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Signals and Systems

Problem Set:
The z-Transform and DT Fourier Transform

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Problem Set

Problem 1 - Transfer functions in MATLAB

A discrete-time, causal LTI system is described by the following difference equation

$$a_0 y[n] + a_2 y[n-2] = b_0 u[n] + b_1 u[n-1],$$

with $a_0 = -1$, $a_2 = 0.5$, $b_0 = 2$, and $b_1 = -1$.

a) Generate the corresponding transfer function $H_1(z)$ using the command `tf(num,den,T)`.

   Hints: type "`help tf`" for the Matlab help. Use `T = -1`, which indicates an undefined sampling time. Use `'Variable','z^-1'` as an option (see *Properties* in the Matlab help for *tf*).

b) Is the system BIBO stable? Verify stability using the command `pole` and `pzmap`.

c) Compute the impulse response $\{h_1[n]\}$ of the system using the command `impulse` and plot it. Given your answer to b), what do you expect to see?

d) Now make the system unstable by changing some coefficients. Compute and plot the impulse response. What do you expect to see? *Hints*: Think about which coefficients affect stability.

Problem 2 - Cascaded systems in MATLAB

A discrete-time, causal LTI system is given by the transfer function

$$H_2(z) = 1 - z^{-1}.$$

The system is cascaded with the system from Problem 1. Compute in Matlab the impulse response $\{h[n]\}$ of the equivalent system, using:

a) Convolution;
   Hint: Use the Matlab command `conv`.

b) Multiplication in the frequency domain (z-domain);

and compare them.

Problem 3 - Properties of the z-Transform

In lecture 3, we listed the following properties of the z-transform:

- Linearity: $a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$
- Time-shifting $\{x[n-1]\} \longleftrightarrow z^{-1}X(z)$
  $\{x[n+1]\} \longleftrightarrow zX(z)$
- Convolution: $\{x_1[n]\} * \{x_2[n]\} \longleftrightarrow X_1(z) \cdot X_2(z)$
- Accumulation: $\{\sum_{k=-\infty}^{n} x[k]\} \longleftrightarrow \frac{z}{z-1}X(z)$

Prove them.

Hint: $\{s[n]\} \longleftrightarrow z/(z-1)$

Problem 4 - Frequency response of a system

Consider the discrete-time, causal, LTI system shown in Fig. 1.

a) Find the frequency response $H(\Omega)$ of the system.

b) Find the impulse response h of the system.

c) Sketch the magnitude response $|H(\Omega)|$ and the phase response $\Theta_H(\Omega)$.

d) Find the frequency $\Omega_{3\mathrm{dB}}$ where the system's magnitude response drops to $1/\sqrt{2}$ of its maximum value.

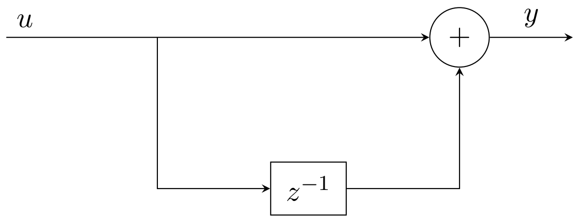
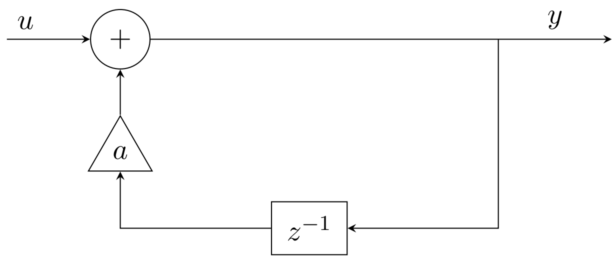


Figure 1

Problem 5 - Frequency response of a system

Consider the discrete-time, causal, LTI system shown below, where a is a constant and $0 < a < 1$.



a) Find the frequency response $H(\Omega)$ of the system.

b) Find the impulse response h of the system.

c) Sketch the magnitude response $|H(\Omega)|$ of the system for $a = 0.9$ and $a = 0.3$.

