Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
IDSC
Institut für Dynamische Systeme und Regelungstechnik

Final Exam **February 9th, 2011**

Signals & Systems (151-0575-01)

Prof. R. D'Andrea

Solutions

Exam Duration:	**150 minutes**
Number of Problems:	**10**
Permitted aids:	One double-sided A4 sheet. Questions can be answered in English or German. Use only the prepared sheets for your solutions. Additional paper is available from the supervisors.

Problem 1 **5 points**

A linear, time invariant system with input $x[n]$ and output $y[n]$ has the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{10 - 2z^{-1}}{a^2 + 2az^{-1} + z^{-2}}$$

a) For what real values of a is the system causal and stable? **(3 points)**

b) Write down a causal difference equation of the above system of the following form: **(2 points)**

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

Solution 1

a) The poles of the system are at

$$\begin{aligned} 0 &= a^2 + 2az^{-1} + z^{-2} \\ &= (a + z^{-1})^2 \\ z^{-1} &= -a \\ z &= -\frac{1}{a} \end{aligned}$$

For the system to be causal and stable, the poles must lie within the unit circle. It follows that $|a| > 1$ must hold.

b)

$$\begin{aligned} \frac{Y(z)}{X(z)} &= \frac{10 - 2z^{-1}}{a^2 + 2az^{-1} + z^{-2}} \\ Y(z)(a^2 + 2az^{-1} + z^{-2}) &= X(z)(10 - 2z^{-1}) \\ a^2 y[n] + 2ay[n-1] + y[n-2] &= 10x[n] - 2x[n-1] \\ y[n] &= \frac{1}{a^2}(10x[n] - 2x[n-1] - 2ay[n-1] - y[n-2]) \end{aligned}$$

Problem 2 — 5 points

A system is governed by the difference equation

$$y[n] = x[n]x[n-1] + \cos\left(3\pi n - \frac{\pi}{3}\right),$$

where $x[n]$ is the input and $y[n]$ is the output. Is the system

a)	linear?	Yes	No
b)	time invariant?	Yes	No
c)	bounded input bounded output (BIBO) stable?	Yes	No
d)	memoryless?	Yes	No
e)	causal?	Yes	No

Please circle either 'Yes' or 'No'. If you change your mind, please cross out both 'Yes' and 'No' and write either 'Yes' or 'No' alongside, or leave it blank. It is not necessary to justify choices.
You can get a maximum of 5 points and a minimum of 0 points for this problem. For each subproblem you get: +1 for a correct answer, -1 for an incorrect answer and 0 for no answer.

Solution 2

a) No
b) No
c) Yes
d) No
e) Yes

Problem 3 **5 points**

The output of a system T is $y[n] = \mathrm{T}\{x[n]\}$. T is a causal, linear, time invariant system. Given the input and output below, calculate the impulse response $h[n]$ of the system T for all n.

n	< 0	0	1	2	3	4	5	> 5
$x[n]$	0	1	-1	-2	0	0	0	0
$y[n]$	0	3	-2	-2	-5	-12	-4	0

Solution 3

Because the system is given to be causal, $h[n] = 0$ for all $n < 0$. One can see from the input output behavior that the system has a finite impulse response (the output goes to zero a finite number of steps after the input goes to zero). The input is zero for $n > 2$, and the output is zero for $n > 5$; therefore the impulse response must be zero for $n > 3$.
$n = 0$:

$$\begin{aligned} y[0] &= h[0]x[0] \\ h[0] &= \frac{y[0]}{x[0]} \\ &= 3 \end{aligned}$$

$n = 1$:

$$\begin{aligned} y[1] &= h[0]x[1] + h[1]x[0] \\ h[1] &= \frac{y[1] - h[0]x[1]}{x[0]} \\ &= 1 \end{aligned}$$

$n = 2$:

$$\begin{aligned} y[2] &= h[0]x[2] + h[1]x[1] + h[2]x[0] \\ h[2] &= \frac{y[2] - h[0]x[2] - h[1]x[1]}{x[0]} \\ &= 5 \end{aligned}$$

