Modeling and Analysis of Dynamic Systems

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Outline

1. Lecture 13: Linear System - Stability Analysis
   - Zero Dynamics: Definitions
   - Zero Dynamics: Analysis
   - Example and Summary

Outline

Zero Dynamics

The dynamic behavior of linear system described as

$$\begin{aligned} \frac{d}{dt}\boldsymbol{x}(t) = \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

can be studied through its poles (eigenvalues of $\boldsymbol{A}$) for the stability of the state vector $\boldsymbol{x}$. See previous lecture.
Let's consider the dynamic behavior when the output equation is considered $\boldsymbol{y}(t)$ through the output matrix $\boldsymbol{C}$.

In the Laplace domain, the relationship between input and output can be represented by a transfer function matrix:

$$\boldsymbol{P}(s) = \boldsymbol{C} \cdot [s\,\boldsymbol{I} - \boldsymbol{A}]^{-1} \cdot \boldsymbol{B} + \boldsymbol{D}$$

SISO Case: Transfer Function

In the SISO case, the transfer function matrix:

$$\boldsymbol{P}(s) = \boldsymbol{C} \cdot [s\,\boldsymbol{I} - \boldsymbol{A}]^{-1} \cdot \boldsymbol{B} + \boldsymbol{D}$$

is a scalar rational transfer function which can always be written in the following form

$$P(s) = \frac{Y(s)}{U(s)} = k \frac{s^{n-r} + b_{n-r-1} s^{n-r-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0}$$

SISO Case: Transfer Function

$$P(s) = \frac{Y(s)}{U(s)} = k\frac{s^{n-r} + b_{n-r-1}s^{n-r-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0}$$

Discussions:

- The order of the highest power of s is n.
- Input gain: k
- The relative degree r:
  - difference between highest power of s at denominator and the highest power of s at numerator.
  - r plays an important role in the discussion of system zeros.

A dynamic system can possess - not only poles - but also zeros.
Question: What is the influence of the zeros?

There is an equivalence between the **transfer function**

$$P(s) = \frac{Y(s)}{U(s)} = k\frac{s^{n-r} + b_{n-r-1}s^{n-r-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0}$$

and its **state-space representation**

$$\frac{d}{dt}\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ k \end{bmatrix} \boldsymbol{u}(t)$$

$$y(t) = \begin{bmatrix} b_0 & \ldots & b_{n-r-1} & 1 & 0 & \ldots & 0 \end{bmatrix} \boldsymbol{x}(t) = C\boldsymbol{x}(t)$$

Remarks:

- controller canonical form with gain k (min. number of parameters)
- the terms involved in the numerator are those of the C output vector ("transmission zeros").

Zero Dynamics - Definition

The Zero Dynamics of a system:

corresponds to its behavior for those special

- non-zero inputs $u^*(t)$
- and initial conditions x^*

for which its output $y(t)$ is identical to zero for a finite interval.

1. Study of the influence of the zeros on the dynamic properties of the system.
2. Study of the "internal dynamics": analyze the stability of the system states, which are not directly controlled by the input $u(t)$.

Zero Dynamics - Problem

In all the reference-tracking control problems, the controller tries to force the error to zero.

$$\epsilon = y_{ref} - y$$

If $y_{ref} = 0$, then $y(t)$ is to be zero for all times, $\Rightarrow$ all its derivatives are to be zero as well.

If a plant has internal dynamics, which are unstable, but not visible at the system's output, problems are to be expected.

Let's see the form of the derivatives.

The relative degree r

is the number of differentiations needed to have the input $u(t)$ explicitly appear in the “output” $y^{(r)}(t)$

$$
\begin{aligned}
y(t) &= \boldsymbol{C}\boldsymbol{x}(t), \\
\dot{y}(t) &= \boldsymbol{C}\dot{\boldsymbol{x}}(t) = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{C}\boldsymbol{B}\boldsymbol{u}(t) = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}(t), \\
&\vdots \\
y^{(r-1)}(t) &= \frac{d}{dt} y^{(r-2)}(t) = \boldsymbol{C}\boldsymbol{A}^{r-1}\boldsymbol{x}(t) + \boldsymbol{C}\boldsymbol{A}^{r-2}\boldsymbol{B}u(t) = \boldsymbol{C}\boldsymbol{A}^{r-1}\boldsymbol{x}(t), \\
y^{(r)}(t) &= \frac{d}{dt} y^{(r-1)}(t) = \boldsymbol{C}\boldsymbol{A}^{r}\boldsymbol{x}(t) + \boldsymbol{C}\boldsymbol{A}^{r-1}\boldsymbol{B}u(t) = \boldsymbol{C}\boldsymbol{A}^{r}\boldsymbol{x}(t) + ku(t)
\end{aligned}
$$

where $k \neq 0$ and $r \leq n$.