Problem 6 - System specifications

The system function $H(z)$ of a causal, discrete-time LTI system is given by

$$H(z) = \frac{b + z^{-1}}{1 - az^{-1}},$$

where a is real and $|a| < 1$. Find the value of b so that the frequency response $H(\Omega)$ of the system satisfies the condition

$$|H(\Omega)| = 1, \quad \text{for all } \Omega$$

Such a system is called an *all-pass* filter.

Problem 7 - System response to sinusoidal inputs (Exam 2013)

Consider a causal, linear, time-invariant system described by the difference equation

$$y[n] = ay[n-1] + bu[n],$$

where $u[n]$ is the input, $y[n]$ is the output for all times n. Let $a = -\sqrt{3}/3$ and $b = 1$ and the input to the system be given by

$$u[n] = 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{3}\right)$$

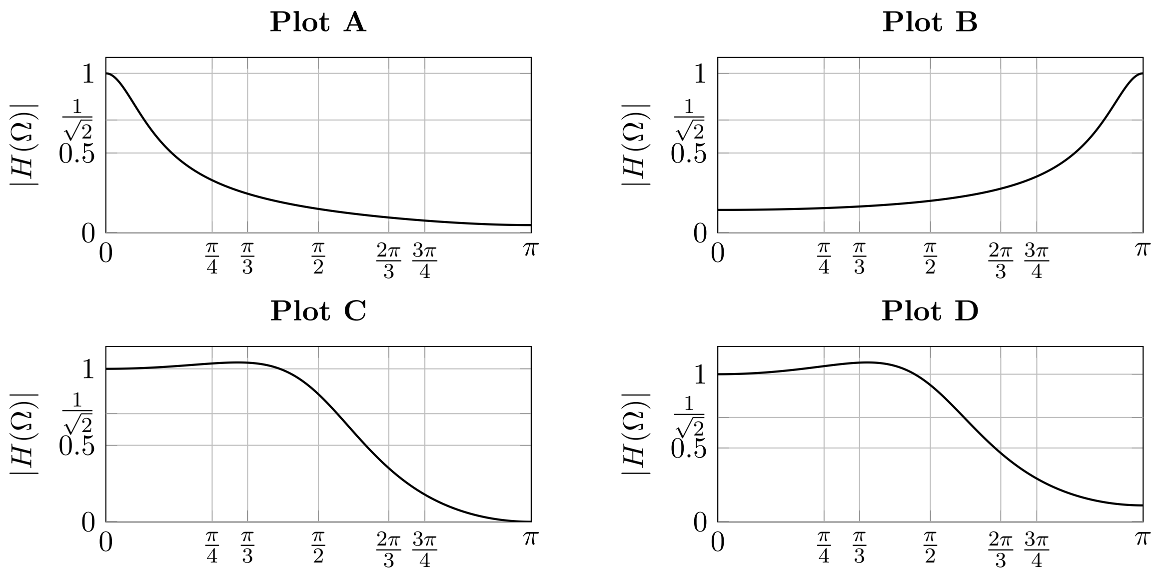
for all n. What is the output $y[n]$ for all n?

Hint: The following trigonometric table might be useful.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-

Problem 8 - Magnitude Response (Quiz 2015)

The magnitude responses of four causal, LTI systems are plotted below on the domain $\Omega = [0, \pi]$:



For the following systems, decide which of the plots shows its magnitude response.

a)

$$H(z) = \frac{1}{4} \cdot \frac{z}{z + 0.75}$$

b)

$$H(z) = \frac{5}{16} \cdot \frac{(z+1)^2}{z^2 + 0.25}$$

Problem 9 - System response to sinusoidal inputs

Consider the causal, discrete-time, linear constant-coefficient difference equation:

$$y[n] - 0.8y[n-1] = u[n].$$

a) Determine the transfer function $H(z)$ of the system.

b) Is the system stable? Why?