$n = 3$:

$$\begin{aligned} y[3] &= h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0] \\ h[3] &= \frac{y[3] - h[0]x[3] - h[1]x[2] - h[2]x[1]}{x[0]} \\ &= 2 \end{aligned}$$

Problem 4 **5 points**

The transfer function of a filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1}}.$$

Calculate the coefficients b_0 and a_1 such that the filter is stable and causal, and such that the frequency response $H(\Omega)$ of the filter fulfills the two criteria

$$H(\Omega = 0) = 1,$$

and

$$\left| H\left(\Omega = \frac{\pi}{2}\right) \right| = \frac{1}{\sqrt{2}}.$$

Solution 4

The first criterion yields

$$1 = \frac{b_0}{1 + a_1 e^{-j0}} = \frac{b_0}{1 + a_1},$$

$$1 + a_1 = b_0.$$

From the second criterion, we obtain

$$\begin{aligned} \frac{1}{\sqrt{2}} &= \left| \frac{b_0}{1 - j a_1} \right| \\ &= \frac{|b_0|}{\sqrt{1 + a_1^2}}. \end{aligned}$$

Combining the two criteria, we obtain

$$\frac{1}{\sqrt{2}} = \frac{|1 + a_1|}{\sqrt{1 + a_1^2}}.$$

It follows that

$$\begin{aligned} 0 &= \frac{(1 + a_1)^2}{1 + a_1^2} - \frac{1}{2}. \\ &= a_1^2 + 4a_1 + 1. \end{aligned}$$

Solving the quadratic equation for a_1, we obtain the two solutions

$$a_1 = -2 \pm \sqrt{3}.$$

Since the filter needs to be stable, causal, and has a pole at $z = -a_1$, the only possible solution is

$$a_1 = -2 + \sqrt{3},$$

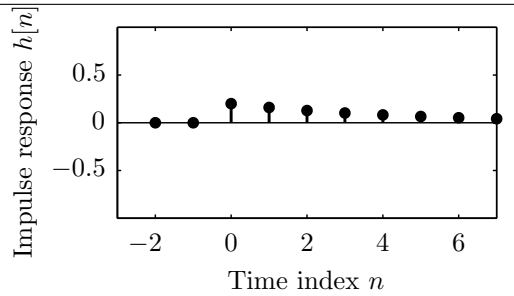
from which follows

$$b_0 = -1 + \sqrt{3}.$$

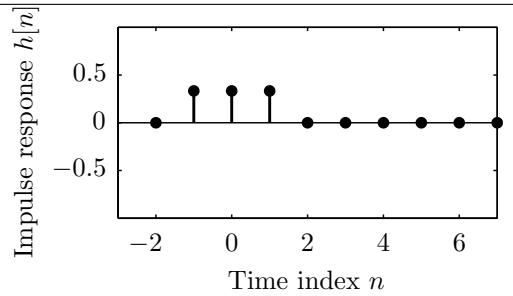
Problem 5 — 5 points

Associate the following impulse responses ((a) to (e)) and frequency responses ((1) to (5)) with the corresponding difference equation by filling out the following table. It is not necessary to justify choices.

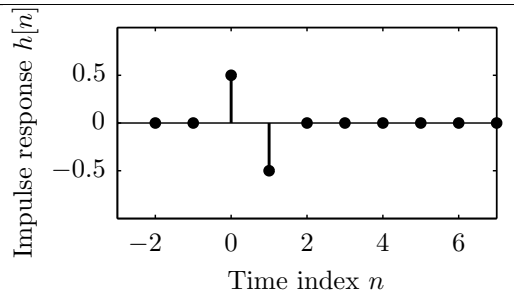
Difference equation	Impulse Response (a) to (e)	Frequency Response (1) to (5)
$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$		
$y[n] = \frac{1}{2}\left(x[n] - x[n-1]\right)$		
$y[n] = 0.8x[n] + 0.2y[n-1]$		
$y[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right)$		
$y[n] = 0.2x[n] + 0.8y[n-1]$		



