Modeling and Analysis of Dynamic Systems

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Outline

1 Lecture 3: Modeling Tools for Mechanical Systems
   - Lagrange Formalism
   - Lagrange Method with Kinematic Constraints
   - Ball on Wheel
Lecture 3: Modeling Tools for Mechanical Systems

Lagrange Formalism
Lagrange Method with Kinematic Constraints
Ball on Wheel

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Lagrange: 1736 -1813
Lagrange Formalism: Recipe

1. Define inputs and outputs
2. Define the generalized coordinates:
   \[ q(t) = [q_1(t), q_2(t), \ldots, q_n(t)] \quad \text{and} \quad \dot{q}(t) = [\dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_n(t)] \]
3. Build the Lagrange function:
   \[ L(q, \dot{q}) = T(q, \dot{q}) - U(q) \]
4. System dynamics equations:
   \[ \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}_k} \right\} - \frac{\partial L}{\partial q_k} = Q_k, \quad k = 1, \ldots, n \]

Notes:
- \( Q_k \) represents the \( k \)-th “generalized force or torque” acting on the \( k \)-th generalized coordinate variable \( q_k \)
- \( n \): number of degrees of freedom in the system
- always \( n \) generalized variables
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Lagrange Equations for Constrained Systems

\[
\frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}_k} \right\} - \frac{\partial L}{\partial q_k} - \sum_{j=1}^{\nu} \mu_j \alpha_{j,k} = Q_k, \quad k = 1, \ldots, n, \quad (1)
\]

Remarks:

- the constraints are included using “Lagrange multipliers”: \( \mu_j \), \( j = 1 \ldots \nu \)
- Number of constraints: \( \nu \) with \( (\nu < n) \)
- \( n \) may be seen as the number of DOF
- In the end, we obtain: \( n + \nu \) coupled equations to be solved for \( \dot{q}_k \) and \( \mu_j \) (usually requires computing the time derivative of the constraints, i.e., \( \dot{\mu} \)).
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wheel: mass moment of inertia around c.o.g. is $\Theta$, radius $R$,
ball: mass $m$, mass moment of inertia around c.o.g. $\vartheta$, radius $r$
Ball on Wheel

Control objective
The ball must be kept on top of the wheel.

Model objectives
Build a model for:
- system analysis (stability, observability, controllability)
- control design

Assumptions
No-slip

no-slip equation
Modeling the ball on the wheel

Step 1: Inputs and outputs

- **Input:** $u(t)$ torque to the wheel
- **Output:** $y(t) = (R + r) \sin \chi$ horizontal distance of the center of the ball w.r.t. the vertical axis of the wheel

\[
u(t) \quad \text{Torque to the wheel} \quad \text{Model of the ball on the wheel} \quad y(t) \quad \text{Horizontal distance to the center of the ball}
\]
Step 2: $n$ generalized coordinates ($n$ DOF) and energies

The system has 3 DOF: rotation of

1. the wheel around its center: $\text{angle } \psi(t)$
2. the ball around the center of the wheel: $\text{angle } \chi(t)$
3. the ball around its own center: $\text{angle } \varphi(t)$
Modeling the ball on the wheel

Step 3: Lagrange function

\[ L(t) = T(t) - U(t) \]

Step 4: Differential equations including constraints

Lagrange equations nonholonomic case \((n = 3, \nu = 1, q_1 = \psi, q_2 = \chi, q_3 = \varphi)\)

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} - \mu \alpha_k = Q_k \]

with \(Q_1 = u(t), Q_2 = Q_3 = 0\) and \(\alpha_1 \dot{q}_1 + \alpha_2 \dot{q}_2 + \alpha_3 \dot{q}_3 = 0\) with \(\alpha_1 = R, \alpha_2 = -(R + r)\) and \(\alpha_3 = r\).

Finally, we get a set of four equations define the four unknown variables \(\{\ddot{\psi}, \ddot{\chi}, \ddot{\varphi}, \mu\}\), where \(\{\ddot{\varphi}, \mu\}\) are easy to eliminate.
Modeling the ball on the wheel

Final results

\[
\begin{bmatrix}
\Theta + \vartheta \frac{R^2}{r^2} & -\vartheta \frac{R(R+r)}{r^2} \\
-\vartheta \frac{R(R+r)}{r^2} & m(R + r)^2 + \vartheta \frac{(R+r)^2}{r^2}
\end{bmatrix}
\begin{bmatrix}
\ddot{\psi} \\
\ddot{\chi}
\end{bmatrix}
= 
\begin{bmatrix}
u \\
mg(R + r) \sin(\chi)
\end{bmatrix}
\]

Mass matrix \( M \) is positive definite. Therefore

\[
\ddot{\psi}(t) = \frac{[(mr^2 + \vartheta)u(t) + mgR\vartheta \sin(\chi(t))]}{\Gamma}
\]

\[
\ddot{\chi}(t) = \frac{[\vartheta Ru(t) + (\Theta r^2 + \vartheta R^2)mg \sin(\chi(t))]}{[\Gamma(r + R)]}
\]

where the scalar \( \Gamma \) is given by

\[
\Gamma = \Theta \vartheta + m(\vartheta R^2 + \Theta r^2)
\]