The relative degree r

$$z_1 = y = \boldsymbol{C}\boldsymbol{x} = [b_0 x_1 + b_1 x_2 + \ldots + b_{n-r-1} x_{n-r} + x_{n-r+1}]$$

$$z_2 = \dot{y} = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x} = [b_0 x_2 + b_1 x_3 + \ldots + b_{n-r-1} x_{n-r+1} + x_{n-r+2}]$$

$$\vdots$$

$$z_r = y^{(r-1)} = \boldsymbol{C}\boldsymbol{A}^{r-1}\boldsymbol{x} = [b_0 x_r + b_1 x_{r+1} + \ldots + b_{n-r-1} x_{n-1} + x_n]$$

$$y^{(r)} = \boldsymbol{C}\boldsymbol{A}^{r}\boldsymbol{x} + ku = [b_0 x_{r+1} + b_1 x_{r+2} + \ldots + b_{n-r} x_n + \dot{x}_n]$$

$\dot{x}_n$ is found from the state-space representation:

$$\frac{d}{dt}\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ k \end{bmatrix} \boldsymbol{u}(t)$$

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Zero Dynamics: Analysis Formulation

The following coordinate transformation $z = \Phi x$ is introduced

$$
\begin{aligned}
z_1 &= \quad y = \boldsymbol{C}\boldsymbol{x} = [b_0 x_1 + b_1 x_2 + \ldots + b_{n-r-1} x_{n-r} + x_{n-r+1}] \\
z_2 &= \quad \dot{y} = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x} = [b_0 x_2 + b_1 x_3 + \ldots + b_{n-r-1} x_{n-r+1} + x_{n-r+2}] \\
&\vdots \\
z_r &= \quad y^{(r-1)} = \boldsymbol{C}\boldsymbol{A}^{r-1}\boldsymbol{x} = [b_0 x_r + b_1 x_{r+1} + \ldots + b_{n-r-1} x_{n-1} + x_n]
\end{aligned}
$$

Zero Dynamics: Analysis Formulation

The remaining $n - r$ coordinates are chosen such that the transformation Φ is regular and such that their derivatives also do not depend on the input u. Obviously the choice

$$\begin{aligned} z_{r+1} &= x_1 \\ z_{r+2} &= x_2 \\ \cdots \\ z_n &= x_{n-r} \end{aligned}$$

satisfies both requirements.
To simplify notation the vector z is partitioned into two subvectors

$$z = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \xi = \begin{bmatrix} z_1 \\ \cdots \\ z_r \end{bmatrix}, \quad \eta = \begin{bmatrix} z_{r+1} \\ \cdots \\ z_n \end{bmatrix}$$

Zero Dynamics: Analysis Formulation

In the new coordinates the system has the form

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ - & - & r^T & - & - & - & - & s^T & - & - \\ \hline 0 & \dots & \dots & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 1 \\ - & - & p^T & - & - & - & - & q^T & - & - \end{array}\right] \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \left[\begin{array}{c} 0 \\ \dots \\ 0 \\ k \\ \hline 0 \\ \dots \\ \dots \\ 0 \end{array}\right] u$$

and obviously $y = z_1$.
In order to have an identically vanishing output it is therefore necessary and sufficient to choose the following control and initial conditions

$$\xi^* = 0, \qquad u^*(t) = -\frac{1}{k} s^T \eta^*(t)$$

where the initial condition $\eta_0^* \neq 0$ can be chosen arbitrarily.

Zero Dynamics: Analysis Formulation

The internal states (zero dynamics states) evolve according to the equation

$$\frac{d}{dt}\eta(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ - & - & q^T & - & - \end{bmatrix} \eta^*(t) = Q\eta^*(t), \qquad \eta^*(0) = \eta_0^*$$

Minimum phase system

- If the matrix Q is asymptotically stable (all eigenvalues with negative real part) $\Rightarrow$ then the system is minimum phase.
- Equivalence: a minimum phase system, is a system whose zeros have all negative real parts.

