Signals and Systems

Lecture 9: Infinite Impulse Response Filters

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Outline

1. Infinite Impulse Response Filters
   - IIR filter : difference equation
   - Transfer function
2. First-Order Low-Pass Filter
   - Definition
   - Properties
   - Design considerations
   - Connection to CT systems
3. IIR Filter Design
   - Methodology
   - CT Butterworth filter design
   - Bilinear transform

Outline

IIR: Difference equation

The class of causal infinite impulse response (IIR) filters can be captured by the difference equation

$$y[n] = \sum_{k=0}^{M-1} b_k u[n-k] - \sum_{k=1}^{N-1} a_k y[n-k],$$

Characteristics :

- M input coefficients $b_k \in \mathbb{R}$,
- $N-1$ output coefficients $a_k \in \mathbb{R}$.
- filter order: is given by $\max(M-1,\ N-1)$ and corresponds to the number of delay elements an implementation of the filter would require;

  it is also the size of the state in a state-space description of the system.

FIR vs. IIR

Key differences :

1. the output of a causal IIR filter is dependent on both the filter's input and on previous outputs (if one or more coefficients a_k are non-zero).
2. Dependence on previous output(s) generally implies that the impulse response has *infinite* length (hence the name: IIR filter).
3. IIR filters are not necessarily stable: the stability depends on the coefficients a_k.

Advantages of IIR filters :

1. they usually meet filter specifications with a lower filter order,
2. this corresponds to lower computation and storage cost compared to FIR filters.

Outline

IIR filter : transfer function

Transfer function and frequency response calculated from difference equation:

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}} \xrightarrow{z=e^{j\Omega}} H(\Omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{N-1} a_k e^{-j\Omega k}}$$

The goal of IIR filter design : find coefficients a_k and b_k such that the filter meets given specifications and is stable.
IIR filter design :

- often employs established continuous-time (CT) filter design methods, for example Butterworth filter design,
- and then transforms the resulting CT filter into DT.

In this lecture, we introduce:

1. the concepts underlying IIR filters;
2. how to design a CT Butterworth filter; and finally,
3. how to convert a CT filter into DT using the bilinear transform.

Outline

IIR 1st order low-pass filter

Consider the causal, first-order, low-pass IIR filter, which has the difference equation

$$y[n] = \alpha\, y[n-1] + (1-\alpha)\, u[n],$$

where $0 \leq \alpha < 1$.

Intuition :

- For $\alpha \neq 0$, this is an infinite impulse response filter.
- If $\alpha = 0$ the output is equal to the input and no filtering occurs.
- As $\alpha \to 1$, the output becomes increasingly constant.

IIR 1st order low-pass filter

Transfer function

$$H(z) = \frac{1-\alpha}{1-\alpha\, z^{-1}}.$$

Stability discussion:

- The filter has a single pole at $z = \alpha$.
- It immediately follows that the filter is stable if $0 \leq \alpha < 1$.

Frequency response

$$H(\Omega) = \frac{1-\alpha}{1-\alpha\, e^{-j\Omega}}.$$

Outline

IIR 1st order low-pass filter : behavior

Low - frequency signals remain unaltered since

$$H(\Omega = 0) = \frac{1-\alpha}{1-\alpha e^{-j0}} = 1.$$

The magnitude response is:

$$|H(\Omega)| = \frac{1-\alpha}{\sqrt{(1-\alpha\cos\Omega)^2 + \alpha^2\sin^2\Omega}}.$$

Furthermore, one can show that the magnitude is monotonically non-increasing:

$$\frac{\mathrm{d}|H(\Omega)|}{\mathrm{d}\Omega} \leq 0, \quad \text{for } 0 \leq \Omega \leq \pi.$$

