Stochastic Differential Equations

Do not worry about your problems with mathematics, I assure you mine are far greater.

Albert Einstein.

Florian Herzog

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Stochastic Differential Equations (SDE)

A ordinary differential equation (ODE)

$$\frac{dx(t)}{dt} = f(t,x)\,, \quad dx(t) = f(t,x)dt\,, \tag{1}$$

with initial conditions $x(0) = x_0$ can be written in integral form

$$x(t) = x_0 + \int_0^t f(s, x(s))ds\,, \tag{2}$$

where $x(t) = x(t, x_0, t_0)$ is the solution with initial conditions $x(t_0) = x_0$. An example is given as

$$\frac{dx(t)}{dt} = a(t)x(t)\,, \quad x(0) = x_0\,. \tag{3}$$

Stochastic Differential Equations (SDE)

When we take the ODE (3) and assume that $a(t)$ is not a deterministic parameter but rather a stochastic parameter, we get a stochastic differential equation (SDE). The stochastic parameter $a(t)$ is given as

$$a(t) = f(t) + h(t)\xi(t)\,, \tag{4}$$

where $\xi(t)$ denotes a white noise process.
Thus, we obtain

$$\frac{dX(t)}{dt} = f(t)X(t) + h(t)X(t)\xi(t)\,. \tag{5}$$

When we write (5) in the differential form and use $dW(t) = \xi(t)dt$, where $dW(t)$ denotes differential form of the Brownian motion,we obtain:

$$dX(t) = f(t)X(t)dt + h(t)X(t)dW(t)\,. \tag{6}$$

Stochastic Differential Equations (SDE)

In general an SDE is given as

$$dX(t,\omega) = f(t, X(t,\omega))dt + g(t, X(t,\omega))dW(t,\omega)\,, \tag{7}$$

where ω denotes that $X = X(t,\omega)$ is a random variable and possesses the initial condition $X(0,\omega) = X_0$ with probability one. As an example we have already encountered

$$dY(t,\omega) = \mu(t)dt + \sigma(t)dW(t,\omega)\,.$$

Furthermore, $f(t, X(t,\omega)) \in \mathbb{R}$, $g(t, X(t,\omega)) \in \mathbb{R}$, and $W(t,\omega) \in \mathbb{R}$. Similar as in (2) we may write (7) as integral equation

$$X(t,\omega) = X_0 + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dW(s,\omega)\,. \tag{8}$$

For the calculation of the stochastic integral $\int_0^T g(t,\omega)dW(t,\omega)$, we assume that $g(t,\omega)$ is only changed at discrete time points $t_i\,(i = 1, 2, 3, \ldots, N-1)$, where $0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N < T$. We define the integral

$$S = \int_0^T g(t,\omega)dW(t,\omega)\,, \tag{9}$$

as the Riemannßum

$$S_N(\omega) = \sum_{i=1}^{N} g(t_{i-1},\omega)\Big(W(t_i,\omega) - W(t_{i-1},\omega)\Big)\,. \tag{10}$$

with $N \to \infty$.

Stochastic Integrals

A random variable S is called the Itô integral of a stochastic process $g(t, \omega)$ with respect to the Brownian motion $W(t, \omega)$ on the interval $[0, T]$ if

$$\lim_{N \to \infty} \mathsf{E}\Big[\Big(S - \sum_{i=1}^{N} g(t_{i-1}, \omega)\Big(W(t_i, \omega) - (W(t_{i-1}, \omega)\Big)\Big] = 0\,, \qquad (11)$$

for each sequence of partitions $(t_0, t_1, \ldots, t_N)$ of the interval $[0, T]$ such that $\max_i(t_i - t_{i-1}) \to 0$. The limit in the above definition converges to the stochastic integral in the mean-square sense. Thus, the stochastic integral is a random variable, the samples of which depend on the individual realizations of the paths $W(., \omega)$.

Stochastic Integrals

The simplest possible example is $g(t) = c$ for all t. This is still a stochastic process, but a simple one. Taking the definition, we actually get

$$
\begin{aligned}
\int_0^T c\, dW(t,\omega) &= c \lim_{N\to\infty} \sum_{i=1}^{N} \Big(W(t_i,\omega) - W(t_{i-1},\omega) \Big) \\
&= c \lim_{N\to\infty} [(W(t_1,\omega) - W(t_0,\omega)) + (W(t_2,\omega) - W(t_1,\omega)) + \ldots \\
&\quad + (W(t_N,\omega) - W(t_{N-1},\omega)) \\
&= c\,(W(T,\omega) - W(0,\omega))\,,
\end{aligned}
$$

where $W(T,\omega)$ and $W(0,\omega)$ are standard Gaussian random variables. With $W(0,\omega) = 0$, the last result becomes

$$\int_0^T c\, dW(t,\omega) = c\, W(T,\omega)\,.$$

Example: $g(t,\omega) = W(t,\omega) \quad \int_0^T W(t,\omega)\, dW(t,\omega) =$

$$
\begin{aligned}
&= \lim_{N\to\infty} \sum_{i=1}^{N} W(t_{i-1},\omega)\Big(W(t_i,\omega) - W(t_{i-1},\omega)\Big) \\
&= \lim_{N\to\infty} \Big[\frac{1}{2}\sum_{i=1}^{N}(W^2(t_i,\omega) - W^2(t_{i-1},\omega)) - \frac{1}{2}\sum_{i=1}^{N}(W(t_i,\omega) - W(t_{i-1},\omega))^2\Big] \\
&= -\frac{1}{2}\lim_{N\to\infty}\sum_{i=1}^{N}(W(t_i,\omega) - W(t_{i-1},\omega))^2 + \frac{1}{2}W^2(T,\omega)\,, \qquad (12)
\end{aligned}
$$

where we have used the following algebraic relationship $y(x-y) = yx - y^2 + \frac{1}{2}x^2 - \frac{1}{2}x^2 = \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}(x-y)^2$.

