \quad \square && \text{(Bounded input: } |u[n]| \leq 1 \text{ for all } n\text{)}
\end{aligned}$$

Because the system's output sequence y is bounded by M, the system is stable. We leave it as an exercise to prove the opposite direction: if an LTI system is stable, its impulse response must satisfy Equation (2.9). □

Finite Impulse Response (FIR) vs. Infinite Impulse Response (IIR)

A causal system is said to have a *finite impulse response* (FIR), if there exists a time $N \in \mathbb{Z}$, such that:

$$h[n] = 0 \text{ for all } n \geq N.$$

In this case, the integer N is a finite upper-bound on the length of the system's impulse response. If a finite N that satisfies the above condition cannot be found, the length of the system's impulse response is unbounded, and the system is said to have an *infinite impulse response* (IIR).

Example
A system with an impulse response of the form:

$$h = \{0, \ldots, 0, \underset{\uparrow}{\alpha_0}, \alpha_1, 0, 0, \ldots\}$$

has an FIR (or is an FIR system) since $h[n] = 0$ for $n \geq 2$. A system with an impulse response

$$h[n] = \begin{cases} \alpha^n \text{ for } n \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

$$h = \{0, \ldots, 0, \underset{\uparrow}{1}, \alpha, \alpha^2, \ldots, \alpha^n, \ldots\}.$$

has an IIR (or is an IIR system).

2.3 Linear Constant-Coefficient Difference Equations

In CT the relationships between different signals are expressed in the form of differential equations. Difference equations, which we now introduce, are their equivalent in DT.

2.3.1 Definition

A Linear Constant-Coefficient Difference Equation (LCCDE) is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k], \quad a_k, b_k \in \mathbb{R}, \tag{2.10}$$

where N and M are non-negative integers.

Recursive definition

Assuming the system is causal (and assuming $a_0 \neq 0$), solving Equation (2.10) for $y[n]$ results in the recursive definition

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k u[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right).$$

2.3.2 Converting from LCCDE to state-space

The state-space (SS) description of a DT system is

$$\begin{aligned} q[n+1] &= Aq[n] + Bu[n], \\ y[n] &= Cq[n] + Du[n] \end{aligned} \tag{2.11}$$

with $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{1\times N}$, and $D \in \mathbb{R}$, and where $u[n] \in \mathbb{R}$ is the system's input at time n, $q[n] \in \mathbb{R}^N$ is the system's state at time n, and $y[n] \in \mathbb{R}$ is the system's output at time n. Note that, in comparison to Lecture 1, we have dropped the subscript d from the system matrices for notational simplicity, as we do not need to distinguish between CT and DT systems. Furthermore, although the SS description supports multiple-input, multiple-output (MIMO) systems, we will mainly consider single-input, single-output (SISO) systems in this class.

We will now show how to obtain a SS description of a DT system from an LCCDE for a special case of Equation (2.10), where $b_k = 0$ for $k > 0$ and $a_0 = 1$. Note that the latter can always be achieved for a causal system by rescaling a_k and b_k, and that the following results can be generalized for arbitrary values of coefficients b_k.
We therefore consider the LCCDE

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 u[n].$$

To calculate $y[n]$ at time n, we need N past outputs as well as the current input $u[n]$. Using the past outputs as the state results in:

$$\left.\begin{aligned} q_1[n] &= y[n-N] \\ q_2[n] &= y[n-(N-1)] = y[n-N+1] \\ &\vdots \\ q_N[n] &= y[n-1] \end{aligned}\right\} q[n] = \begin{bmatrix} q_1[n] \\ \vdots \\ q_N[n] \end{bmatrix}.$$

We therefore have

$$\begin{aligned} q_1[n+1] &= q_2[n], \;\; q_2[n+1] = q_3[n], \; \ldots, q_{N-1}[n+1] = q_N[n] \\ q_N[n+1] &= y[n] = b_0 u[n] - a_N q_1[n] - \cdots - a_1 q_N[n]. \end{aligned}$$

Now that we have defined the state $q[n]$, we need to find matrices A, B, C, D, which satisfy Equation (2.11). This is achieved by the following:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \qquad D = \begin{bmatrix} b_0 \end{bmatrix}.$$

The impulse response of a DT LTI system with a state-space description

The state-space description of a DT LTI system (2.11) can be solved to obtain the system's impulse response. Solving recursively yields:

$$\begin{aligned} q[1] &= Aq[0] + Bu[0] \\ q[2] &= Aq[1] + Bu[1] = A^2 q[0] + ABu[0] + Bu[1] \\ &\vdots \\ q[n] &= A^n q[0] + \sum_{k=0}^{n-1} A^{n-k-1} Bu[k], \quad n \geq 0 \\ y[n] &= Cq[n] + Du[n] = CA^n q[0] + C \sum_{k=0}^{n-1} A^{n-k-1} Bu[k] + Du[n], \quad n \geq 0. \end{aligned}$$

Assuming that the system has zero initial conditions ($q[n] = 0$ for $n \leq 0$), and using a unit impulse input $u[n] = \delta[n]$ for all n, we can read off the impulse response h for $n \geq 0$:

$$h = \{D, CB, CAB, \ldots, CA^{n-1}B, \ldots\}.$$

Note that the lack of arrow in the above sequence implies that the first term of the sequence (in this case D) occurs at time $n = 0$.

2.3.3 Relation between LCCDE and FIR/IIR LTI systems

It can be shown that an LTI system that can be written in the non-recursive form

$$y[n] = \sum_{k=0}^{M} b_k u[n-k], \text{ for some integer } M, \tag{2.12}$$

has an FIR.

Example (FIR)
Consider the system described by the LCCDE:

$$y[n] = b_0 u[n] + b_1 u[n-1] \text{ for all } n,$$

which can be expressed in the form of (2.12) with $M = 1$. The system's output can be computed non-recursively, the system therefore has a finite impulse response. One can verify this by computing the impulse response

$$h = \{\dots, 0, \underset{\uparrow}{b_0}, b_1, 0, \dots\},$$

and noting that it has a finite length.

Example (IIR)
Consider the system described by the LCCDE:

$$y[n] = a_1 y[n-1] + u[n] \text{ for all } n,$$

which cannot be expressed in the form of (2.12). We calculate the system's impulse response recursively, assuming $y[n] = 0$ for $n < 0$:

$$h = \{\dots, 0, \underset{\uparrow}{1}, a_1, a_1^2, \dots, a_1^n, \dots\},$$

and note that it has an infinite length, thus implying the system has an infinite impulse response.