Modeling and Analysis of Dynamic Systems

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Outline

1 Lecture 10: Analysis of Linear Systems
   • Normalization
   • Linearization
   • Solution of Linear ODE
   • Stability of Linear Systems

2 Lecture 10: Reachability and Observability
   • Reachability Conditions
   • Observability Conditions
   • Example: Ball on Wheel

3 Lecture 10: Balanced Realization and Order Reduction
   • Introduction / Example
   • Gramian Matrices
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   - Introduction / Example
   - Gramian Matrices
Remark: (from script p.89) For those students who followed the courses “Control Systems I and II,” most of the material presented in this chapter will be a repetition. However, since several points are discussed in more detail and since additional material is introduced, it is not recommended to skip these classes.

Motivations

Once the modeling of a system is done, we are interested in analysing this system for:

- stability
- controllability, observability
- performance

Methodology?
Introduction

Models are usually nonlinear with physical ("non-normalized"):
- states $x$, inputs $u$, and outputs $y$.

According to dynamic and output equations:

$$\frac{d}{dt} x(t) = f(x(t), u(t), t), \quad y(t) = g(x(t), u(t), t)$$

However, difficulties arise for systems which:

1. are nonlinear: difficult to analyze
2. have non-normalized variables:
   - are prone to numerical problems
   - the state variables are hard to compare to one another.

Solution for analysis: Normalize and linearize around a chosen setpoint.
Questions

1. Assuming known input signals and initial conditions: what output signals are to be expected?

2. Which points in the state space may be:
   - reached by appropriate input signals?
   - observed by analyzing the associated output signals?

3. What parts of a dynamic system are relevant for the input/output characteristics?
Consider the scaling factors: $x_{i,0}, u_{j,0}, y_{k,0}$ such that

$$
\overline{x}_i(t) = \frac{x_i(t)}{x_{i,0}}, \quad \overline{u}_j(t) = \frac{u_j(t)}{u_{j,0}}, \quad \overline{y}_k(t) = \frac{y_k(t)}{y_{k,0}},
$$

The new normalized variables $\overline{x}_i(t), \overline{u}_j(t)$ and $\overline{y}_k(t)$ will have no physical units, and are usually close to 1. Since $x_{i,0}$ is constant

$$
\frac{d}{dt}x_i(t) = x_{i,0}\frac{d}{dt}\overline{x}_i(t)
$$

→ no change in the dynamics (only scaled).

The normalization may be expressed in vector notation as

$$
x = T \cdot \overline{x}, \quad T = \text{diag}\{x_{1,0}, \ldots, x_{n,0}\}
$$

“Similarity transformation” matrix does not change the systems characteristics (stability, input-output behavior, etc.).
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The system after normalization has the form

1. Dynamics equation

\[
\frac{d}{dt} \overline{x}(t) = \dot{x}(t) = f_0(\overline{x}(t), \overline{u}(t), t),
\]

2. Output equation

\[
\overline{y}(t) = g_0(\overline{x}(t), \overline{u}(t), t)
\]

with \( \overline{x}(t) \in \mathbb{R}^n, \overline{u}(t) \in \mathbb{R}^m, \overline{y}(t) \in \mathbb{R}^p, \)

and \( f_0 \) and \( g_0 \) nonlinear normalized functions.

Remark: here we can use, the true variable or the normalized. In the following we will simply use the notations \( x, y, u. \)
System dynamics in vector form

\[ \dot{x}(t) = f(x, u, t) \]

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n \\
\end{pmatrix} =
\begin{pmatrix}
f_1(x, u) \\
f_2(x, u) \\
\vdots \\
f_n(x, u) \\
\end{pmatrix}
\]
Notion of Neighborhood

Linearization behavior in a “small” neighborhood

1 operating point \( \{x_o, u_o\} \)

\[ B_r := \{x \in \mathbb{R}^n \mid ||x - x_o||^2 + ||u - u_o||^2 \leq r\} \]