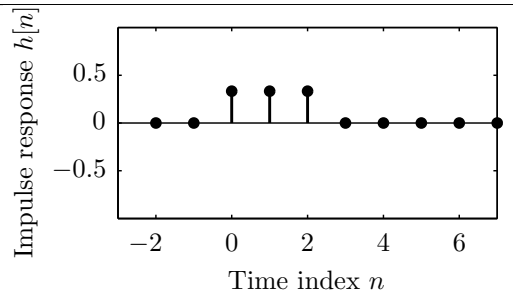
(a)



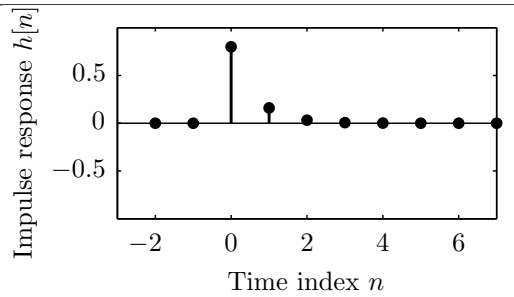
(b)



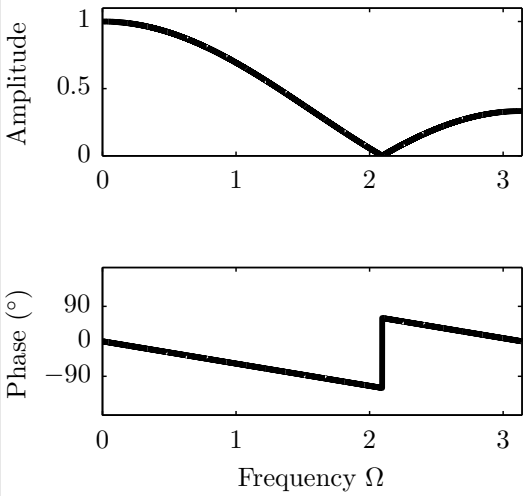
(c)



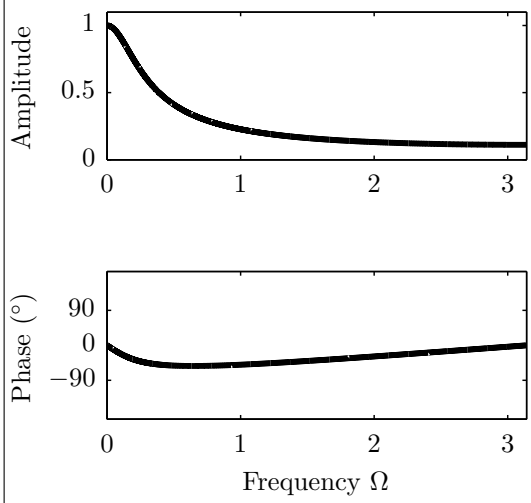
(d)



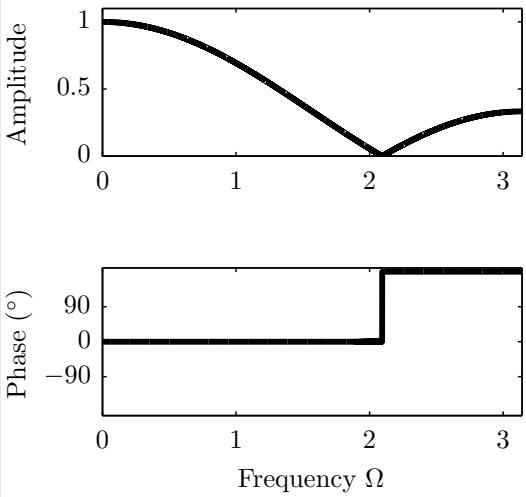
(e)