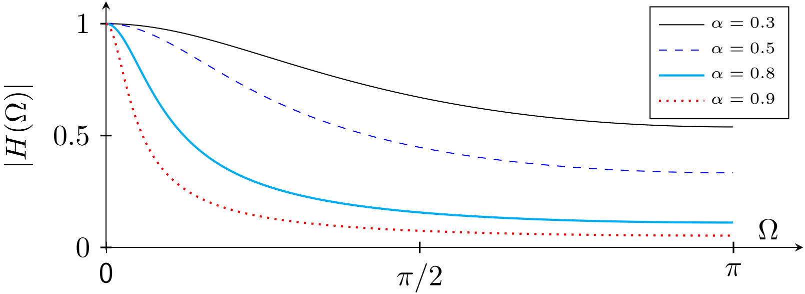
The phase is

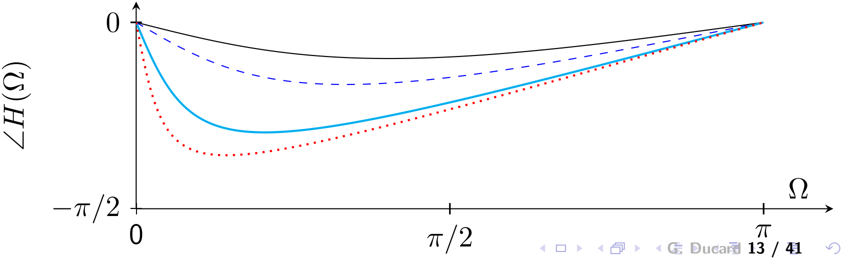
$$\angle H(\Omega) = \arctan\left(\frac{\overbrace{-\alpha\sin\Omega}^{\text{Always negative}}}{\underbrace{1-\alpha\cos\Omega}_{\text{Always positive}}}\right), \quad \text{for } 0 \leq \Omega \leq \pi.$$

Therefore

$$-\frac{\pi}{2} < \angle H(\Omega) \leq 0.$$

A plot of the magnitude and phase response follows:





Outline

Choice of parameter α

To choose n , let :

- T_s be the sampling time,
- and T_0 be the desired time for the continuous process to decay to e^{-1}, i.e., $T_0 = n\,T_s$.

$$n = \frac{T_0}{T_s} \quad \Rightarrow \alpha = e^{-\frac{1}{n}} = e^{-\frac{T_s}{T_0}},$$

we assume that T_0 is an integer multiple of T_s (if this is not the case, n may be rounded).

Example :

- sampling time $T_s = 0.01\,\text{s}$
- chosen decay time $T_0 = 1\,\text{s}$,

$$\alpha = \exp(-0.01) \approx 0.99.$$

Usually, T_0 is large relative to T_s and we may use a first-order approximation to obtain

$$\alpha \approx 1 - \frac{T_s}{T_0}.$$

Outline

A connection to CT systems can be made: In CT, the transfer function of a first-order low-pass filter is given by

$$H(s) = \frac{1}{\tau s + 1}.$$

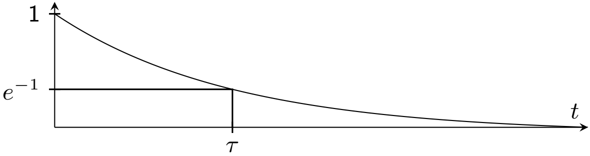
The differential equation for the output is

$$\dot{y}(t) = -\frac{1}{\tau}\left(y(t) - u(t)\right). \quad (1)$$

Assuming $u(t) = 0$ for $t \geq 0$, we obtain the time-domain system response

$$y(t) = y(0)e^{-\frac{t}{\tau}}.$$

Choosing $y(0) = 1$, we obtain the system response:



Therefore, the time to reach e^{-1} is τ.

Another connection to CT can be made by discretizing

$$\dot{y}(t) = -\frac{1}{\tau}\left(y(t) - u(t)\right)$$

assuming the input u is constant over a sample period T_s, i.e. a zero-order hold device is used:

$$\begin{bmatrix} \dot{y} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} & \frac{1}{\tau} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \quad 0 \leq t < T_s,$$

which we solve using the matrix exponential and obtain

$$\begin{bmatrix} y(T_s^-) \\ u(T_s^-) \end{bmatrix} = \exp\left(\begin{bmatrix} -\frac{T_s}{\tau} & \frac{T_s}{\tau} \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} y(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} e^{-\frac{T_s}{\tau}} & 1 - e^{-\frac{T_s}{\tau}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ u(0) \end{bmatrix}.$$

As discussed in Lecture 1, this solution is valid on any time interval because the system is time invariant. Substituting the decay time T_0 for the time constant τ, the resulting difference equation becomes

$$y[n] = e^{-\frac{T_s}{T_0}} y[n-1] + (1 - e^{-\frac{T_s}{T_0}}) u[n-1] = \alpha y[n-1] + (1-\alpha) u[n-1]$$

which closely resembles the first-order, low-pass IIR filter, except that the input is delayed by one sample.

Outline

IIR filter design

⚠ FIR filter design only makes sense in DT.

In contrast,

1. IIR filters are often designed in CT using an established method, for example
   - a Butterworth,
   - or Chebyshev method.
2. they are then converted to DT using the *bilinear transform* (sometimes also called Tustin method).

We will see in the following section,

💡 the bilinear transform has properties that make it a useful tool for the above design procedure.