2 equilibrium point \( \{x_e, u_e\} \)

\[ B_r := \{x \in \mathbb{R}^n \mid ||x - x_e||^2 + ||u - u_e||^2 \leq r\} \]

around a chosen equilibrium point \( \{x_e, u_e\} \), \( f_0(x_e, u_e, t) = 0 \).
In the 1-D case, the dynamics of the system are \( \dot{x} = f(x, u) \)
Geometric Interpretation

\[
\dot{x} = f(x, u) = f(x_0, u_0) + \delta_x \left( \left. \frac{\partial f}{\partial x} \right|_{x_0, u_0} (x - x_0) \right) + \delta_u \left( \left. \frac{\partial f}{\partial u} \right|_{x_0, u_0} (u - u_0) \right)
\]

\[
+ \frac{1}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_0, u_0} (x - x_0)^2 + \ldots
\]
Geometric Interpretation

\[
\dot{x} = f(x, u) - f(x_0, u_0) \approx \frac{\partial f}{\partial x} \bigg|_{x_0, u_0} (x - x_0) + \frac{\partial f}{\partial u} \bigg|_{x_0, u_0} (u - u_0)
\]
Deviation Around Equilibrium Point

\[ \tilde{x}(t) = x(t) - x_e, \]

\[ \tilde{u}(t) = u(t) - x_e, \]

\[ \tilde{y}(t) = y(t) - g_0(x_e, u_e, t) \]

such that in the new coordinates the ODEs are

\[ \frac{d}{dt} \tilde{x}(t) = \tilde{f}_0(\tilde{x}(t), \tilde{u}(t), t), \]

\[ \tilde{y}(t) = \tilde{g}_0(\tilde{x}(t), \tilde{u}(t), t) \]

with \( \tilde{f}_0(0, 0, t) = 0. \)
Only small deviations are considered. The following new variables are introduced

\[ x_i(t) = x_e + \delta x_i(t) \text{ with } |\delta x_i| \ll 1, \]
\[ u_i(t) = u_e + \delta u_i(t) \text{ with } |\delta u_i| \ll 1, \]
\[ y_i(t) = y_e + \delta y_i(t) \text{ with } |\delta y_i| \ll 1 \]

Taylor series neglecting all terms of second and higher order yields

\[ \frac{d}{dt} \delta x(t) = \frac{\partial f_0}{\partial x} \bigg|_{x_e,u_e} \delta x(t) + \frac{\partial f_0}{\partial u} \bigg|_{x_e,u_e} \delta u(t) \]

and

\[ \delta y(t) = \frac{\partial g_0}{\partial x} \bigg|_{x_e,u_e} \delta x(t) + \frac{\partial g_0}{\partial u} \bigg|_{x_e,u_e} \delta u(t) \]
Notation

\[ \frac{d}{dt} x(t) = Ax(t) + Bu(t) \]

\[ y(t) = Cx(t) + Du(t) \]

System can be described using (infinitely) many other coordinate systems (similarity transformations)

\[ x = T\tilde{x}, \quad T \in \mathbb{R}^{n \times n}, \quad \det(T) \neq 0 \]

The columns of \( T \) are the unit vectors of the new coordinate frame expressed in the old coordinate system.

In the new coordinates

\[ \frac{d}{dt} \tilde{x}(t) = T^{-1}AT\tilde{x}(t) + T^{-1}Bu(t) \]

\[ y(t) = CT\tilde{x}(t) + Du(t) \]
The input-output behavior (i.e., the transfer function) is not affected by the specific choice of coordinates

\[
\tilde{P}(s) = CT[sI - T^{-1}AT]^{-1}T^{-1}B + D = C[sI - A]^{-1}B + D = P(s)
\]

Fundamental system properties (stability, controllability, etc.) are independent of the coordinates chosen. However, if the systems ODEs are derived by physical arguments there are reasons for sticking to these “natural” coordinates.
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Objective:
prediction of the state \( x \) and the output \( y \) of a linear system described by its matrices \( \{A, B, C, D\} \),

\[
\frac{d}{dt} x(t) = \dot{x}(t) = A x(t) + B u(t)
\]

\[
y(t) = C x(t) + D u(t)
\]

with
- initial conditions \( x(0) \),
- control signal \( u \),

are known.
Matrix exponential

\[ e^{At} = I + \frac{1}{1!}At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots + \frac{1}{n!}(At)^n + \cdots \]