c) Substitute $z = e^{j\Omega} = \cos(\Omega) + j\sin(\Omega)$, and solve for:

  i) The magnitude response: $|H(\Omega)|$ and

  ii) The phase response: $\Theta_H(\Omega)$

Given the input $u[n] = \cos((\pi/10)n)$ for all n:

d) Calculate the output $y[n]$ for all n.

e) Define the transfer function in MATLAB.
Hint: use the `tf` command, as shown in a previous exercise

f) Now, using the `freqresp(H,Omega)` command, compute the magnitude and phase response to the input.

g) Simulate the system in MATLAB using the `lsim` command for one period of the input. Why does the simulated output not reflect your expectations?

h) Try simulating the system for multiple periods of the input. What happens now? What do you expect to happen in the limit?

Sample Solutions

Problem 1 - Solution

Sample solutions in the form of Matlab files can be found in the corresponding zip file on the website.

Problem 2 - Solution

Sample solutions in the form of Matlab files can be found in the corresponding zip file on the website.

Problem 3 - Solution

For the following proofs, let

$$\begin{aligned}
&\{x_1[n]\} \longleftrightarrow X_1(z), \\
&\{x_2[n]\} \longleftrightarrow X_2(z), \text{ and} \\
&\{x_3[n]\} \longleftrightarrow X_3(z).
\end{aligned}$$

- Linearity

  Let $\{x_3[n]\} = \alpha_1\{x_1[n]\} + \alpha_2\{x_2[n]\}$, then

  $$\begin{aligned}
  X_3(z) &= \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \sum_{n=-\infty}^{\infty} (\alpha_1 x_1[n] + \alpha_2 x_2[n])\, z^{-n} \\
  &= \alpha_1 \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} + \alpha_2 \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = \alpha_1 X_1(z) + \alpha_2 X_2(z)
  \end{aligned}$$

  Thus

  $$a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1 X_1(z) + a_2 X_2(z).$$

- Time-shifting

  Let $\{x_2[n]\} = \{x_1[n+k]\}$, then

  $$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = \sum_{n=-\infty}^{\infty} x_1[n+k]z^{-n}$$

  By a change of variables $l = n + k$, we obtain

  $$X_2(z) = \sum_{l=-\infty}^{\infty} x_1[l]z^{k-l} = z^k \sum_{l=-\infty}^{\infty} x_1[l]z^{-l} = z^k X_1(z).$$

  Thus

  $$\{x[n+k]\} \longleftrightarrow z^k X(z).$$

- Convolution

  Let $\{x_3[n]\} = \{x_1[n]\} * \{x_2[n]\}$, then

$$X_3(z) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right) z^{-n} = \sum_{k=-\infty}^{\infty} x_1[k] \left(\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \right)$$
$$= \sum_{k=-\infty}^{\infty} x_1[k] \left(z^{-k} X_2(z) \right) = X_1(z)X_2(z).$$

  Thus

$$\{x_1[n]\} * \{x_2[n]\} \longleftrightarrow X_1(z) \cdot X_2(z).$$

- Accumulation

  Let

$$\{x_2[n]\} = \{ \sum_{k=-\infty}^{n} x_1[k]\} = \{ \sum_{k=-\infty}^{\infty} x_1[k]s[n-k]\} = \sum_{k=-\infty}^{\infty} x_1[k]\{s[n-k]\}$$
$$= \{x_1[n]\} * \{s[n]\}$$

  By using the convolution property from above, we have

$$X_2(z) = X_1(z) \cdot S(z) = X_1(z) \left(\frac{z}{z-1} \right).$$

  Thus

$$\{ \sum_{k=-\infty}^{n} x[k]\} \longleftrightarrow \frac{z}{z-1} X(z).$$

Problem 4 - Solution

a) From Fig. 1, we have

$$y[n] = u[n] + u[n-1].$$

The frequency response can then be read directly from the LCCDE:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = 1 + e^{-j\Omega}$$
$$= e^{-j\Omega/2} \left(e^{j\Omega/2} + e^{-j\Omega/2} \right)$$
$$= 2e^{-j\Omega/2} \cos(\Omega/2), \quad -\pi < \Omega \leq \pi.$$

