Random Processes

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Random Process

Definition 1. Random process
A random (stochastic) process $\{X_t, t \in T\}$ *is a collection of random variables on the same probability space* $(\Omega, \mathcal{F}, P)$*. The index set* T *is usually representing time and can be either an interval* $[t_1, t_2]$ *or a discrete set. Therefore, the stochastic process* X *can be written as a function:*

$$X : \mathbb{R} \times \Omega \to \mathbb{R}, \quad (t, \omega) \mapsto X(t, \omega)$$

Filtration

Observations:

- The amount of information increases with time.
- We use the concept of sigma algebras.
- We assume that information is not lost with increasing time
- Therefore the corresponding σ-algebras will increase over time when there is more information available.
- $\Rightarrow$ This concept is called filtration.

Definition 2. Filtration/adapted process
A collection $\{\mathcal{F}_t\}_{t\geq 0}$ *of sub* σ*-algebras is called filtration if, for every* $s \leq t$*, we have* $\mathcal{F}_s \subseteq \mathcal{F}_t$*. The random variables* $\{X_t : 0 \leq t \leq \infty\}$ *are called adapted to the filtration* $\mathcal{F}_t$ *if, for every t,* X_t *is measurable with respect to* $\mathcal{F}_t$*.*

Filtration

Example 1. *Suppose we have a sample space of four elements:* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$*. At time zero, we don't have any information about which* ω *has been chosen. At time* $\frac{T}{2}$ *we know wether we will have* $\{\omega_1, \omega_2\}$ *or* $\{\omega_3, \omega_4\}$*.*

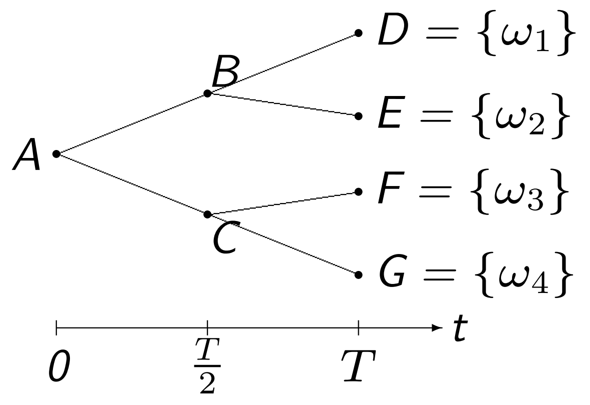

Abbildung 1: Example of a filtration

Difference between Random process and smooth functions

Let $x(\cdot)$ be a real, continuously differentiable function defined on the interval $[0,T]$. Its continuous differentiability implies both a bounded total variation and a vanishing "sum of squared increments":

1. Total variation:

$$\int_0^T \left| \frac{dx(t)}{dt} \right| dt < \infty$$

2. "Sum of squares":

$$\lim_{N\to\infty} \sum_{k=1}^{N} \left(x\left(k\frac{T}{N}\right) - x\left((k-1)\frac{T}{N}\right) \right)^2 = 0$$

Random processes do not have either of these nice smoothness properties in general. This allows the desired "wild" and "random" behavior of the (sample) "noise signals".

Markov Process

A Markov process has the property that only the instantaneous value $X(t)$ is relevant for the future evolution of X.

Past and future of a Markov process have no direct relation.

Definition 3. Markov process
A continuous time stochastic process $X(t),\ t \in T$*, is called a Markov process if for any finite parameter set* $\{t_i : t_i < t_{i+1}\} \in T$ *we have*

$$P(X(t_{n+1}) \in B | X(t_1), \ldots, X(t_n)) = P(X(t_{n+1}) \in B | X(t_n))$$

Transition probability

Definition 1. *Let $X(t)$ be a Markov process. The function $P(s, X(s), t, B)$ is the conditional probability $P(X(t) \in B \mid X(s))$ called $transition\ probability$ or $transition\ function.$*

This means it is the probability that the process $X(t)$ will be found inside the area B at time t, if at time $s < t$ it was observed at state $X(s)$.

Transition probability

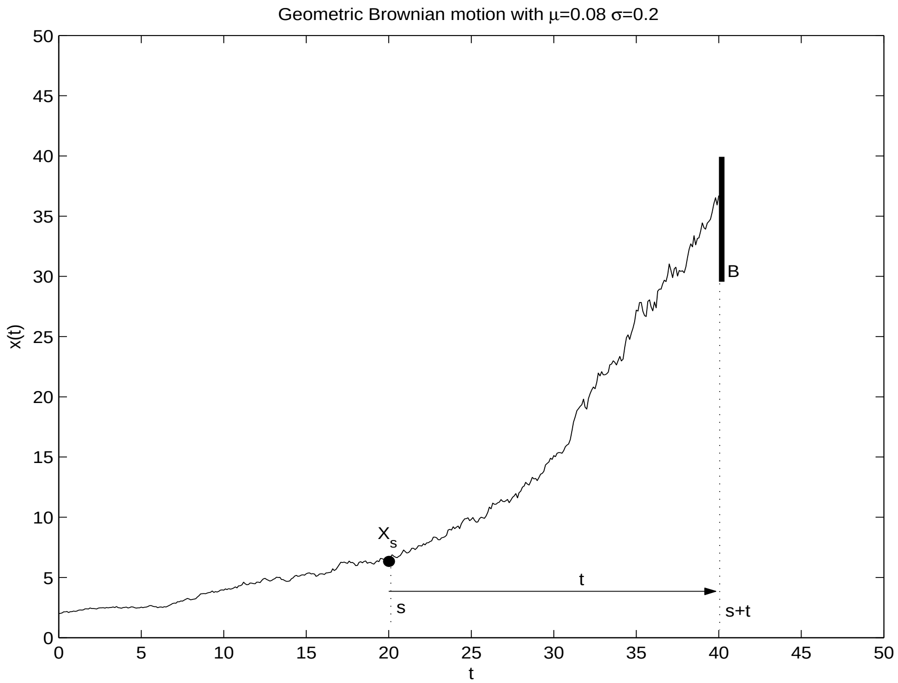

Abbildung 2: Transition probability $P(s, x, t + s, B)$.

Gaussian Process

- A stochastic process is called Gaussian if all its joint probability distributions are Gaussian.

- If $X(t)$ is a Gaussian process, then $X(t) \sim \mathcal{N}(\mu(t), \sigma^2(t))$ for all t.

- A Gaussian process is fully characterized by its mean and covariance function.

- Performing linear operations on a Gaussian process still results in a Gaussian process.

- Derivatives and integrals of Gaussian processes are Gaussian processes themselves (note: stochastic integration and differentiation).

Martingale Process

A stochastic process $X(t)$ is a martingale relative to $(\{\mathcal{F}_t\}_{t\geq 0}, P)$ if the following conditions hold:

- $X(t)$ is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, $E[|X(t)|] < \infty$ for all $t \geq 0$.
- E$[X(t)|\mathcal{F}_s] =$ X(s) a.s. $(0 \leq s \leq t)$.

- The best prediction of a martingale process is its current value.

- Fair game model

Diffusions

A diffusion is a Markov process with continuous trajectories such that for each time t and state $X(t)$ the following limits exist

$$\begin{aligned}
\mu(t, X(t)) &:= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} E[X(t+\Delta t) - X(t) | X(t)], \\
\sigma^2(t, X(t)) &:= \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} E[\{X(t+\Delta t) - X(t)\}^2 | X(t)].
\end{aligned}$$

For these limits, $\mu(t, X(t))$ is called drift and $\sigma^2(t, X(t))$ is called the diffusion coefficient.