Main property

\[ \frac{de^{At}}{dt} = A e^{At} = e^{At} A \quad \text{(commute)} \]

Properties

1. In general, \( e^A \cdot e^B \neq e^{A+B} \).
2. Only if \( A \) and \( B \) commute (\( AB = BA \)), \( e^A \cdot e^B = e^{A+B} \).
3. Accordingly, \( e^{At} e^{-At} = e^{A(t-t)} = e^0 = I \Rightarrow (e^{At})^{-1} = e^{-At} \).
Solution to the initial value problem

\[ \dot{x}(t) = A \ x(t) + B \ u(t), \quad x(0) = x_0 \]

Multiplying (1) on the left-hand side by the term \( e^{-At} \) yields

\[ e^{-At} \dot{x}(t) = e^{-At} A x(t) + e^{-At} B u(t), \]

\[ \iff \quad e^{-At} \dot{x}(t) - e^{-At} A x(t) = e^{-At} B u(t). \] (1)

The left-hand side of (1) is equal to \( \frac{d(e^{-At} x(t))}{dt} \), therefore (1) is rewritten as

\[ \frac{d(e^{-At} x(t))}{dt} = e^{-At} B u(t). \] (2)
Integrating the above equation on the time interval $[0, t]$ yields

\[
\int_0^t \frac{d(e^{-A\tau} x(\tau))}{d\tau} \ d\tau = \int_0^t e^{-A\sigma} Bu(\sigma) d\sigma,
\]

\[
e^{-At} x(t) - x(0) = \int_0^t e^{-A\sigma} Bu(\sigma) d\sigma,
\]

\[
e^{-At} x(t) = x(0) + \int_0^t e^{-A\sigma} Bu(\sigma) d\sigma,
\]

\[
x(t) = e^{At} x(0) + \int_0^t e^{A(t-\sigma)} Bu(\sigma) d\sigma.
\]
\[ x(t) = e^{At}x(0) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma \]

The \textbf{continuous-time transition matrix} is defined as follows:

\[ \Phi(t) = e^{At} = I + At + \frac{(At)^2}{2!} + \ldots + \frac{(At)^n}{n!} + \ldots \]

and finally the continuous solution for \( x(t) \) is written as

\[ x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\sigma)Bu(\sigma)d\sigma \]

\[ y(t) = Cx(t) + Du(t) \]
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Introduction

**Stability** = most important concept in analysis of dynamic systems

Stability is always connected to a “metric,” i.e., the size of vectors is important.

⇒ The symbol $\|x\|$ will be used to denote this operation, and any norm will be acceptable.

For instance the Euclidean length:

$$x \in \mathbb{R}^n, \quad ||x||^2 := \sum_{i=1}^{n} x_i^2$$
Lyapunov Stability

Linear and time-invariant systems:

\[
\frac{d}{dt} x(t) = A \cdot x(t), \quad x(0) = x_0, \quad 0 < \|x_0\| < \infty \tag{3}
\]

The input \(u(t) = 0\) for stability analysis, only \(A\) is important.

For stability: three possible cases. The system (3) is

1. **asymptotically stable** if \(\lim_{t \to \infty} \|x(t)\| = 0\);

2. **stable** if \(\|x(t)\| < \infty \quad \forall \ t \in [0, \infty]\); and

3. **unstable** if \(\lim_{t \to \infty} \|x(t)\| = \infty\),

where the norm is the Euclidean length (defined before).

**Remark:** A general definition for nonlinear systems will be seen later.
For linear systems, the stability of the system

\[ \frac{d}{dt} x(t) = A x(t), \quad x(0) = x_0 \neq 0 \]

is tightly related to the eigenvalues of the matrix \( A \).