Amplitude
1
0.5
0
0
1
2
3
Phase (°)
90
0
−90
0
1
2
3
Frequency Ω

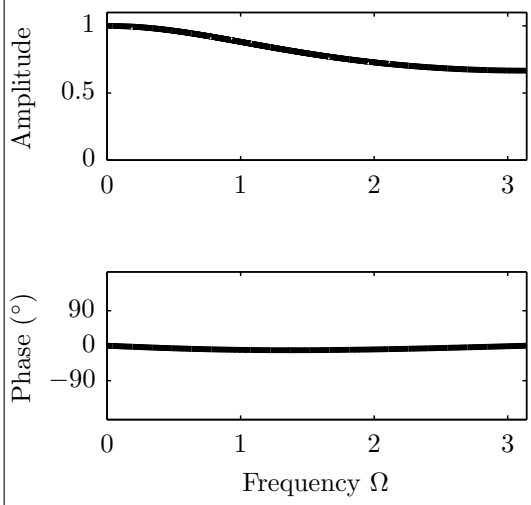
(1)


Amplitude
1
0.5
0
0
1
2
3
Phase (°)
90
0
−90
0
1
2
3
Frequency Ω

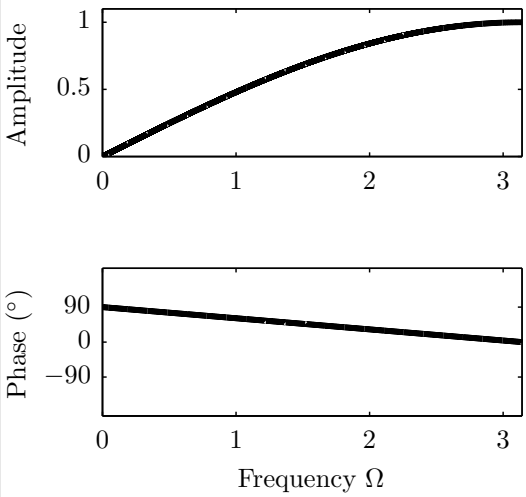
(2)


Amplitude
1
0.5
0
0
1
2
3
Phase (°)
90
0
−90
0
1
2
3
Frequency Ω

(3)


Amplitude
1
0.5
0
0
1
2
3
Phase (°)
90
0
−90
0
1
2
3
Frequency Ω

(4)


Amplitude
1
0.5
0
0
1
2
3
Phase (°)
90
0
−90
0
1
2
3
Frequency Ω

(5)

Solution 5

Difference equation	Impulse Response (a) to (e)	Frequency Response (1) to (5)
$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$	d	1
$y[n] = \frac{1}{2}\left(x[n] - x[n-1]\right)$	c	5
$y[n] = 0.8x[n] + 0.2y[n-1]$	e	4
$y[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right)$	b	3
$y[n] = 0.2x[n] + 0.8y[n-1]$	a	2

Problem 6 **5 points**

For a zero mean signal $x[n]$,

$$R_{xx}[0] = \sigma_x^2,$$

where R_{xx} is the autocorrelation function and σ_x^2 is the variance of the signal $x[n]$. The autocorrelation function is defined as

$$R_{xx}[k] := \mathrm{E}\left(x[n]x[n-k]\right),$$

where $\mathrm{E}(\cdot)$ is the expected value operator. Let $x[n]$ be white noise with variance $\sigma_x^2 = 2$. The signal $x[n]$ is applied to a linear time invariant system T to generate the output

$$y[n] = \mathrm{T}\left\{x[n]\right\}.$$

The frequency response $H(\Omega)$ of T can be approximated by

$$H(\Omega) = \begin{cases} \frac{1}{2}e^{-j\Omega}, & 0 \leq |\Omega| \leq \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}.$$

Calculate the frequency Ω_c such that the variance of the output signal $y[n]$ is $\sigma_y^2 = 0.4$.