Stochastic Integrals

We take now a detailed look at :$\lim_{N\to\infty}\sum_{i=1}^{N}(W(t_i,\omega)-W(t_{i-1},\omega))^2$.

$$
\begin{aligned}
\mathsf{E}[\lim_{N\to\infty}\sum_{i=1}^{N}(W(t_i,\omega)-W(t_{i-1},\omega))^2] &= \lim_{N\to\infty}\sum_{i=1}^{N}\mathsf{E}[(W(t_i,\omega)-W(t_{i-1},\omega))^2] \\
&= \lim_{N\to\infty}\sum_{i=1}^{N}(t_i-t_{i-1}) \\
&= T \\
\mathsf{Var}[\lim_{N\to\infty}\sum_{i=1}^{N}(W(t_i,\omega)-W(t_{i-1},\omega))^2] &= \lim_{N\to\infty}\sum_{i=1}^{N}\mathsf{Var}[(W(t_i,\omega)-W(t_{i-1},\omega))^2] \\
&= 2\lim_{N\to\infty}\sum_{i=1}^{N}(t_i-t_{i-1})^2 .
\end{aligned}
$$

By reducing the partition, the variance becomes zero,

$$\begin{aligned}
\lim_{N\to\infty}\sum_{i=1}^{N}(t_i - t_{i-1})^2 &\leq \max_i(t_i - t_{i-1}) \lim_{N\to\infty}\sum_{i=1}^{N}(t_i - t_{i-1}) \\
&= \max_i(t_i - t_{i-1})\, T \\
&= 0\,, \qquad (13)
\end{aligned}$$

since $t_{i-1} - t_i \to 0$. Since the expected value of $\sum_{i=1}^{N}(t_i - t_{i-1})^2$ is T and the variance becomes zero, we get

$$\sum_{i=1}^{N}(W(t_i,\omega) - W(t_{i-1},\omega))^2 = T \qquad (14)$$

Stochastic Integrals

The stochastic integral has the solution

$$\int_0^T W(t,\omega)\,dW(t,\omega) = \frac{1}{2}W^2(T,\omega) - \frac{1}{2}T \tag{15}$$

This is in contrast to our intuition from standard calculus. In the case of a deterministic integral $\int_0^T x(t)dx(t) = \frac{1}{2}x^2(t)$, whereas the Itô integral differs by the term $-\frac{1}{2}T$. — This example shows that the rules of differentiation (in particular the chain rule) and integration need to be re-formulated in the stochastic calculus.

Stochastic Integrals

Properties of Itô Integrals.

$$E[\int_0^T g(t,\omega)\, dW(t,\omega)] = 0\,.$$

Proof:

$$\begin{aligned} E[\int_0^T g(t,\omega) dW(t,\omega)] &= E[\lim_{N\to\infty} \sum_{i=1}^{N} g(t_{i-1},\omega)\Big(W(t_i,\omega) - W(t_{i-1},\omega)\Big)] \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} E[g(t_{i-1},\omega)]\, E[\Big(W(t_i,\omega) - W(t_{i-1},\omega)\Big)] \\ &= 0\,. \end{aligned}$$

The expectation of stochastic integrals is zero. This is what we would expect anyway.

Stochastic Integrals

Properties of Itô Integrals.

$$\text{Var}\Big[\int_0^T g(t,\omega)dW(t,\omega)\Big] = \int_0^T E[g^2(t,\omega)]dt\,. \tag{16}$$

Proof:

$$\begin{aligned}\text{Var}\Big[\int_0^T g(t,\omega)dW(t,\omega)\Big] &= E\Big[(\int_0^T g(t,\omega)dW(t,\omega))^2\Big] \\ &= E\Big[\Big(\lim_{N\to\infty}\sum_{i=1}^N g(t_{i-1},\omega)\Big(W(t_i,\omega)-W(t_{i-1},\omega)\Big)\Big)^2\Big]\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} E[g(t_{i-1},\omega)g(t_{j-1},\omega) \\
&\quad (W(t_i,\omega) - W(t_{i-1},\omega))(W(t_j,\omega) - W(t_{j-1},\omega))] \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} E[g^2(t_{i-1},\omega)]\, E[\Big(W(t_i,\omega) - W(t_{i-1},\omega)\Big)^2] \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} E[g^2(t_{i-1},\omega)]\, (t_i - t_{i-1}) \\
&= \int_0^T E[g^2(t,\omega)]dt\,. \qquad (17)
\end{aligned}
$$

Stochastic Integrals

The calculation of the variance of the Itô Integrals shows two important properties:

- $\mathsf{E}\Big[\Big(\int_0^T g(t,\omega)dW(t,\omega)\Big)^2\Big] = \int_0^T \mathsf{E}\Big[g^2(t,\omega)\Big]dt$
- $\int_0^T \mathsf{E}[g^2(t,\omega)]dt < \infty$

The second property is the condition of existence for Itô integrals. The next property is the linearity of Itô integrals:

$$\int_0^T [a_1\, g_1(t,\omega) + a_2\, g_2(t,\omega)]dW(t,\omega)$$

$$= a_1 \int_0^T g_1(t,\omega)dW(t,\omega) + a_2 \int_0^T g_2(t,\omega)dW(t,\omega)\,, \tag{18}$$

for numbers a_1, a_2 and stochastic functions $g_1(t,\omega)$, $g_2(t,\omega)$.

Itô's lemma

As mentioned shown in the second example, the rules of classical calculus are not valid for stochastic integrals and differential equations. It is the equivalent to the chain rule in classical calculus. The problem can be stated as follows:
Given a stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \; , \tag{19}$$

and another process $Y(t)$ which is a function of $X(t)$,

$$Y(t) = \phi(t, X(t)) \; ,$$

where the function $\phi(t, X(t))$ is continuously differentiable in t and twice continuously differentiable in X, find the stochastic differential equation for the process $Y(t)$:

$$dY(t) = \tilde{f}(t, X(t))dt + \tilde{g}(t, X(t))dW(t) \; .$$

Itô's lemma

In the case when we assume that $g(t, X(t)) = 0$, we know the result: the chain rule for standard calculus. The result is given by

$$dy(t) = (\phi_t(t, x) + \phi_x(t, x) f(t, x))dt\,. \tag{20}$$

In the case of stochastic problems, we reason as follows: The Taylor expansion of $\phi(t, X(t))$ yields

$$\begin{aligned} dY(t) &= \phi_t(t, X)dt + \frac{1}{2}\phi_{tt}(t, X)dt^2 + \phi_x(t, X)dX(t) \\ &\quad + \frac{1}{2}\phi_{xx}(t, X)(dX(t))^2 + \text{ h.o.t}\,. \end{aligned} \tag{21}$$

Itô's lemma

We use (19) for $dX(t)$ and get

$$\begin{aligned} dY(t) &= \phi_t(t,X)dt + \phi_x(t,X)[f(t,X(t))dt + g(t,X(t))dW(t)] \\ &\quad +\phi_{tt}(t,X)dt^2 + \frac{1}{2}\phi_{xx}(t,X)\Big(f^2(t,X(t))dt^2 + g^2(t,X(t))dW^2(t) \\ &\quad +2f(t,X(t))g(t,X(t))dt\,dW(t)\Big) + \text{ h.o.t}\,. \end{aligned} \tag{22}$$