Recall Matrix \( A \) characterizes linear mapping \( \mathbb{R}^n \to \mathbb{R}^n \)

\[ y = A x, \quad x, y \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n} \]

An eigenvector \( v_i \) of \( A \) is a vector which is mapped onto itself, through the scaling parameter \( \lambda_i \) called the eigenvalue of \( v_i \), according to:

\[ A v_i = \lambda_i v_i \]

Remark: Even for real matrices \( A \in \mathbb{R}^{n \times n} \), \( \lambda_i \) and \( v_i \) can be complex (always complex conjugate pairs).
If \( n \) linearly independent eigenvectors exist, then the matrix

\[
T = [v_1 \ldots v_n]
\]

which – by definition – diagonalizes \( A \), i.e.,

\[
AT = T \Lambda \quad \Rightarrow \quad T^{-1}AT = \Lambda
\]

where

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & \lambda_n
\end{bmatrix}
\]

Remark:
If all eigenvalues are distinct, i.e., \( \lambda_i \neq \lambda_j \) for all \( i \neq j \), then \( n \) independent eigenvectors always exist.
If there are multiple (same) eigenvalues \( \Rightarrow \) situation more complex.

Two scalars are important:

- **multiplicity** of eigenvalue \( \lambda_i \): \( r_i \)
- **rank loss** of the matrix \([\lambda_i I - A] \): \( \rho_i \), associated with the eigenvalue \( \lambda_i \).

In the general case, three distinct situations arise:

1. \( \rho_i = 1 \), \( \Rightarrow \) only 1 eigenvector exists for the \((r_i > 1)\) identical eigenvalues \( \lambda_i \),

2. \( \rho_i < r_i \), \( \Rightarrow \) there are less than \( r_i \) independent eigenvectors,

3. \( \rho_i = r_i \), \( \Rightarrow \) exactly \( r \) independent eigenvectors exist.

The third case is similar to the regular case.

**Remark:** When all eigenvalues are distinct: \( r_i = \rho_i = 1 \) for all \( i = 1, \ldots, n \) and the matrix \( A \) is “diagonal.”
In the first and in the second case, $A$ is not diagonalizable but can be brought to what is known as a “Jordan Form”

$$J_i = \begin{bmatrix}
\lambda_i & 1 & 0 & \ldots & \ldots & 0 \\
0 & \lambda_i & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & \lambda_i & 1 \\
0 & \ldots & \ldots & 0 & \lambda_i & \ldots \\
\end{bmatrix}$$
Examples:

$T^{-1}AT = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

- $\lambda_i \neq \lambda_j$
  - yes
  - no
  - $\rho_1 = r_1 = 1$
  - $r_i > 1$
  - $\rho_i = 1$

$T^{-1}AT = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

- only one EV exists
- full $i$-th Jordan block
- $A$ is cyclic

$T^{-1}AT = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

- fewer than $r_i$ EV exist
- $i$-th Jordan block mixed
- $A$ neither diagonal nor cyclic

$T^{-1}AT = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

- $r_i$ EV exist
- $i$-th Jordan block empty
- $A$ is diagonal
If the matrix $\Lambda$ is diagonal three situations can be encountered:

1. all eigenvalues have negative real parts $\sigma_i < 0 \Rightarrow$ system is asymptotically stable;

2. some eigenvalues have zero real parts $\sigma_i = 0$, but none has a positive real part $\Rightarrow$ system is Lyapunov stable,

3. At least one eigenvalue has a positive real part $\sigma_i > 0 \Rightarrow$ system is unstable.

Remark: For cyclic or mixed matrices $A$, this is no longer true. In fact, systems with multiple eigenvalues on the imaginary axis can be stable only if all Jordan blocks associated to multiple eigenvalues with real part zero are diagonal.
Example: “series double integrator structure”

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

The **continuous-time transition matrix** is defined as follows:

\[
\Phi(t) = e^{At} = I + At + \frac{(At)^2}{2!} + \ldots + \frac{(At)^n}{n!} + \ldots
\]

and finally the continuous solution for \(x(t)\) is written as

\[
x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\sigma)B \ u(\sigma) d\sigma
\]

Transition matrix (stability analysis \(u = 0\)).

\[
x(t) = \left\{ I + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \right\} x(0) = \begin{bmatrix} 1 \\ \frac{1}{t^2} \\ 1 \end{bmatrix} x(0)
\]
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Reachability of Linear Systems

\[ x(t) = e^{At} x(0) + \int_0^t e^{A(t-\sigma)} B u(\sigma) d\sigma \]