Solution 6

The power spectral density function of $y[n]$ is

$$S_{yy}(\Omega) = |H(\Omega)|^2 S_{xx}(\Omega) = \begin{cases} \frac{1}{4}\sigma_x^2 = \frac{1}{2}, & 0 \leq |\Omega| \leq \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}.$$

The system T is linear, therefore the zero mean input $x[n]$ results in a zero mean output $y[n]$. We calculate the variance of $y[n]$ using the inverse Fourier transform

$$\begin{aligned} \sigma_y^2 = R_{yy}[k=0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\Omega)\, e^{j\Omega\cdot 0} d\Omega \\ &= \frac{1}{\pi}\int_0^{\Omega_c} \frac{1}{2} d\Omega = \frac{\Omega_c}{2\pi}. \end{aligned}$$

Solving for Ω_c with $\sigma_y^2 = 0.4$, we obtain

$$\Omega_c = \frac{4\pi}{5}.$$

Problem 7 **5 points**

Consider the continuous time signal

$$x(t) = 4\cos\left(6\pi t + \frac{\pi}{3}\right) + 2\cos(8\pi t + \pi) - 2.$$

a) Determine the largest possible sampling time T_s in seconds to sample the signal without aliasing effects. **(1 point)**

b) Sample the given continuous time signal with the sampling time you obtained in a), starting at $t = 0$: **(1 point)**

$$x[0] = x(0), \quad x[1] = x(T_s), \quad \ldots, \quad x[n] = x(nT_s).$$

Determine the fundamental period N_0 of the discrete time signal $x[n]$.

c) State the Fourier series coefficients c_k given by **(3 points)**

$$c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\frac{j2\pi k}{N_0}n}, \qquad k = 0, 1, ..., N_0 - 1.$$

Solution 7

a) The largest possible sampling time is $T_s = 1/8\,\text{s}$.

b) The sampled signal is

$$x[n] = 4\cos\left(\frac{3}{4}\pi n + \frac{\pi}{3}\right) + 2\cos(\pi n + \pi) - 2.$$

The fundamental period of the signal is $N_0 = 8$.

c) The coefficients are

k	0	1	2	3	4	5	6	7
c_k	-2	0	0	$2e^{j\frac{\pi}{3}}$	-2	$2e^{-j\frac{\pi}{3}}$	0	0

.

For a detailed solution, please refer to the solution of problem 4 of the 2009 final exam or the recitation notes posted on Nov. 5 on the class website.

Problem 8 **5 points**

A continuous time, linear, time invariant system with input $x(t)$ and output $y(t)$ is given by the state space description

$$\begin{aligned} \dot{q}(t) &= Aq(t) + Bx(t) \\ y(t) &= Cq(t) + Dx(t), \end{aligned}$$

where the state vector has two elements $q(t) = \{q_1(t), q_2(t)\}^\mathsf{T}$ and the matrices are

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix} & D &= [0.5] \end{aligned}$$

Assume the input $x(t)$ is piece-wise constant

$$x(t) = x[k] \qquad kT_s \leq t < (k+1)T_s$$

with sampling time T_s. Compute A_d, B_d, C_d, D_d of a discrete time state space description of the system

$$\begin{aligned} q[k+1] &= A_d\ q[k] + B_d\ x[k] \\ y[k] &= C_d\ q[k] + D_d\ x[k], \end{aligned}$$

when the output $y[k]$ is defined as

$$y[k] := y(t = kT_s).$$

Solution 8

We solve this problem using the matrix exponential (see lecture notes for derivation). We define:

$$M := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The discrete time matrices A_d, B_d are then obtained by calculating

$$\begin{bmatrix} A_d & B_d \\ 0 & 1 \end{bmatrix} = e^{MT_s}.$$

Since the matrix M is nilpotent, i.e. $M^3 = 0$, it is straightforward to calculate the matrix exponential:

$$\begin{bmatrix} A_d & B_d \\ 0 & 1 \end{bmatrix} = I + MT_s + \frac{1}{2}M^2T_s^2,$$

where I is the identity matrix. The matrices are therefore

$$\begin{array}{ll} A_d = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} & B_d = \begin{bmatrix} \frac{1}{2}T_s^2 \\ T_s \end{bmatrix} \\ C_d = C & D_d = D \end{array}.$$