The differentials of higher order (dt, dW) become fast zero, $dt^2 \to 0$ and $dtdW(t) \to 0$. The stochastic term $dW^2(t)$ according to the rules of Brownian motion is given as

$$dW^2(t,\omega) = dt\,. \tag{23}$$

Itô's lemma

Omitting higher order terms and using the properties of Brownian motion, we arrive at

$$
\begin{aligned}
dY(t) &= [\phi_t(t,X) + \phi_x(t,X)f(t,X(t)) + \frac{1}{2}\phi_{xx}(t,X)g^2(t,X(t))]dt \\
&\quad +\phi_x(t,X)g(t,X(t))dW(t)\,. \qquad (24)
\end{aligned}
$$

Reordering the terms yields the scalar version of Itô's Lemma:

$$
\begin{aligned}
dY(t) &= \tilde{f}(t,X(t))dt + \tilde{g}(t,X(t))dW(t)\,, \qquad (25)\\
\tilde{f}(t,X(t)) &= \phi_t(t,X) + \phi_x(t,X)f(t,X(t)) \\
&\quad +\frac{1}{2}\phi_{xx}(t,X)g^2(t,X(t))\,, \qquad (26)\\
\tilde{g}(t,X(t)) &= \phi_x(t,X)g(t,X(t))\,. \qquad (27)
\end{aligned}
$$

Itô's lemma

The term $\frac{1}{2}\phi_{xx}(t, X)g^2(t, X(t))$ is often called the *Itô corretion term*, since this does not occur in the det. case.
We apply Itôs formula for the following problem: $\phi(t, X) = X^2$ with the SDE $dX(t) = dW(t)$. From the SDE, we get $X(t) = W(t)$ and calculate the partial derivatives of $\frac{\partial\phi(t,X)}{\partial X} = 2X$, $\frac{\partial^2\phi(t,X)}{\partial X^2} = 2$, and $\frac{\partial\phi(t,X)}{\partial t} = 0$. The Itô lemma yields

$$d(W^2(t)) = 1dt + 2W(t)dW(t)\,. \tag{28}$$

We rewrite the equation and use $W(0) = 0$

$$\begin{aligned} W^2(t) &= 1t + 2\int_0^t W(t)dW(t)\,, \\ \int_0^t W(t)dW(t) &= \frac{1}{2}W^2(t) - \frac{1}{2}t\,. \end{aligned} \tag{29}$$

Itô's lemma

We now allow that the process $X(t)$ is in $\mathbb{R}^n$. We let $W(t)$ be an m-dimensional standard Brownian motion and $f(t, X(t)) \in \mathbb{R}^n$ and $g(t, X(t)) \in \mathbb{R}^{n \times m}$. Consider a scalar process $Y(t)$ defined by $Y(t) = \phi(t, X(t))$, where $\phi(t, X)$ is a scalar function which is continuously differentiable with respect to t and twice continuously differentiable with respect to X. The Itô formula can be written in vector notation as follows:

$$
\begin{aligned}
dY(t) &= \tilde{f}(t, X(t))dt + \tilde{g}(t, X(t))dW(t)\,, && (30) \\
\tilde{f}(t, X(t)) &= \phi_t(t, X(t)) + \phi_x(t, X(t)) \cdot f(t, X(t)) \\
&\quad + \frac{1}{2} tr\Big(\phi_{xx}(t, X(t))g(t, X(t))g^T(t, X(t))\Big)\,, && (31) \\
\tilde{g}(t, X(t)) &= \phi_x(t, X(t)) \cdot g(t, X(t))\,, && (32)
\end{aligned}
$$

where "tr" denotes the trace operator.

Itô's lemma

Consider the following stochastic differential equation:

$$dS(t) = \mu \, S(t)dt + \sigma \, S(t)dW(t) \, , \tag{33}$$

We want to find the SDE for the process Y related to S as follows: $Y(t) = \phi(t, S) = \ln(S(t))$. The partial derivatives are: $\frac{\partial \phi(t,S)}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 \phi(t,S)}{\partial S^2} = -\frac{1}{S^2}$, and $\frac{\partial \phi(t,S)}{\partial t} = 0$. Therefore, according to Itô we get,

$$\begin{aligned} dY(t) &= \Big(\frac{\partial \phi(t,S)}{\partial t} + \frac{\partial \phi(t,S)}{\partial S}\mu S(t) + \frac{1}{2}\frac{\partial^2 \phi(t,S)}{\partial S^2}\sigma^2 S^2(t)\Big)dt \\ &\quad + \Big(\frac{\partial \phi(t,S)}{\partial S}\sigma S(t)\Big)dW(t) \, , \end{aligned} \tag{34}$$

$$dY(t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t) \, . \tag{35}$$

Since the right hand side of (35) is independent of $Y(t)$, we are able to compute the stochastic integral:

$$Y(t) = Y_0 + \int_0^t (\mu - \frac{1}{2}\sigma^2)dt + \int_0^t \sigma dW\,, \quad (36)$$

$$Y(t) = Y_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\,. \quad (37)$$

Since $Y(t) = \ln S(t)$ we have found a solution for $S(t)$:

$$\ln(S(t)) = \ln(S(0)) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\,, \quad (38)$$

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}\,, \quad (39)$$

where $W(t)$ is a standard BM.

Itô's lemma

Show for $U(t) = X_1(t)X_2(t)$ with

$$\begin{aligned} dX_1(t) &= f_1(t, X_1)dt + g_1(t, X_1)dW(t) , \\ dX_2(t) &= f_2(t, X_2)dt + g_2(t, X_2)dW(t) , \end{aligned}$$

that following formula is valid:

$$dU(t) = dX_1(t)X_2(t) + X_1(t)dX_2(t) + g_1(t, X_1)g_2(t, X_2)dt \quad (40)$$

We show that we obtain the same result as in the previous formula by apply Itô's lemma. By (40) liefert

$$\begin{aligned} dU(t) &= [X_2(t)f_1(t, X_1) + X_1(t)f_2(t, X_2) + g_1(t, X_1)g_2(t, X_2)]dt \\ &\quad +[X_2(t)g_1(t, X_1) + X_1(t)g_2(t, X_2)]dW(t) \end{aligned}$$

Itô's lemma

The partial derivatives of U are : $\frac{\partial U}{\partial X} = (X_2(t), X_1(t))^T$, $\frac{\partial^2 U}{\partial X^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\frac{\partial U}{\partial t} = 0$.

$$\begin{aligned}
dU(t) &= [\frac{\partial U}{\partial t} + \frac{\partial U}{\partial X}[f_1(t, X_1), f_2(t, X_2)]^T \\
&\quad + \frac{1}{2} tr\Big(\frac{\partial^2 U}{\partial X^2} \begin{bmatrix} g_1(t, X_1)^2 & g_1(t, X_1) g_2(t, X_2) \\ g_1(t, X_1) g_2(t, X_2) & g_2(t, X_2)^2 \end{bmatrix}\Big)] dt \\
&\quad + \frac{\partial U}{\partial X}[g_1(t, X_1), g_2(t, X_2)]^T dW(t) \\
&= [X_2(t) f_1(t, X_1) + X_1(t) f_2(t, X_2) + g_1(t, X_1) g_2(t, X_2)] dt \\
&\quad + [X_2(t) g_1(t, X_1) + X_1(t) g_2(t, X_2)] dW(t)
\end{aligned}$$