Investigate all states that can be reached at time \( \tau \) starting at \( x(0) = 0 \)

\[ x(\tau) = e^{A\tau} \int_0^\tau e^{-A\sigma} B u(\sigma) d\sigma \]

Matrix \( e^{A\tau} \) regular for all \( \tau < \infty \), therefore use \( x^*(\tau) = e^{-A\tau} x(\tau) \)

\[ x^*(\tau) = \int_0^\tau e^{-A\sigma} B u(\sigma) d\sigma \]

Using the definition of the matrix exponential

\[ x^*(\tau) = \int_0^\tau \left\{ I - \frac{1}{1!} A\sigma + \frac{1}{2!} (A\sigma)^2 - \frac{1}{3!} (A\sigma)^3 + \cdots \right\} B u(\sigma) d\sigma \]
and from that

\[ x^*(\tau) = B \int_0^\tau u(\sigma) d\sigma - AB \int_0^\tau \frac{1}{1!} \sigma u(\sigma) d\sigma + A^2 B \int_0^\tau \frac{1}{2!} \sigma^2 u(\sigma) d\sigma \]

Let’s define the integral basis as:

\[ v_i = \int_0^\tau \frac{1}{i!} \sigma^i u(\sigma) d\sigma, \quad i = 0, 1, \ldots \]

The reachable states \( x^* \) are then defined by the linear equation

\[ x^*(\tau) = \begin{bmatrix} B & AB & A^2 B & A^3 B & \cdots \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \end{bmatrix} = R v \]
Define

\[
R_n = \begin{bmatrix}
B & AB & A^2B & A^3B & \cdots & A^{n-1}B
\end{bmatrix}
\]

Therefore, condition \( \text{rank}(R_n) = n \) guarantees that for arbitrary \( x(0) = x^* \neq 0 \) a suitable control signal \( u^*(x^*) \) exists. The system is said to be reachable.

Read page 105 in the script, last 2 paragraphs.
Observability of Linear Systems
Is it possible to reconstruct \( x(0) \) using the output signal \( y(t) \) only?
Input \( u(t) = 0 \) for all times \( t \), therefore

\[
y(t) = Cx(t) \quad \frac{d}{dt}y(t) = CAx(t) \quad \frac{d^2}{dt^2}y(t) = CA^2x(t) \quad \text{etc.}
\]

Evaluating this equation for \( t = 0 \) yields the linear equation

\[
\begin{bmatrix}
y(t) \\
\frac{d}{dt}y(t) \\
\frac{d^2}{dt^2}y(t) \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots
\end{bmatrix}
\begin{bmatrix}
x(0) \\
x(0) \\
x(0) \\
\vdots
\end{bmatrix}
\begin{bmatrix}
w(0) \\
w(0) \\
w(0) \\
w(0)
\end{bmatrix}
\]

By “measuring” the vector \( w(0) \) unknown \( x(0) \) unique if and only if kernel of \( O \) is zero. ( \( x(0) \) can be found uniquely if \( O \) invertible.)
In general

\[ w(0) = O x_1 \quad (4) \]
\[ w(0) = O x_2 \]

and therefore the difference \( \Delta x = x_1 - x_2 \) satisfies

\[ 0 = O(x_1 - x_2) = O \Delta x \Rightarrow \Delta x \in Ker \{O\} \]

The initial condition \( x(0) \) is uniquely determined by (4) only if the rank of \( O \) is full (i.e., the kernel of \( O \) contains only the element 0).
Cayley-Hamilton theorem

\[ O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \]

If \( \text{rank}(O_n = n) \), the system is observable.
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Ball on Wheel