These two definitions are consistent: see definition of the vector q^T

$$q^T = [-b_0, \;\; -b_1, \;\; \dots, \;\; -b_{n-r-2}, \;\; -b_{n-r-1}]$$

Unstable Zero dynamics

As soon as there is a zero with positive real part:

- system is non-minimum phase,
- system has its zero dynamics unstable,
- its internal states η can diverge without $y(t)$ being affected.

Consequences:

- the input $u(t)$ may not be chosen such that the output $y(t)$ is (almost) zero before the states η associated with the zero dynamics are (almost) zero.
- Feedback control more difficult to design.
- This imposes a constraint of the bandwidth of the closed-loop system:
  $\Rightarrow$ significantly slower than the "slowest" non-minimum phase zero.

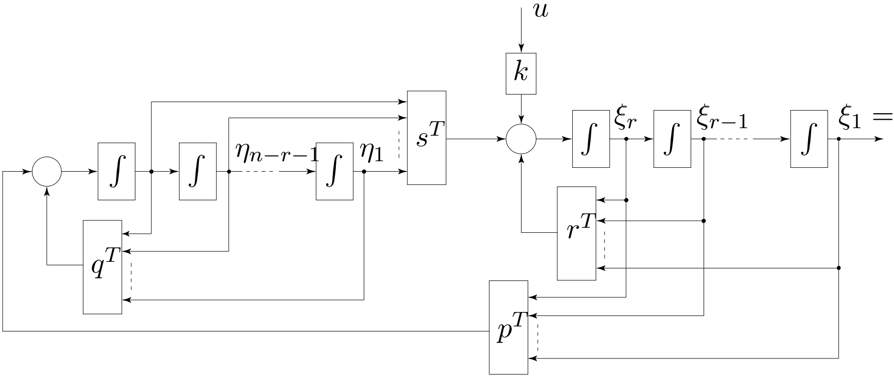
Zero Dynamics: Analysis Formulation

This equation can be derived using the definition of z_n, the original system equation, the definition of z_1 (1), again the coordinate transformation,

$$
\begin{aligned}
\dot{z}_n &= \dot{x}_{n-r} \\
&= x_{n-r+1} \\
&= z_1 - b_0 x_1 \ldots - b_{n-r-1} x_{n-r} \\
&= z_1 - b_0 z_{r+1} \ldots - b_{n-r-1} z_n \\
&= z_1 + q^T \eta
\end{aligned}
$$

Therefore, the eigenvalues of Q coincide with the transmission zeros of the original system and with the roots of the numerator of its transfer function.

Zero Dynamics: Analysis Formulation



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Zero Dynamics Analysis on a Small SISO System

$$P(s) = \frac{Y(s)}{U(s)} = k\frac{b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Step 1: Convert the plant's transfer function into a state-space controller canonical form

number of states $n = 4$, relative degree $r = 2$.

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & 1 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \end{bmatrix} \cdot u(t)$$

Step 2: Coordinate transformation

Relative degree $r = 2$, therefore

$$\begin{aligned} y(t) &= b_0 x_1(t) + b_1 x_2(t) + x_3(t) \\ \dot{y}(t) &= b_0 x_2(t) + b_1 x_3(t) + x_4(t) \\ \ddot{y}(t) &= -a_0 x_1(t) - a_1 x_2(t) + (b_0 - a_2) x_3(t) + (b_1 - a_3) x_4(t) + k\, u(t) \end{aligned}$$

The coordinate transformation $z = \Phi^{-1} \cdot x$ has the form

$$\begin{aligned} z_1 &= y = b_0 x_1 + b_1 x_2 + x_3 \\ z_2 &= \dot{y} = b_0 x_2 + b_1 x_3 + x_4 \\ z_3 &= x_1 \\ z_4 &= x_2 \end{aligned}$$

Step 3: Find the transformation matrices $\mathbf{\Phi}^{-1}$, such that $z = \mathbf{\Phi}^{-1} \cdot x$ and then compute $\mathbf{\Phi}$

$$\mathbf{\Phi}^{-1} = \begin{bmatrix} b_0 & b_1 & 1 & 0 \\ 0 & b_0 & b_1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and}$$