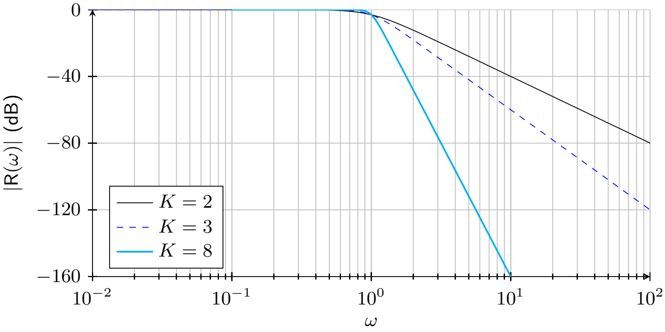
Outline

CT design: Butterworth filter

It is a general purpose low-pass filter. The starting point is the desired frequency response, with corner frequency 1 rad/sec:

$$R(\omega) = \frac{1}{\sqrt{1 + \omega^{2K}}},$$

where K is the order of the filter.



Butterworth filter design: properties

This frequency response has two very desirable properties:

1. it has no ripples,
2. and is maximally flat.

$$R(\omega) = \frac{1}{\sqrt{1+\omega^{2K}}} = (1+\omega^{2K})^{-\frac{1}{2}},$$

First, let us calculate the derivative $dR/d\omega$.

$$\begin{aligned} dR(\omega)/d\omega &= -\frac{1}{2}(1+\omega^{2K})^{-\frac{3}{2}}2K\omega^{2K-1} \\ &= -KR^3\omega^{2K-1} \leq 0 \text{ for all } \omega \geq 0. \end{aligned}$$

Conclusion

This means that the Butterworth filter has no ripples.
In other words, all derivatives of R up to $2K-1$ are 0 at 0 and the filter is said to be *maximally flat*.

Transfer function

Let $H(s)$ be the transfer function of a filter with frequency response $R(\omega)$. We then have that

$$|H(j\omega)|^2 = R(\omega)^2 = (1 + \omega^{2K})^{-1}.$$

The only stable transfer function that achieves this is

$$H(s) = \frac{1}{\prod_{k=1}^{K} (s - s_k)},$$

where $s_k = e^{\frac{j(2k+K-1)\pi}{2K}}, \quad k = 1, \dots, K.$

Filters with a transfer function of this structure are known as Butterworth filters.

Second-order Butterworth low-pass filter

Consider one of the most common Butterworth filters: a second-order $(K = 2)$ low-pass. We have,

$$s_1 = e^{j3\pi/4} = \frac{-1+j}{\sqrt{2}}, \quad (135^\circ)$$

$$s_2 = e^{j5\pi/4} = \frac{-1-j}{\sqrt{2}}, \quad (225^\circ)$$

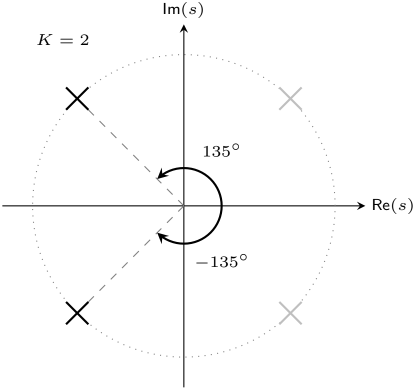
which results in the transfer function

$$H(s) = \frac{1}{(s + \frac{1-j}{\sqrt{2}})(s + \frac{1+j}{\sqrt{2}})} = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

Second-order Butterworth low-pass filter

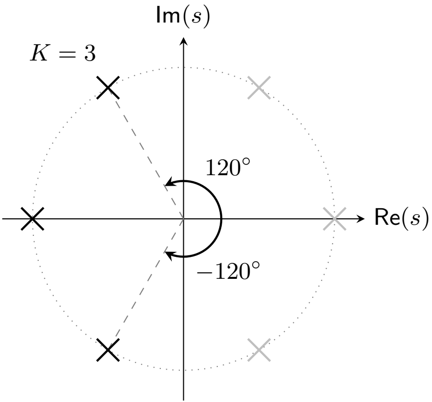
The poles of the CT filter lie on the unit circle in the s-plane (not to be confused with the z-plane) and are represented by the black crosses below.

Remark: The gray crosses represent the poles of $H(-s)$ and are useful to visualize the pole-placement pattern.



Third-order Butterworth low-pass filter

If, instead, a third-order low-pass filter is chosen ($K = 3$), the pole plot looks as follows:



Corner frequency specification

The design method introduced above assumed a corner frequency of 1 rad/sec. However, other corner frequencies ω_c can be chosen.