Stochastic Differential Equations (SDE)

We classify SDEs into two large groups, linear SDEs and non-linear SDEs. Furthermore, we distinguish between scalar linear and vector-valued linear SDEs.
We start with the easy case, the scalar linear linear SDEs. An SDE

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) , \tag{41}$$

for a one-dimensional stochastic process $X(t)$ is called a linear (scalar) SDE if and only if the functions $f(t, X(t))$ and $g(t, X(t))$ are affine functions of $X(t) \in \mathbb{R}$ and thus

$$\begin{aligned} f(t, X(t)) &= A(t)X(t) + a(t) , \\ g(t, X(t)) &= [B_1(t)X(t) + b_1(t), \cdots , B_m(t)X(t) + b_m(t)] , \end{aligned}$$

where $A(t), a(t) \in \mathbb{R}$, $W(t) \in \mathbb{R}^m$ is an m-dimensional Brownian motion, and $B_i(t), b_i(t) \in \mathbb{R}$, $i = 1, \cdots , m$. Hence, $f(t, X(t)) \in \mathbb{R}$ and $g(t, X(t)) \in \mathbb{R}^{1\times m}$.

The linear SDE possesses the following solution

$$X(t) = \Phi(t)\Big(x_0 + \int_0^t \Phi^{-1}(s)\Big[a(s) - \sum_{i=1}^m B_i(s)b_i(s)\Big]ds + \sum_{i=1}^m \int_0^t \Phi^{-1}(s)b_i(s)dW_i(s)\Big) , \qquad (42)$$

where we denote $\Phi(t)$ as the fundamental matrix, which we obtain from

$$\Phi(t) = \exp\Big(\int_0^t \Big[A(s) - \sum_{i=1}^m \frac{B_i^2(s)}{2}\Big]ds + \sum_{i=1}^m \int_0^t B_i(s)dW_i(s)\Big) , \qquad (43)$$

The solution is similar to the solution of ODEs.

Stochastic Differential Equations (SDE)

Let us assume that $W(t) \in \mathbb{R}$, $a(t) = 0$, $b(t) = 0$, $A(t) = A$, $B(t) = B$. We want to compute the solution of the SDE

$$dX(t) = AX(t)dt + BX(t)dW(t)\,, \quad X(t) = x_0\,, \tag{44}$$

We can solve it using (42) and (43):

$$\Phi(t) = e^{(A - \frac{1}{2}B^2)t + BW(t)}\,, \tag{45}$$

and (42) is easy to calculate since

$$x(t) = \Phi(t)x_0 = x_0 e^{(A - \frac{1}{2}B^2)t + BW(t)}\,. \tag{46}$$

Stochastic Differential Equations (SDE)

The expectation $m(t) = \mathsf{E}[X(t)]$and the second moment $P(t) = \mathsf{E}[X^2(t)]$ for

$$dX(t) = (A(t)X(t) + a(t))dt + \sum_{i=1}^{m}(B_i(t)X(t) + b(t))dW_i(t)\,. \tag{47}$$

can be calculated by solving the following system of ODEs:

$$\dot{m}(t) = A(t)m(t) + a(t)\,, \quad m(0) = x_0\,, \tag{48}$$

$$\dot{P}(t) = \Big(2A(t) + \sum_{i=1}^{m} B_i^2(t)\Big)P(t) + 2m(t)\Big(a(t) + \sum_{i=1}^{m} B_i(t)b_i(t)\Big) + \sum_{i=1}^{m} b_i^2(t)\Big)\,, \quad P(0) = x_0^2\,. \tag{49}$$

Stochastic Differential Equations (SDE)

The ODE for the expectation is derived by applying the expectation operator on both sides of (42).

$$
\begin{aligned}
\mathsf{E}[dX(t)] &= \mathsf{E}\Big[(A(t)X(t) + a(t))dt + \sum_{i=1}^{m}(B_i(t)X(t) + b_i(t))dW_i(t)\Big] \\
\underbrace{\mathsf{E}[dX(t)]}_{dm(t)} &= (A(t)\underbrace{\mathsf{E}[X(t)]}_{=m(t)} + a(t))dt \\
&\quad + \sum_{i=1}^{m}\mathsf{E}[(B_i(t)X(t) + b_i(t))]\underbrace{\mathsf{E}[dW_i(t)]}_{=0} \\
dm(t) &= (A(t)m(t) + a(t))dt\,. \qquad (50)
\end{aligned}
$$

Stochastic Differential Equations (SDE)

In order to compute the second moment, we need to derive the SDE for $Y(t) = X^2(t)$:

$$
\begin{aligned}
dY(t) &= \Big[2X(t)(A(t)X(t) + a(t)) + \sum_{i=1}^{m} \Big(B_i(t)X(t) + b_i(t)\Big)^2\Big] dt \\
&\quad +2X(t)\sum_{i=1}^{m} \Big(B_i(t)X(t) + b_i(t)\Big) dW_i(t) \qquad (51) \\
dY(t) &= \Big[2A(t)X^2(t) + 2X(t)a(t) + \sum_{i=1}^{m} \Big(B_i^2(t)X^2(t) + 2B_i(t)b_i(t)X(t) \\
&\quad +b_i^2(t)\Big)\Big] dt + 2X(t)\sum_{i=1}^{m} \Big(B_i(t)X(t) + b_i(t)\Big) dW_i(t) \qquad (52)
\end{aligned}
$$

Stochastic Differential Equations (SDE)

Furthermore, we apply the expectation operator to (52) and use $P(t) = \mathsf{E}[X^2(t)] = \mathsf{E}[Y(t)]$ and $m(t) = \mathsf{E}[X(t)]$.

$$
\begin{aligned}
\mathsf{E}[dY(t)] &= \Big[2A(t)\mathsf{E}[X^2(t)] + 2a(t)\mathsf{E}[X(t)] + \sum_{i=1}^{m}\Big(B_i^2(t)\mathsf{E}[X^2(t)] \\
&\quad + 2B_i(t)b_i(t)\mathsf{E}[X(t)] + b_i^2(t)\Big)\Big]dt \\
&\quad + \mathsf{E}\Big[2X(t)\sum_{i=1}^{m}\Big(B_i(t)X(t) + b_i(t)\Big)dW_i(t)\Big] \\
dP(t) &= \Big[2A(t)P(t) + 2a(t)m(t) \\
&\quad + \sum_{i=1}^{m}\Big(B_i^2(t)P(t) + 2B_i(t)b_i(t)m(t) + b_i^2(t)\Big)\Big]dt
\end{aligned}
$$