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -b_0 & -b_1 \\ -b_1 & 1 & b_0\, b_1 & b_1^2 - b_0 \end{bmatrix}$$

Remark: Notice that, by construction, $\det(\Phi) = \det(\Phi^{-1}) = 1$

Step 4: Build a new state-space representation in $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix}$

$$\boldsymbol{\xi} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} z_3 \\ z_4 \end{bmatrix}$$

in the new coordinates the system

$$\frac{d}{dt}\boldsymbol{z}(t) = \boldsymbol{\Phi}^{-1}\boldsymbol{A}\,\boldsymbol{\Phi}\;\boldsymbol{z}(t) + \boldsymbol{\Phi}^{-1}\boldsymbol{B}\,\boldsymbol{u}(t), \quad y(t) = \boldsymbol{C}\,\boldsymbol{\Phi}\;\boldsymbol{z}\,(t)$$

$$\frac{d}{dt}\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ r_1 & r_2 & s_1 & s_2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -b_0 & -b_1 \end{bmatrix} \cdot \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \eta_1(t) \\ \eta_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ k \\ 0 \\ 0 \end{bmatrix} \cdot u(t)$$

The coefficients r_1, r_2, s_1, s_2 are listed below

$$
\begin{aligned}
r_1 &= b_0 - a_2 - b_1(b_1 - a_3) \\
r_2 &= b_1 - a_3 \\
s_1 &= b_0\, b_1(b_1 - a_3) - a_0 - b_0(b_0 - a_2) \\
s_2 &= (b_1 - a_3)(b_1^2 - b_0) - a_1 - (b_0 - a_2)b_1
\end{aligned}
$$

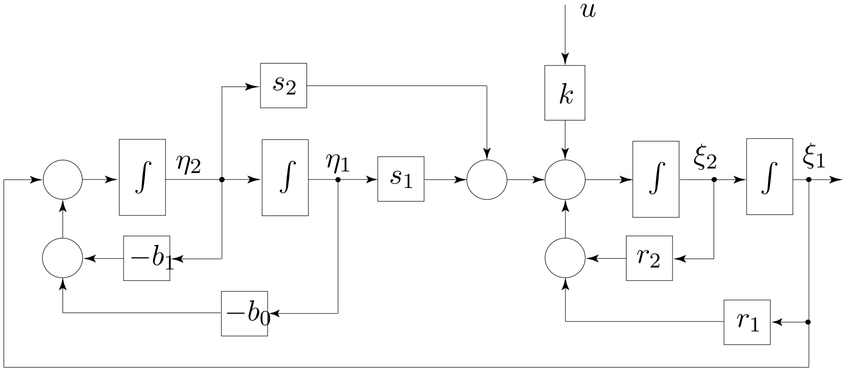

Figure: System structure of the example's zero dynamics.

Step 5: Study the submatrix $\boldsymbol{Q}$ of $\tilde{\boldsymbol{A}} = \boldsymbol{\Phi}^{-1}\boldsymbol{A}\boldsymbol{\Phi}$ corresponding to the zero-dynamics vector $\boldsymbol{\eta}$

Choosing the following initial conditions $\xi_1^*(0) = \xi_2^*(0) = 0$ and control signal $u^*(t) = -\frac{1}{k}\left[s_1\eta_1^*(t) + s_2\eta_2^*(t)\right]$ yields a zero output $y(t) = 0$ for all $t \geq 0$. The initial conditions $\eta_1^*(0) \neq 0$ and $\eta_2^*(0) \neq 0$ may be chosen arbitrarily.

The trajectories of state variables $\eta_1(t)$ and $\eta_2(t)$, in this case, are defined by the equations

$$\frac{d}{dt}\eta^*(t) = \begin{bmatrix} 0 & 1 \\ -b_0 & -b_1 \end{bmatrix} \cdot \eta^*(t) = \boldsymbol{Q} \cdot \eta^*(t)$$

Step 6: Conclude on the conditions to have $\boldsymbol{Q}$ asymptotically stable

$$\frac{d}{dt}\eta^*(t) = \begin{bmatrix} 0 & 1 \\ -b_0 & -b_1 \end{bmatrix} \cdot \eta^*(t) = \boldsymbol{Q} \cdot \eta^*(t)$$