- wheel: mass moment of inertia around c.o.g. is $\Theta$, radius $R$,
- ball: mass $m$, mass moment of inertia around c.o.g. $\vartheta$, radius $r$
Ball-on-Wheel
Linearizing system at equilibrium $\psi = \dot{\psi} = \chi = \dot{\chi} = 0$ yields

$$\delta \dot{x}(t) = A \cdot \delta x(t) + b \cdot \delta u(t), \quad \delta y(t) = c \cdot \delta x(t)$$

with

$$\delta x(t) = \begin{bmatrix} \delta \psi(t) \\ \delta \dot{\psi}(t) \\ \delta \chi(t) \\ \delta \dot{\chi}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix}, \quad c^T = \begin{bmatrix} 0 \\ 0 \\ c_1 \end{bmatrix}$$
Parameters defined by

\[ a_1 = \frac{mgR\dot{\vartheta}}{\Gamma}, \quad a_2 = \frac{mg(R^2\dot{\vartheta} + r^2\Theta)}{(R + r)\Gamma} \]

\[ b_1 = \frac{mr^2 + \vartheta}{\Gamma}, \quad b_2 = \frac{R\dot{\vartheta}}{(R + r)\Gamma}, \quad c_1 = R + r \]

To simplify the notation

\[ \Theta = 5\vartheta, \quad R = 3r, \quad \vartheta = r^2m, \quad g = 10 \text{ m}^2/\text{s}, \quad r = 0.2 \text{ m}, \quad m = 1 \text{ kg} \]

Controllability matrix

\[
\mathcal{R}_n = \begin{bmatrix}
0 & \frac{50}{19} & 0 & \frac{75}{76} \\
\frac{50}{19} & 0 & \frac{75}{76} & 0 \\
0 & \frac{5625}{722} & 0 & \frac{13125}{1444} \\
\frac{5625}{722} & 0 & \frac{13125}{1444} & 0
\end{bmatrix},
\]

\[ \det(\mathcal{R}_n) = 263.444 \neq 0, \text{ not rank deficient, i.e., the system is completely controllable.} \]
Observability matrix

\[
O_n = \begin{bmatrix}
0 & 0 & \frac{4}{5} & 0 \\
0 & 0 & 0 & \frac{4}{5} \\
0 & 0 & \frac{140}{19} & 0 \\
0 & 0 & 0 & \frac{140}{19}
\end{bmatrix}
\]

\[\det(O_n) = 0\] (first 2 columns are equal to 0).

For this special case it is easy to see that the two state variables \(\psi\) and \(\dot{\psi}\) cannot be estimated using the output signal \(y\) as the only information.
The linearized system is unstable. Its eigenvalues are

\[ \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = -\lambda_4 = 3.03488 \]

According to the Lyapunov principle, the nonlinear system must be unstable as well. However, this does not mean that the system cannot meet the desired objective, i.e., to balance the ball at the equilibrium position!

The question is now, can the system be stabilized by an appropriate feedback action?

⇒ As it will be shown below, this is not possible.
Transfer function from $u \rightarrow y$

$$P(s) = \frac{15}{19 s^2 - 175}$$

- The two observable states are those whose dynamics are described by the two eigenvalues $\lambda_{3,4}$.
- The dynamics described by the two eigenvalues $\lambda_{1,2}$, cannot be influenced by feedback action. This subsystem has a double eigenvalue at zero with rank loss only 1.

Therefore, this subsystem is unstable (intuition confirmed)!

For simulations use simple feedback (lead-lag) controller

$$C(s) = \frac{k_2 s + k_1}{\tau s + 1} = \frac{2.2417 s + 17.05}{0.01 s + 1}$$
closed-loop step response $y(t)$; solid=nonlinear, dashed=linear

Figure: Reference step responses of the closed-loop "ball-on-wheel" system; all initial conditions equal to zero.
Figure: Behavior of the “hidden state variables” for the same reference step as illustrated in Figure 1.
Figure: Behavior of the closed-loop system for an initial condition of $\chi(0) = \pi/12$ rad, $\dot{\chi}(0) = \pi/12$ rad/s and reference values $\chi_r = \dot{\chi}_r = 0$. 
1. Lecture 10: Analysis of Linear Systems
   - Normalization
   - Linearization
   - Solution of Linear ODE
   - Stability of Linear Systems

2. Lecture 10: Reachability and Observability
   - Reachability Conditions
   - Observability Conditions
   - Example: Ball on Wheel

3. Lecture 10: Balanced Realization and Order Reduction
   - Introduction / Example
   - Gramian Matrices
Balanced Realization and Order Reduction

With the matrices $R$ and $O$ one obtains “yes/no” answers. What about quantitative criteria?
Is a specific subspace “well or hardly” reachable?
Closely linked to that question is the notion of system order reduction.