Stochastic Differential Equations (SDE)

In the case that $B_i(t) = 0, \; i = 1, \ldots, m$, we are able to directly compute the distribution. The scalar linear SDE

$$dX(t) = (A(t)X(t) + a(t))dt + \sum_{i=1}^{m} b_i(t)dW_i(t), \tag{53}$$

with $X(0) = x_0$ is normaly distributed
$P(X(t)|x_0) \sim \mathcal{N}(m(t), V(t))$ with expected value $m(t)$ and variance $V(t)$, which are solutions of the following ODEs,

$$\dot{m}(t) = A(t)m(t) + a(t)\,, \quad m(0) = x_0\,, \tag{54}$$

$$\dot{V}(t) = 2A(t)V(t) + \sum_{i=1}^{m} b_i^2(t)\,, \quad V(0) = 0\,. \tag{55}$$

Stochastic Differential Equations (SDE)

There are some specific scalar linear SDEs which are found to be quite useful in practice. The simplest case of SDE is where the drift and the diffusion coefficients are independent of the information received over time

$$dS(t) = \mu dt + \sigma dW(t)\,, \quad S(0) = S_0\,. \tag{56}$$

This model has been used to simulate commodity prices, such as metals or agricultural products.
The mean is $\mathsf{E}[S(t)] = \mu t + S_0$ and the variance $\mathsf{Var}[S(t)] = \sigma^2 t$. $S(t)$ possesses a behavior of fluctuations around the straight line $S_0 + \mu t$.The process is normally distributed with the given mean and variance.

Stochastic Differential Equations (SDE)

The standard model of stock prices is the geometric Brownian motion as given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t,\omega)\,, \quad S(0) = S_0\,.$$

The mean is given by $\mathsf{E}[S(t)] = S_0 e^{\mu t}$ and its variance by $\mathsf{Var}[S(t)] = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$. This model forms the starting point for the famous Black-Scholes formula for option pricing. The geometric Brownian motion has two main features which make it popular for stock

The first property is that $S(t) > 0$ for all $t \in [0, T]$ and the second is that all returns are in scale with the current price. This process has a log-normal probability density function.

Stochastic Differential Equations (SDE)

Another very popular class of SDEs are mean reverting linear SDEs. The model is obtained by

$$dS(t) = \kappa[\mu - S(t)]dt + \sigma\, dW(t,\omega)\,, \quad S(0) = S_0\,. \tag{57}$$

A special case of this SDE where $\mu = 0$ is called *Ohrnstein-Uhlenbeck* process. Equation (57) models a process which naturally falls back to its equilibrium level of μ. The expected price is $\mathsf{E}[S(t)] = \mu - (\mu - S_0)e^{-\kappa\, t}$ and the variance is

$$\mathsf{Var}[S(t)] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\, t}\right).$$

Stochastic Differential Equations (SDE)

In the long run, the following (unconditional) approximations are valid

$$\lim_{t \to \infty} \mathsf{E}[S(t)] = \mu$$

and

$$\lim_{t \to \infty} \mathsf{Var}[S(t)] = \frac{\sigma^2}{2\kappa}.$$

This analysis shows that the process fluctuates around μ and has a variance of $\frac{\sigma^2}{2\kappa}$ which depends on the parameter κ: the higher κ, the lower the variance.
This is obvious since the higher κ, the faster the process reverts back to its mean value.

This process is a stationary process which is normally distributed.

Stochastic Differential Equations (SDE)

A popular extension is where the diffusion term is in scale with the current value, i.e., the geometric mean reverting process:

$$dS(t) = \kappa[\mu - S(t)]dt + \sigma S(t)dW(t, \omega)\,, \quad S(0) = S_0\,.$$

In this model $S(t) \geq 0$, if $S_0 \geq 0$, $\mu > 0$, and $\kappa > 0$.

The first mean reversion model(57) may produce negative values even for $\mu > 0$.

Since the second mean-reversion model has always positive realizations, it is also called log-normal mean reversion. This type of model is used to model interest rate or volatilities.

Stochastic Differential Equations (SDE)

In control engineering science, the most important (scalar) case is

$$dX(t) = (A(t)X(t) + C(t)u(t))\,dt + \sum_{i=1}^{m} b_i(t)\,dW_i\,. \tag{58}$$

In this equation, $X(t)$ is normally distributed because the Brownian motion is just multiplied by time-dependent factors.

When we compute an optimal control law for this SDE, the deterministic optimal control law (ignoring the Brownian motion) and the stochastic optimal control law are the same.

This feature is called *certainty equivalence*. For this reason, the stochastics are often ignored in control engineering.

Stochastic Differential Equations (SDE)

The logical extension of scalar SDEs is to allow $X(t) \in \mathbb{R}^n$ to be a vector. The rest of this section proceeds in a similar fashion as for scalar linear SDEs. A stochastic vector differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t)$$

with the initial condition $X(0) = x_0 \in \mathbb{R}^n$ for an n-dimensional stochastic process $X(t)$ is called a linear SDE if the functions $f(t, X(t)) \in \mathbb{R}^n$ and $g(t, X(t)) \in \mathbb{R}^{n\times m}$ are affine functions of $X(t)$ and thus

$$\begin{aligned} f(t, X(t)) &= A(t)X(t) + a(t)\,, \\ g(t, X(t)) &= [B_1(t)X(t) + b_1(t), \cdots, B_m(t)X(t) + b_m(t)]\,, \end{aligned}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $a(t) \in \mathbb{R}^n$, $W(t) \in \mathbb{R}^m$ is an m-dimensional Brownian motion, and $B_i(t) \in \mathbb{R}^{n\times n}$, $b_i(t) \in \mathbb{R}^n$.

Stochastic Differential Equations (SDE)

Alternatively, the vector-valued linear SDE can be written as

$$dX(t) = (A(t)X(t) + a(t))dt + \sum_{i=1}^{m}(B_i(t)X(t) + b_i(t))dW_i(t)\,. \qquad (59)$$

A common extension of the above equation is the following form of a controlled stochastic differential equation as given by

$$\begin{aligned} dX(t) \quad &= \quad (A(t)X(t) + C(t)u(t) + a(t))\,dt \\ &\quad + \sum_{i=1}^{m}(B_i(t)X(t) + D_i(t)u(t) + b_i(t))\,dW_i\,, \end{aligned} \qquad (60)$$

where $u(t) \in \mathbb{R}^k$, $C(t) \in \mathbb{R}^{n\times k}$, $D_i(t) \in \mathbb{R}^{n\times k}$.