b) The system has a finite impulse response

$$h = \{\dots, 0, \underset{\uparrow}{1}, 1, 0, \dots\},$$

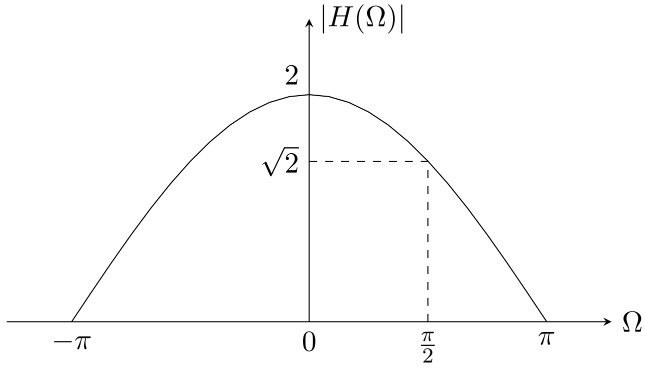
which can be written as $\{h[n]\} = \{\delta[n]\} + \{\delta[n-1]\}$.

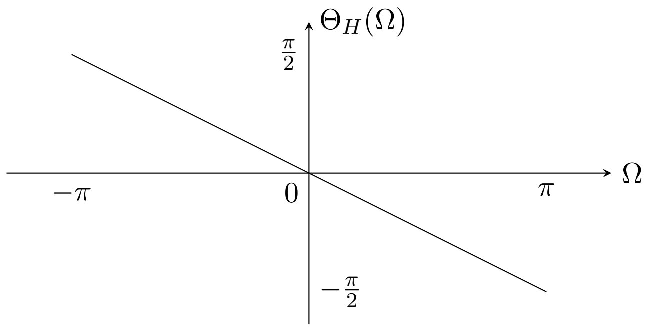
c) Recall that a complex number z can also be written in cartesian form as $z = a + jb$. The magnitude is then given by $|z| = \sqrt{a^2 + b^2}$ and the phase by $\angle z = \texttt{atan2}(b, a)$, where $\texttt{atan2}$ is the quadrant arc-tangent function.

In this case, however, magnitude and phase can be directly read:

$$|H(\Omega)| = 2\cos(\Omega/2), \quad -\pi < \Omega \leq \pi.$$
$$\Theta_H(\Omega) = -\frac{\Omega}{2}, \quad -\pi < \Omega \leq \pi.$$

These are sketched below.





d) Let $\Omega_{3\text{dB}}$ be the frequency for which the following relation holds:

$$|H(\Omega_{3\text{dB}})| = \frac{1}{\sqrt{2}} \max_{\Omega} |H(\Omega)|$$

Inserting our particular $H(\Omega)$ we obtain

$$2\cos\left(\frac{\Omega_{3\text{dB}}}{2}\right) = \frac{1}{\sqrt{2}} \cdot 2$$

and therefore $\Omega_{3\text{dB}} = \frac{\pi}{2}$.

Problem 5 - Solution

a) From the Figure we have

$$y[n] - ay[n-1] = u[n] \tag{1}$$

The frequency response can then be read directly from the LCCDE:

$$H(\Omega) = \frac{Y(\Omega)}{U(\Omega)} = \frac{1}{1 - ae^{-j\Omega}}.$$

b) The transfer function of the system is

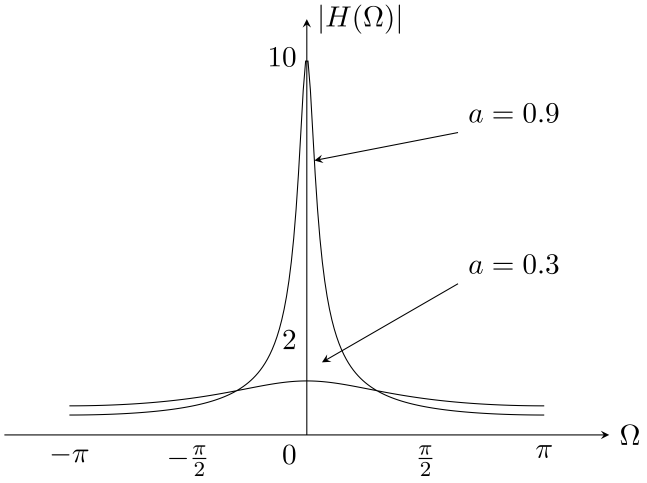
$$H(z) = \frac{z}{z-a}.$$

Since the system is causal, its impulse response is given by $h[n] = a^n s[n]$.

c) Using Euler's formula we obtain

$$\begin{aligned} H(\Omega) &= \frac{1}{1 - ae^{-j\Omega}} = \frac{1}{1 - a\cos\Omega + ja\sin\Omega} \\ |H(\Omega)| &= \frac{1}{\sqrt{(1 - a\cos\Omega)^2 + (a\sin\Omega)^2}} = \frac{1}{\sqrt{1 + a^2 - 2a\cos\Omega}} \end{aligned}$$

which is sketched below for $a = 0.9$ and $a = 0.3$. We see that the system is a discrete-time low-pass infinite impulse response (IIR) filter.



Problem 6 - Solution

The frequency response of the system is

$$H(\Omega) = H(z)\Big|_{z=e^{j\Omega}} = \frac{b + e^{-j\Omega}}{1 - ae^{-j\Omega}}$$

With the condition

$$|H(\Omega)| = \left|\frac{b + e^{-j\Omega}}{1 - ae^{-j\Omega}}\right| = 1$$

we have:

$$\begin{aligned} |b + e^{-j\Omega}| &= |1 - ae^{-j\Omega}| \\ |b + \cos\Omega - j\sin\Omega| &= |1 - a\cos\Omega + ja\sin\Omega| \\ 1 + b^2 + 2b\cos\Omega &= 1 + a^2 - 2a\cos\Omega \end{aligned}$$

For $b = -a$ the above relation holds for all Ω.

Problem 7 - Solution

The output of an LTI system to a sinusoidal input can be obtained by analyzing the frequency response $H(\Omega)$ of the system. If an input

$$u[n] = A\cos(\Omega_0 n + \Theta)$$

is applied to the system for all times n, the output $y[n]$ is given by

$$y[n] = |H(\Omega_0)|\, A\cos\left(\Omega_0 n + \Theta + \Theta_H(\Omega_0)\right)$$

where $|H(\Omega_0)|$ and $\Theta_H(\Omega_0)$ are respectively the magnitude and phase response at frequency Ω_0. The given input has the frequency $\Omega_0 = \pi/2$. Therefore,

$$H\left(\frac{\pi}{2}\right) = \frac{b}{1 - ae^{-j\pi/2}} = \frac{b}{1 + j\,a}.$$

Evaluating for $a = -\sqrt{3}/3$ and $b = 1$ yields the following magnitude and phase response, respectively:

$$|H\left(\frac{\pi}{2}\right)| = \frac{|b|}{\sqrt{1+a^2}} = \frac{\sqrt{3}}{2}$$
$$\Theta_H(\Omega_0) = \angle H(\frac{\pi}{2}) = \angle(b) - \angle(1 + ja) = 0 - \arctan\left(\frac{a}{1}\right) = \frac{\pi}{6}.$$

It follows that the output $y[n]$ for the given input $u[n]$ is, for all times n, given by

$$y[n] = \sqrt{3}\cos\left(\frac{\pi}{2}n + \frac{\pi}{2}\right).$$

Problem 8 - Solution

The solution can be found by looking at the magnitude of $H(\Omega)$ at $\Omega = 0$ and $\Omega = \pi$.