System order reduction consists in:

1. identifying those states that do not influence significantly the input/output behavior of the system,
2. finding a smart way eliminate those “not so important” states.
Outline

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Gramian Matrices

Controllability Gramian

\[ W_R = \int_0^{\infty} e^{A\sigma} B B^T e^{A^T \sigma} \, d\sigma \]

The closer \( W_R \) is to a singular matrix (det close to 0), the less controllable the corresponding system will be.

Observability Gramian

\[ W_O = \int_0^{\infty} e^{A^T \sigma} C^T C e^{A\sigma} \, d\sigma \]

The closer \( W_O \) is to a singular matrix, the less observable the corresponding system will be.
Gramian Matrices

The computation of the two Gramians appears to be difficult. The following result, however, simplifies those steps considerably.

If the system is Hurwitz, the two Gramian matrices are the solutions of the two Lyapunov equations

\[ AW_R + W_R A^T = -B B^T \]

and

\[ A^T W_O + W_O A = -C^T C \]
Bad idea:

simply delete those system parts that do not significantly contribute to the controllability or observability Gramian matrices (poor controllability compensated by excellent observability).

Better idea:

look for a coordinate transformation $T \cdot x_b = x$, which transforms the original system into a system whose controllability and observability Gramians are equal and diagonal, i.e.,

$$W_{R,b} = W_{O,b} = \text{diag}(\sigma_i), \quad i = 1, \ldots, n$$

Proper normalization is very important for this procedure to work correctly!
Assume: the last $\nu$ elements $\sigma_j$ with $j = n - \nu + 1, \ldots, n$ are substantially smaller than the other first $n - \nu$ elements $\sigma_i$ with $i = 1, \ldots n - \nu$. Then the contribution of the last $\nu$ balanced modes to the system’s IO behavior may be neglected.

First step: system partitioning

\[
\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)
\]

\[
y(t) = [C_1 \ C_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t)
\]

where $x_1 \in \mathbb{R}^{n-\nu}$ and $x_2 \in \mathbb{R}^\nu.$
Since the contribution of the last $\nu$ states is small, the system can be reduced by simply omitting the corresponding elements, i.e.,

\[
\frac{d}{dt} x_1(t) = A_{11} x_1(t) + B_1 u(t) \\
\text{y}(t) = C_1 x_1(t) + D u(t)
\]

Typically, this will yield good agreement in the frequency domain but, in general the DC gains of the original system (5) and the reduced-order system (5) will be different.
If this is to be avoided, a “singular perturbation” approach is better suited where:

1. the dynamics of the last $\nu$ states is neglected,
2. but not their DC contributions.

The details of that approach are

$$\frac{d}{dt} x_2(t) = 0 \rightarrow x_2(t) = -A_{2,2}^{-1}[A_{2,1}x_1(t) + B_2u(t)]$$

and

$$\frac{d}{dt} x_1(t) = [A_{11} - A_{1,2}A_{2,2}^{-1}A_{2,1}]x_1(t) + [B_1 - A_{1,2}A_{2,2}^{-1}B_2]u(t)$$

$$y(t) = [C_1 - C_2A_{2,2}^{-1}A_{2,1}]x_1(t) + [D - C_2A_{2,2}^{-1}B_2]u(t)$$

Always feasible if the original system was asymptotically stable (not restrictive, otherwise Gramians do not exist).

The next figure shows the step responses for increasing $\nu$, red = original and blue = reduced-order outputs, system order $n = 7$. 
Step Response
\( \nu = 3 \)

Amplitude

Time (sec)

Step Response
\( \nu = 4 \)

Amplitude

Time (sec)

Step Response
\( \nu = 5 \)

Amplitude

Time (sec)

Step Response
\( \nu = 6 \)

Amplitude

Time (sec)
Next lecture + Upcoming Exercise

Next lecture

- Case Study

Next exercises:

- Linearizing
- System balancing and order reduction