Stochastic Differential Equations (SDE)

The linear SDE (59) has the following solution:

$$X(t) = \Phi(t)\Big(x_0 + \int_0^t \Phi^{-1}(s)\Big[a(s) - \sum_{i=1}^m B_i(s)b_i(s)\Big]ds + \sum_{i=1}^m \int_0^t \Phi^{-1}(s)b_i(s)dW_i(s)\Big)\,, \qquad (61)$$

where the fundamental matrix $\Phi(t) \in \mathbb{R}^{n\times n}$ is the solution of the homogenous stochastic differential equation.

Stochastic Differential Equations (SDE)

The fundamental matrix $\Phi(t) \in \mathbb{R}^{n\times n}$ is the solution of the homogenous stochastic differential equation:

$$d\Phi(t) = A(t)\Phi(t)dt + \sum_{i=1}^{m} B_i(t)\Phi(t)dW_i(t)\,, \tag{62}$$

with initial condition $\Phi(0) = I$, $I \in \mathbb{R}^{n\times n}$ e now prove that (61) and (62) are solutions of (59). We rewrite (61) as

$$\begin{aligned} X(t) &= \Phi(t)\Big(x_0 + \int_0^t \Phi^{-1}(t)dY(t)\Big) \\ dY(t) &= \Big[a(t) - \sum_{i=1}^{m} B_i(t)b_i(t)\Big]dt + \sum_{i=1}^{m} b_i(t)dW_i(t)\,. \end{aligned}$$

Stochastic Differential Equations (SDE)

$$\begin{aligned} X(t) &= \Phi(t)Z(t)\,, \quad Z(t) = \left(x_0 + \int_0^t \Phi^{-1}(t)dY(t)\right) \\ dZ(t) &= \Phi^{-1}(t)dY(t) \end{aligned}$$

We use the Itô formula to calculate $X(t) = \Phi(t)Z(t)$:

$$\begin{aligned} dX(t) &= \Phi(t)dZ(t) + d\Phi(t)Z(t) + \sum_{i=1}^{m} B_i(t)\Phi(t)\Phi(t)^{-1}b_i(t)dt \\ &= dY(t) + A(t)\Phi(t)Z(t)dt + \sum_{i=1}^{m} B_i(t)\Phi(t)Z(t)dW_i(t) + \sum_{i=1}^{m} B_i(t)b_i(t)dt \end{aligned}$$

Stochastic Differential Equations (SDE)

Noting that $Z(t) = \Phi^{-1}(t)X(t)$ and using the SDE for $Y(t)$, we get

$$\begin{aligned} dX(t) &= dY(t) + A(t)\Phi(t)Z(t)dt + \sum_{i=1}^{m} B_i(t)\Phi(t)Z(t)dW_i(t) + \sum_{i=1}^{m} B_i(t)b_i(t)dt \\ &= \Big[a(t) - \sum_{i=1}^{m} B_i(t)b_i(t)\Big]dt + \sum_{i=1}^{m} b_i(t)dW_i(t) + A(t)X(t)dt \\ &\quad + \sum_{i=1}^{m} B_i(t)X(t)dW_i(t) + \sum_{i=1}^{m} B_i(t)b_i(t)dt \\ &= [a(t) + A(t)X(t)]dt + \sum_{i=1}^{m}(B_i(t)X(t) + b_i(t))dW_i(t)\,. \end{aligned}$$

This completes the proof.

Stochastic Differential Equations (SDE)

The expectation $m(t) = \mathsf{E}[X(t)] \in \mathbb{R}^n$ and the second moment matrix $P(t) = \mathsf{E}[X(t)X^T(t)] \in \mathbb{R}^{n \times n}$ can be computed as follows:

$$\dot{m}(t) = A(t)m(t) + a(t)\,, \quad m(0) = x_0\,, \tag{63}$$

$$\begin{aligned}\dot{P}(t) &= A(t)P(t) + P(t)A^T(t) + a(t)m^T(t) + m(t)a^T(t) \\ &+ \sum_{i=1}^{m} [B_i(t)P(t)B_i^T(t) + B_i(t)m(t)b_i^T(t) \\ &+ b_i(t)m^T(t)B_i^T(t) + b_i(t)b_i(t)^T]\,, \quad P(0) = x_0 x_0^T\,. \end{aligned} \tag{64}$$

The covariance matrix for the system of linear SDEs is given by als

$$V(t) = \mathsf{Var}\{x(t)\} = P(t) - m(t)m^T(t)\,. \tag{65}$$

Stochastic Differential Equations (SDE)

The special case

$$dX(t) = (A(t)X(t) + a(t))dt + \sum_{i=1}^{m} b_i(t)dW_i(t)$$

with the initial condition $X(0) = x_0 \in \mathbb{R}^n$ is normally distributed, i.e.,

$$P(X(t)|x_0) \sim \mathcal{N}(m(t), V(t))$$

where

$$\begin{aligned}\dot{m}(t) &= A(t)m(t) + a(t) \quad m(0) = x_0 \\ \dot{V}(t) &= A(t)V(t) + V(t)A^T(t) + \sum_{i=1}^{m} b_i b_i^T(t) \quad V(0) = 0\,.\end{aligned}$$

Stochastic Differential Equations (SDE)

As first example of a linear vector valued SDE, we consider a two dimensional geometric Brownian motion:

$$dS_1(t) = \mu_1 S_1(t)dt + S_1(t)\Big(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)\Big)\,, \tag{66}$$

$$dS_2(t) = \mu_2 S_2(t)dt + S_2(t)\Big(\sigma_{21}dW_1(t) + \sigma_{22}dW_2(t)\Big)\,. \tag{67}$$

Written in matrix form $S = (S_1, S_2)^T$, the same SDE is given as:

$$A(t) = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \; a(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; B_1(t) = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{21} \end{pmatrix} \; B_2(t) = \begin{pmatrix} \sigma_{12} & 0 \\ 0 & \sigma_{22} \end{pmatrix}$$

Both processes $S_1(t)$ and $S_2(t)$ are correlated if $\sigma_{12} = \sigma_{21} \neq 0$. This model can be easily extended to n processes.

Stochastic Differential Equations (SDE)

The observed volatility for real existing price processes, such as stocks or bonds is itself a stochastic process. The following model describes this observation:

$$
\begin{aligned}
dP(t) &= \mu dt + \sigma(t) dW_1(t) \,, \quad P(0) = P_0 \,, \\
d\sigma(t) &= \kappa(\theta - \sigma(t))dt + \sigma(t)\sigma_1 dW_2(t) \,, \quad \sigma(0) = \sigma_0 \,.
\end{aligned}
$$

where θ is the average volatility, σ_1 a volatility, and κ the mean reversion rate of the volatility process $\sigma(t)$. If this model is used for stock prices, the transformation $P(t) = \ln(S(t))$ is useful. The two Brownian motions $dW_1(t)$ and $dW_2(t)$ are correlated, hence corr$[dW_1(t), dW_2(t)] = \rho$. This model captures the behavior of real existing prices better and its distribution of returns shows "fatter tails".