a) Computing the values $|H(\Omega = 0)|$ and $|H(\Omega = \pi)|$ yields

$$|H(\Omega = 0)| = |H(z = 1)| = \left|\frac{1}{4}\cdot\frac{1}{1 + 0.75}\right| = \frac{1}{7}$$
$$|H(\Omega = \pi)| = |H(z = -1)| = \left|\frac{1}{4}\cdot\frac{-1}{-1 + 0.75}\right| = 1.$$

Comparing with the plots given in the problem description, the correct answer is Plot B.

b) Computing the values $|H(0)|$ and $|H(\pi)|$ yields

$$|H(\Omega = 0)| = |H(z = 1)| = \left|\frac{5}{16}\cdot\frac{(1+1)^2}{1^2 + 0.25}\right| = 1$$
$$|H(\Omega = \pi)| = |H(z = -1)| = \left|\frac{5}{16}\cdot\frac{(-1+1)^2}{(-1)^2 + 0.25}\right| = 0.$$

Comparing with the plots given in the problem description, the correct answer is Plot C.

Problem 9 - Solution

Given the LTI system $y[n] - 0.8y[n-1] = u[n]$:

a) Find the transfer function:

$$
\begin{aligned}
y[n] - 0.8y[n-1] &= u[n] \\
Y(z) - 0.8z^{-1}Y(z) &= U(z) \\
(1 - 0.8z^{-1})Y(z) &= U(z) \\
H(z) &= \frac{1}{1 - 0.8z^{-1}} \\
H(z) &= \frac{z}{z - 0.8}
\end{aligned}
$$

b) Yes the system is stable, the only pole is at z=0.8, which is inside the unit circle.

c) We now substitute $z = e^{j\Omega} = \cos(\Omega) + j\sin(\Omega)$ and solve for the magnitude and the phase response.

$$
\begin{aligned}
H(z) &= \frac{1}{1 - 0.8e^{-j\Omega}} \\
H(\Omega) &= \frac{1}{1 - 0.8(\cos(\Omega) - j\sin(\Omega))} \\
&= \frac{1}{(1 - 0.8\cos(\Omega)) + (j0.8\sin(\Omega))}
\end{aligned}
$$

i) Calculating the magnitude response:

$$
\begin{aligned}
|H(\Omega)| &= \frac{|1|}{|(1 - 0.8\cos(\Omega)) + (j0.8\sin(\Omega))|} \\
&= \frac{1}{\sqrt{(1 - 0.8\cos(\Omega))^2 + (j0.8\sin(\Omega))^2}} \\
&= \frac{1}{\sqrt{1.64 - 1.6\cos(\Omega)}}
\end{aligned}
$$

ii) Calculating the phase response:

$$
\begin{aligned}
\Theta_H(\Omega) &= \tan^{-1}\left(\frac{1}{(1 - 0.8\cos(\Omega)) + (j0.8\sin(\Omega))}\right) \\
&= 0 - \tan^{-1}\left(\frac{0.8\sin(\Omega)}{1 - 0.8\cos(\Omega)}\right)
\end{aligned}
$$

d) From the results in part c) we conclude that a sinusoidal input with $\Omega = \pi/10$ applied for all time, will result in a sinusoidal output of equal frequency, with $|H(\Omega = \pi/10)| = 2.9073$ times the magnitude, and with a phase shift of $\Theta_H(\Omega = \pi/10) = -0.8020$ radians. Therefore, the input

$$u[n] = \cos((\pi/10)n) \text{ for all } n \tag{2}$$

will result in a steady state output of

$$y[n] = 2.9073\cos((\pi/10)n - 0.8020) \text{ for all } n. \tag{3}$$

e) See MATLAB solution

f) See MATLAB solution

g) We see the beginning of the simulated response does not correspond with our expectation. This is due to the system's *transient* response. In the lecture, we have seen that the above result holds when the input is applied *for all time*. When simulating the system using `lsim`, the input is applied starting at time $n = 0$, and hence the transient response affects the system's output!

h) When simulating for multiple periods, the system's output converges to the expected value, as the transient response converges to 0. We will look at this in more detail in future lectures.