In that case, we proceed with the following change of variable

$$s \to \frac{s}{\omega_c}.$$

For example, the second order filter becomes

$$H(s) = \frac{{\omega_c}^2}{s^2 + \sqrt{2}\omega_c s + {\omega_c}^2},$$

which has the same response as a mass-spring-damper system with sub-critical damping $1/\sqrt{2}$ and natural frequency ω_c.

Outline

Bilinear transform or Tustin's method

Once a CT filter has been designed, ⇒ the *bilinear transform* (also known as Tustin's method) can be used to convert it into a DT filter.

The bilinear transform uses the substitution that

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right),$$

where T_s is the sampling time.

Bilinear transform or Tustin's method

Recall that z can be given the interpretation of a DT shift operator. If

$$Y(z) = zU(z),$$

then

$$y[n] = u[n+1].$$

Similarly, in CT, e^{sT_s} can be given the interpretation of a time shift operator. If

$$Y(s) = e^{sT_s}U(s)$$

where $Y(s)$ and $U(s)$ are the Laplace transform of $y(t)$ and $u(t)$, respectively, then

$$y(t) = u(t+T_s).$$

Therefore, the two operators are equivalent:

$$z = e^{sT_s}.$$

Bilinear transform or Tustin's method

A rational approximation for the relation between z and s will map a rational CT transfer function to a rational DT transfer function. This is equivalent to converting differential equations to difference equations. We therefore use the approximation

$$e^{sT_s} = \frac{e^{s\frac{T_s}{2}}}{e^{-s\frac{T_s}{2}}} \approx \frac{1 + s\frac{T_s}{2}}{1 - s\frac{T_s}{2}},$$

which is valid if sT_s is small. We call

$$z = \frac{1 + s\frac{T_s}{2}}{1 - s\frac{T_s}{2}}$$

the *bilinear transform*. The inverse is

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$$

and is straightforward to verify by substitution.

DC- CT mapping

We now evaluate the bilinear transform along the imaginary axis of the s-plane; that is, let $s = j\omega$. Using the bilinear transform, this point maps to

$$z = \frac{1 + j\omega\frac{T_s}{2}}{1 - j\omega\frac{T_s}{2}}.$$

Note that

$$|z| = \left|\frac{1 + j\omega\frac{T_s}{2}}{1 - j\omega\frac{T_s}{2}}\right| = 1.$$

Conclusion

The bilinear transform therefore maps the imaginary axis of the s-plane to the unit circle in the z-plane.

DC- CT mapping

We therefore write

$$z = e^{j\Omega} = \frac{1 + j\omega\frac{T_s}{2}}{1 - j\omega\frac{T_s}{2}},$$

and calculate the mapping of a CT frequency ω to a DT frequency Ω as:

$$\begin{aligned} \angle e^{j\Omega} &= \angle(1 + j\omega\tfrac{T_s}{2}) - \angle(1 - j\omega\tfrac{T_s}{2}) \\ \Omega &= \arctan(\omega\tfrac{T_s}{2}) - \arctan(-\omega\tfrac{T_s}{2}) \\ &= 2\arctan(\omega\tfrac{T_s}{2}). \end{aligned}$$

Conclusion

The frequency response of the CT system at ω (the CT transfer function evaluated on the imaginary axis at $s = j\omega$)

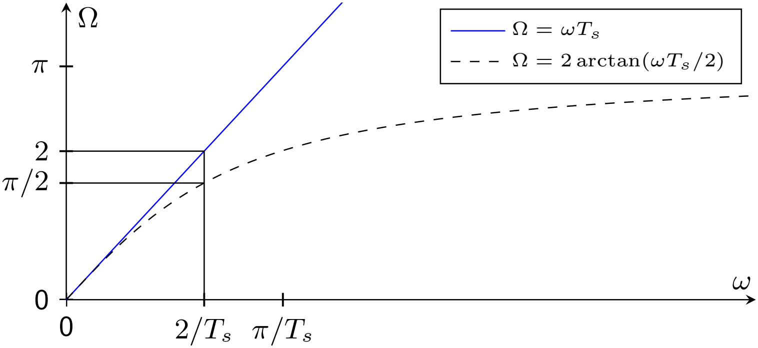
directly corresponds to

the frequency response of the resulting DT system at $\Omega = 2\arctan(\omega\frac{T_s}{2})$ (the DT transfer function evaluated on the unit circle at $z = e^{j\Omega}$).