Problem 9 **5 points**

You are given the state space description of a system:

$$q[n+1] = \begin{bmatrix} 2 & -\frac{1}{4} \\ 25 & -3 \end{bmatrix} q[n] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} x[n]$$
$$y[n] = \begin{bmatrix} -1 & 1 \end{bmatrix} q[n] + \begin{bmatrix} 4 \end{bmatrix} x[n]$$

a) Is the system bounded input bounded output stable? **(2 points)**

b) Is the system controllable? **(1 point)**

c) Calculate the step response of the system for $n = 0, 1, 2$. **(1 point)**

d) Consider the second system **(1 point)**

$$q[n+1] = \begin{bmatrix} -\frac{1}{4} & 2 \\ -3 & 25 \end{bmatrix} q[n] + \begin{bmatrix} -1 \\ 2 \end{bmatrix} x[n]$$
$$y[n] = \begin{bmatrix} 1 & -1 \end{bmatrix} q[n] + \begin{bmatrix} 3 \end{bmatrix} x[n]$$

is the input output behavior of this system identical to the first one?

Solution 9

a) The eigenvalues of the matrix A are $\lambda_1 = \lambda_2 = -\frac{1}{2}$. The system is therefore stable.

b) The controllability matrix is $[B\ AB] = \begin{bmatrix} 2 & \frac{17}{4} \\ -1 & 53 \end{bmatrix}$. The matrix has full row rank and the system is therefore controllable.

c) $n = 0$:

$$\begin{aligned} q[1] &= \begin{bmatrix} 2 & -\frac{1}{4} \\ 25 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} 1 \\ &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ y[0] &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \end{bmatrix} 1 \\ &= 4 \end{aligned}$$

$n = 1$:

$$\begin{aligned} q[2] &= \begin{bmatrix} 2 & -\frac{1}{4} \\ 25 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} 1 \\ &= \begin{bmatrix} \frac{25}{4} \\ 52 \end{bmatrix} \\ y[1] &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \end{bmatrix} 1 \\ &= 1 \end{aligned}$$

$n = 2$:

$$\begin{aligned} y[2] &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{25}{4} \\ 52 \end{bmatrix} + \begin{bmatrix} 4 \end{bmatrix} 1 \\ &= \frac{199}{4} \end{aligned}$$

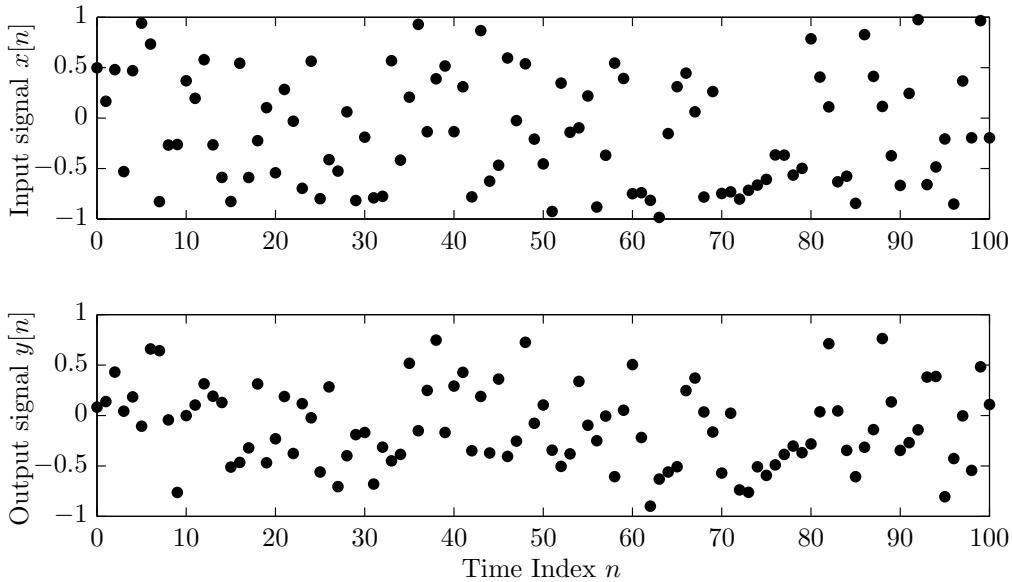
d) No. An easy way to see this is the first value of the step response: The state of both systems is $q[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The output of the first system is therefore $y[0] = 4$, while the output of the second system is $y[0] = 3$.

Problem 10 — 5 points

Assume that you are identifying a linear time invariant system T with input $x[n]$ and output $y[n]$:

$$y[n] = \mathrm{T}\{x[n]\}$$

You applied the input $x[n]$ and measured the output $y[n]$ for 101 samples, i.e. $0 \leq n \leq 100$, as shown below:



You want to use this data to identify a model of the form

$$\widehat{H}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

in the frequency domain.

a) Let $Y(\Omega)$ be the Discrete Fourier Transform of the measured output $y[n]$. How many frequency points will the Transform have in the range $0 \leq \Omega \leq \pi$? **(1 point)**

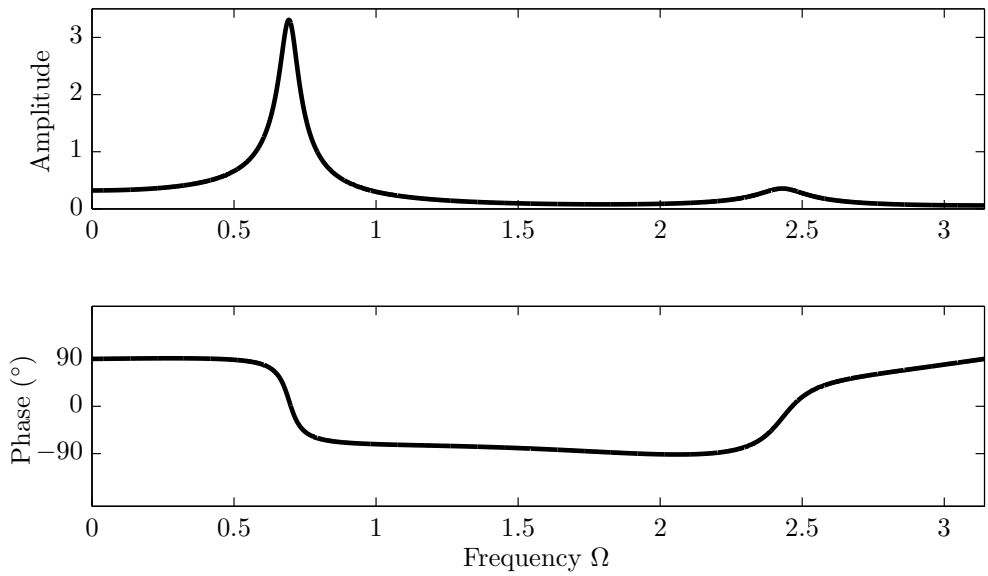
Now assume that you have calculated the system transfer function

$$H(\Omega_k) = \frac{Y(\Omega_k)}{X(\Omega_k)}$$

at each frequency point Ω_k.

b) When fitting a model to the system transfer function, what are the highest possible model orders N and M that you can identify using only the given measurement data? **(2 points)**

c) A colleague has identified a model from the given measurement data. A plot of the frequency response of his estimated model $\widehat{H}(z)$ is shown below. Is this result plausible, and why? **(2 points)**



Solution 10

a) There will be 51 points. The Discrete Fourier Transform will result in 101 frequency domain points in a 2π interval. This means that the spacing between frequency points will be

$$\Delta_\Omega = \frac{2\pi}{101}.$$

There will be one point at $\Omega = 0$. There will be 50 more points in the interval, because $50\Delta_\Omega < \pi$ and $51\Delta_\Omega > \pi$.

b) There are 101 pieces of information (the point at $\Omega = 0$ is real, and all points at $\Omega \neq 0$ are complex.). Therefore,

$$N + M + 1 \leq 101$$
$$N + M \leq 100.$$

c) No. The system has a phase of 90° at $\Omega = 0$, which means that the system has a complex gain. This is clearly not the case, as the input-output data is real.