Stochastic Differential Equations (SDE)

Die system (68) can be rewritten as linear SDE:

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & -\kappa \end{pmatrix} \; a(t) = \begin{pmatrix} \mu \\ \kappa\theta \end{pmatrix} \; B_1(t) = \begin{pmatrix} 0 & 1 \\ 0 & \sigma_1\rho \end{pmatrix}$$

$$B_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1\sqrt{1-\rho^2} \end{pmatrix},$$

wobei $x(t) = (P(t),\ \sigma(t))^T$. The system (68) has the property, that the variance of $P(t)$ depends on the initial condition σ_0 For the parameters $\mu = 0.1$, $\kappa = 2$, $\theta = 0.2$, $\sigma_1 = 0.5$ and $\rho = 0.5$, we calculate the standard deviation of $P(t)$ with $\sigma_0 = 0.1$ and alternatively with $\sigma_0 = 0.8$. The expected value of $\sigma(t)$ has the following evaluation over time $m(t) = \theta + (\sigma_0 - \theta)e^{-\kappa t}$ and thus the variance of $P(t)$ depends on σ_0.

Stochastic Differential Equations (SDE)

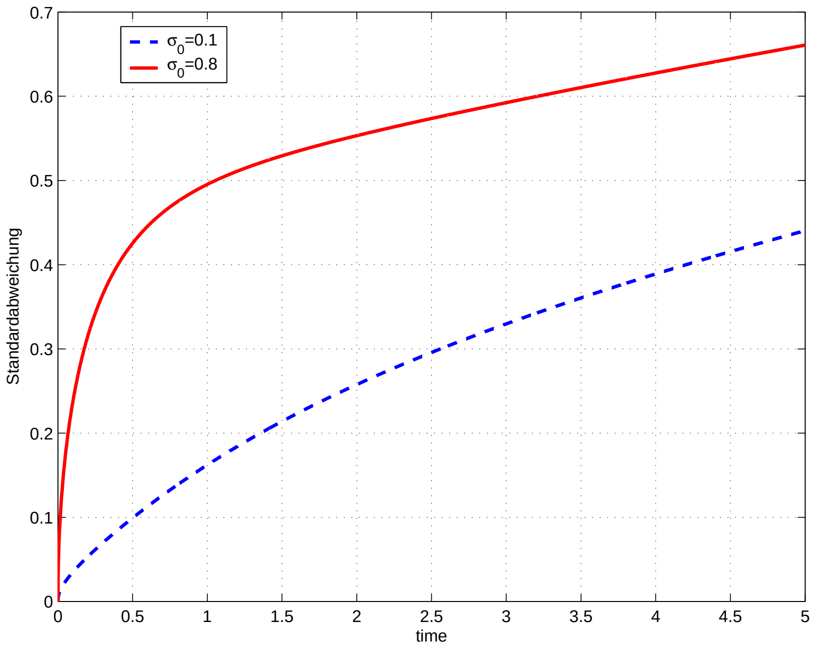

Abbildung 1: Stand. dev. of $P(t)$ for different initial conditions of $\sigma(t)$

Stochastic Differential Equations (SDE)

In comparison with linear SDEs, nonlinear SDEs are less well understood. No general solution theory exists. And there are no explicit formulae for calculating the moments. In this section, we show some examples of nonlinear SDEs and their properties.
In general, a scalar square root process can be written as

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t)$$

with

$$\begin{aligned} f(t, X(t)) &= A(t)X(t) + a(t) \\ g(t, X(t)) &= B(t)\sqrt{X(t)}\,, \end{aligned}$$

where $A(t)$, $a(t)$, and $B(t)$ are real scalars. The nonlinear mean reverting SDEs differ from the linear scalar equations by their nonlinear diffusion term. For this process, the distribution and moments can be calculated.

Stochastic Differential Equations (SDE)

For a specific square root process with $A(t) = 0$, $a(t) = 1$ and $B(t) = 2$ we are able to derive the analytical solution: The SDE

$$dX(t) = 1dt + 2\sqrt{X(t)}dW(t)\,, \quad X(0) = x_o\,,$$

has the solution $X(t) = (W(t) + x_0)^2$We verify the solution using Itô formula. We use $\Phi(t) = X(t) = (Y(t) + x_0)^2$ and $dY(t) = dW(t)$. The partial derivatives are $\Phi_t = 0$, $\Phi_Y = 2(Y(t) + x_0)$, and $\Phi_{YY} = 2$. Thus

$$\begin{aligned} d\Phi(t) &= [\Phi_t + \Phi_Y \cdot 0 + \frac{1}{2}\Phi_{YY} \cdot 1]dt + \Phi_Y \cdot 1dW(t)\,, \\ d\Phi(t) &= 1dt + 2(Y(t) + x_0)dW(t)\,, \Rightarrow dX(t) = 1dt + 2\sqrt{X(t)}dW(t)\,, \end{aligned}$$

since $\sqrt{X(t)} = Y(t) + x_0$.

Stochastic Differential Equations (SDE)

Another widely used mean reversion model is obtained by

$$dS(t) \quad = \quad \kappa[\mu - S(t)]dt + \sigma\sqrt{S(t)}dW(t)\,, \quad S(0) = S_0\,. \tag{68}$$

This model is also known as the Cox-Ross-Ingersol processes.The process shows a less volatile behavior than its linear geometric counterpart and it has a non-central chi-square distribution. The process is often used to model short-term interest rates or stochastic volatility processes for stock prices. Another often used square root process is similar to the geometric Brownian motion, but with a square root diffusion term instead of the linear diffusion term. Its model is given by

$$dS(t) = \mu S(t)dt + \sigma\sqrt{S(t)}dW(t)\,, \quad S(0) = S_0\,. \tag{69}$$

Stochastic Differential Equations (SDE)

The expected value for (69) is $\mathsf{E}[S(t)] = S_0 e^{\mu t}$ and the variance is obtained by $\mathsf{Var}[S(t)] = \frac{\sigma^2 S_0}{\mu}\Big(e^{2\mu t} - e^{\mu t}\Big)$.
Another widely used mean reversion model is obtained by

$$dS(t) = \kappa S(t)[\mu - \ln(S(t))]dt + S(t)\sigma dW(t)\,. \tag{70}$$

Using the transformation $P(t) = \ln(S(t))$ yields the linear mean reverting and normally distributed process $P(t)$:

$$dP(t) = \kappa[(\mu - \frac{\sigma^2}{2\kappa}) - P(t)]dt + \sigma dW(t)\,, \tag{71}$$

Because of the transformation, $S(t)$ is log-normally distributed. This model is used to model stock prices, stochastic volatilities, and electricity prices. Because $S(t)$ is log-normally distributed, $S(t)$ is always positive.

Stochastic Differential Equations (SDE)

In this part, we introduce three major methods to compute solution of SDEs.

- The first method is based on the Itô integral and has already been used for linear solutions.
- We introduce numerical methods to compute path-wise solutions of SDEs.
- The third method is based on partial differential equations, where the problem of finding the probability density function of the solution is transformed into solving a partial differential equation.

Stochastic Differential Equations (SDE)

The stochastic process $X(t)$ governed by the stochastic differential equation

$$
\begin{aligned}
dX(t) &= f(t, X(t))dt + g(t, X(t))dW(t) \\
X(0) &= X_0
\end{aligned}
$$

is explicitly described by the integral form

$$X(t, \omega) = X_0 + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s)\,,$$

where the first integral is a path-wise Riemann integral and the second integral is an Itô integral.
In this definition, it is assumed that the functions $f(t, X(t))$ and $g(t, X(t))$ are sufficiently smooth in order to guarantee the existence of the solution $X(t)$.

Stochastic Differential Equations (SDE)

There are several ways of finding analytical solutions. One way is to guess a solution and use the Itô calculus to verify that it is a solution for the SDE under consideration.

We assume that the following nonlinear SDE

$$dX(t) = dt + 2\sqrt{X(t)}\, dW(t) \,,$$

has the solution

$$X(t) = (W(t) + \sqrt{X_0})^2 \,.$$

In order to verify this claim, we use the Itô calculus. We have $X(t) = \phi(W)$ where $\phi(W) = (W(t) + \sqrt{X_0})^2$, so that $\phi'(W) = 2(W(t) + \sqrt{X_0})$ and $\phi''(W) = 2$.

Stochastic Differential Equations (SDE)

Using Itô's rule, we get

$$\begin{aligned} dX(t) &= \widetilde{f}(t,X)dt + \widetilde{g}(t,X)dW(t) \\ \widetilde{f}(t,X) &= \phi(W)'1 + \frac{1}{2}\phi''(W)(2\sqrt{X(t)})^2 = 1 \\ \widetilde{g}(T,X) &= \phi'(W)(2\sqrt{X(t)}) = 2(W(t) + \sqrt{X_0}) \,. \end{aligned}$$

Since $X(t) = (W(t) + \sqrt{X_0})^2$ we know that $(W(t) + \sqrt{X_0}) = \sqrt{X(t)}$ and thus the Itô calculation generated the original SDE where we started at.

Stochastic Differential Equations (SDE)

For some classes of SDEs, analytical formulas exist to find the solution, e.g. consider the following SDE:

$$dX(t) = f(t, X(t))dt + \sigma(t)dW(t)\,, \quad X(0) = x_0 \tag{72}$$

where $X(t) \in \mathbb{R}^n$, $f(t, X(t)) \in \mathbb{R}^n$ is an arbitrary function, $\sigma(t) \in \mathbb{R}^{n\times m}$ and $dW(t) \in \mathbb{R}^m$. This class of SDEs has the following general solution:

$$X(t) = Y(t) + F(t) \tag{73}$$

$$dY(t) = f(t, Y(t) + F(t))dt\,, \quad Y(0) = x_0 \tag{74}$$

$$dF(t) = \sigma(t)dW(t)\,, \quad F(0) = 0\,. \tag{75}$$

The SDE for $F(t)$ can be integrated, i.e. $F(t) = \int_0^t \sigma(s)dW(s)$. When $\sigma(t) = \sigma$ than $F(t) = \sigma W(t)$.

Stochastic Differential Equations (SDE)

Since$F(t)$ is know,, we are able to solve for $Y(t)$ in in function of $F(t)$.
Using Itô lemman, we show that $X(t) = Y(t) + F(t)$ and this solves the SDE

$$\begin{aligned} dX(t) &= dY(t) + dF(t) = f(t, Y(t) + F(t))dt + \sigma(t)dW(t) \\ &= f(t, X(t))dt + \sigma(t)dW(t) \end{aligned} \tag{76}$$

This solution is not very suprising, since $X(t)$ is the sum of the process of $Y(t)$ and the BM of $F(t)$.

Stochastic Differential Equations (SDE)

For another class of SDEs, exist an analytical formula for their solution:

$$dX(t) = f(t, X(t))dt + c(t)X(t)dW(t)\,, \quad X(0) = x_0\,, \tag{77}$$

where $f(t, X(t)) \in \mathbb{R}$, $c(t) \in \mathbb{R}$ and $dW \in \mathbb{R}$. DThe solution can be derived as follows:

$$X(t) = F^{-1}(t)Y(t) \tag{78}$$

$$dF(t) = F(t)c^2(t)dt - F(t)c(t)dW(t)\,, \quad F(0) = 1 \tag{79}$$

$$dY(t) = F(t)f(t, F^{-1}Y(t))dt \tag{80}$$

The proof is similar to the first case, sice the diffusion is linear.

Stochastic Differential Equations (SDE)

Calculate the analytical solution for

$$\begin{aligned}
dX(t) &= \frac{dt}{X(t)} + \alpha X(t)dW(t)\,, \quad X(0) = x_0\,. \\
F(t) &= e^{\frac{1}{2}\alpha^2 t - \alpha W(t)}\,, \quad dY(t) = \frac{F(t)}{F^{-1}(t)Y}dt = \frac{F^2(t)}{Y}dt \\
dY(t)Y(t) &= F^2(t)dt\,, \quad \frac{1}{2}Y^2(t) = \int_0^t F^2(s)ds + C_0 \\
Y(t) &= \left(x_0^2 + 2\int_0^t e^{\alpha^2 s - 2\alpha W(s)}ds\right)^{\frac{1}{2}} \\
X(t) &= e^{-\frac{1}{2}\alpha^2 t + \alpha W(t)}\left(x_0^2 + 2\int_0^t e^{\alpha^2 s - 2\alpha W(s)}ds\right)^{\frac{1}{2}}
\end{aligned}$$

Stochastic Differential Equations (SDE)

However, most SDEs, especially nonlinear SDEs, do not have analytical solutions so that one has to resort to numerical approximation schemes in order to simulate sample paths of solutions to the given equation.
The simplest scheme is obtained by using a first-order approximation. This is called the Euler scheme

$$X(t_k) = X(t_{k-1}) + f(t_{k-1}, X(t_{k-1}))\Delta t + g(t_{k-1}, X(t_{k-1}))\Delta W(t_k)\,.$$

The Brownian motion term can be approximated as follows:

$$\Delta W(t_k) = \epsilon(t_k)\sqrt{\Delta t}\,,$$

where the $\epsilon(.)$ is a discrete-time Gaussian white process with mean 0 and standard deviation 1.