This is a desirable property, as we will later see.

DC- CT mapping

For small ωT_s, the DT frequency is approximately $\Omega \approx 2(\omega \frac{T_s}{2}) = \omega T_s$. This is also evident when we plot the mapping of CT frequencies to DT frequencies for $z = e^{sT_s}$ and the bilinear transform:



Note how Ω asymptotically converges to π as $\omega \to \infty$: The bilinear transform compresses the imaginary axis of the s-plane onto the unit circle.

Frequency warping

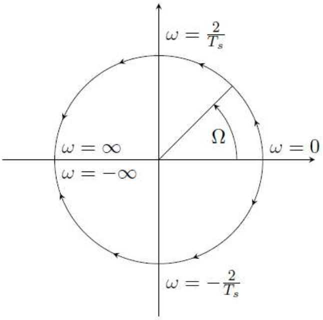
The underlying nonlinear relation between ω and Ω is called *frequency warping*. A few common values are:

$$\omega = 0 \quad \Rightarrow \quad \Omega = 0$$

$$\omega = \infty \quad \Rightarrow \quad \Omega = \pi$$

$$\omega = \frac{2}{T_s} \quad \Rightarrow \quad \Omega = \frac{\pi}{2}$$

$$s = j\omega \Rightarrow z = e^{j\Omega}.$$

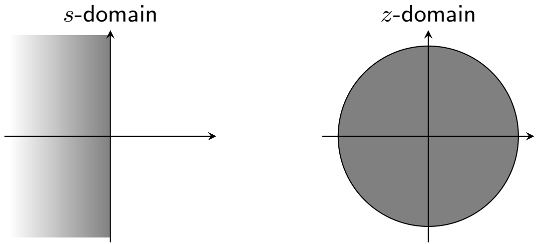


Other points are:

$$z = 0 \Rightarrow s = -\frac{2}{T_s}, \qquad z = \infty \Rightarrow s = \frac{2}{T_s}.$$

Region of stability mapping

Stable poles in the continuous domain are mapped to stable poles in the discrete domain:



This is desirable, as it means that a stable CT filter is transformed into a stable DT filter.

Summary
The bilinear transform preserves stability and maps the imaginary axis in the s-plane to the unit circle in the z-plane by compressing the CT frequencies $-\infty < \omega < \infty$ to DT frequencies $-\pi < \Omega < \pi$.

Example: Converting a CT first-order low-pass filter

Consider the CT low-pass filter with time constant τ

$$H(s) = \frac{1}{\tau s + 1}.$$

Using the bilinear transform, we obtain

$$H(z) = \frac{1}{1 + \tau \frac{2}{T_s} \left(\frac{z-1}{z+1} \right)} = \frac{1-\alpha}{1-\alpha z^{-1}} \frac{1+z^{-1}}{2}$$

with

$$\alpha = \frac{1 - \frac{T_s}{2\tau}}{1 + \frac{T_s}{2\tau}}.$$

Example: Converting a CT first-order low-pass filter

For small values of $\frac{T_s}{\tau}$, $\alpha \approx 1 - \frac{T_s}{\tau}$, as before.

Let us compare different discretization methods:

Method	Transfer function	Filter parameter
Direct	$H(z) = \frac{1-\alpha}{1-\alpha z^{-1}}$	$\alpha = e^{-\frac{T_s}{\tau}}$ (decay time τ)
Sample and Hold	$H(z) = \frac{(1-\alpha)z^{-1}}{1-\alpha z^{-1}}$	$\alpha = e^{-\frac{T_s}{\tau}}$ (time constant τ)
Bilinear	$H(z) = \frac{(1-\alpha)(\frac{1+z^{-1}}{2})}{1-\alpha z^{-1}}$	$\alpha = \frac{1-\frac{T_s}{2\tau}}{1+\frac{T_s}{2\tau}}$ (time constant τ)

Example: Converting a CT first-order low-pass filter

A nice property of the bilinear transform, for this example: is that

$$H(z=-1)=0$$

This corresponds to

$$z=-1=e^{-j\pi}$$

which is the highest possible DT frequency .

Remark: Observe that the CT first-order low-pass has

$$\lim_{s\to j\infty} H(s)=0$$

Conclusion :
the frequency responses at the highest possible frequencies are the same for the CT and the DT system when using the bilinear transform.
⇒ It follows that their high-frequency behavior is similar, which is one of the advantages of the bilinear transform.

Example: Converting a CT first-order low-pass filter

This can be seen by looking at the frequency response plots of the resulting filters for $T_s = 1$ and $\tau = 2$: