Problem Solutions

Vehicle Energy and Fuel Consumption

Vehicle Energy Losses and Performance Analysis

Problem 2.1

For a vehicle with $m_v = 1500$ kg, $A_f \cdot c_d = 0.7\,\mathrm{m}^2$, $c_r = 0.012$, a vehicle speed $v = 120\,\mathrm{km/h}$ and an acceleration $a = 0.027\,\mathrm{g}$, calculate the traction torque required at the wheels and the corresponding rotational speed level (tires 195/65/15T). Calculate the road slope that is equivalent to that acceleration.

- Solution

  Assume $\rho_a = 1.20 kg/m^3, g = 9.81 m/s^2$.

a) Traction torque required at wheels:

$$F_t = m_v \cdot c_r \cdot g + 1/2 \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 + m_v \cdot a =$$

$$= 1500 \cdot 0.012 \cdot 9.81 + \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot \left(\frac{120}{3.6}\right)^2 + 1500 \cdot 0.027 \cdot 9.81 = 1041 \text{ N}$$

The information about the tires is explained by
newline

$$\underbrace{195}_{\substack{\text{width of the}\\ \text{tire in [cm]}}} \; / \; \underbrace{65}_{\substack{\text{ratio of sidewall}\\ \text{height to tire}\\ \text{width in [\%]}}} \; / \; \underbrace{15}_{\substack{\text{wheel diameter}\\ \text{in [inch]}}} \quad \underbrace{\text{T}}_{\text{max. 190 [km/h]}} \; .$$

Thus

$$r_w = \frac{d_w}{2} + h_{\mathrm{sw}} = \frac{15”}{2} + 0.65 \cdot 0.195 = 15 \cdot \frac{0.0254}{2} + 0.65 \cdot 0.195 = 0.317 \text{ m}$$
$$T_t = r_w \cdot F_t = 0.317 \cdot 1041 = 330 \text{ Nm}$$

b) Rotational speed level:

$$\omega_w = \frac{v}{r_w} = \frac{120/3.6}{0.317} \cdot 0.317 = 105.2 \text{ rad/s} = 1004 \text{ rpm}$$

c) Acceleration-equivalent road slope:

$$\alpha = \arcsin((\frac{a}{g} = 0.027) \text{ rad}$$
$$\alpha_{\%} = 100 \cdot \tan(0.027) = 100 \cdot 0.027 = 2.7 \text{ \%}$$

For the requested velocity, a traction torque of 330 Nm at a rotational speed of 1004 rpm is required. This is equivalent to the acceleration caused by a slope of 2.7%.

Problem 2.2

Find the road slope α that is equivalent to a step of height h for a car with (a rigid) wheel radius r_w on a flat terrain. Calculate the result for $h/2r_w = \{0.01, 0.02, 0.05, 0.1, 0.2\}$.

- Solution

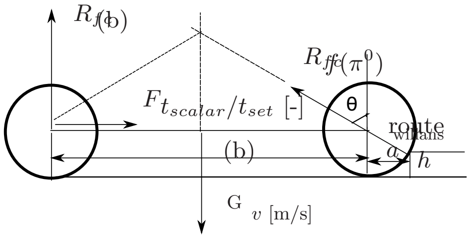

Fig. 10.7. Force and torque balance of the problem.

Assume the weight of vehicle distributed uniformly along the wheel base b, and the reaction force of the front wheel is from the contact point to the centre of the wheel.

a) Force balance
When in contact with the step, the car wheel will rotate around the contact point. Thus the front wheel's reaction force R_f will be directed from the contact point to the wheel center (neglect slip here) and the reaction force at the contact point on the terrain becomes null. Front wheel reaction force R_f is balanced by the weight G_v, back wheel reaction force R_b and the traction force F_t, all of which have to be calculated. One can write two equations for force balance and one equation for torque balance.

$$\begin{aligned}
&x: \ F_t = R_f \ \sin\theta \\
&y: \ m_v \, g = R_f \ \cos\theta + R_b \\
&z: \ F_t \times (r_w - h) + R_b \times b = m_v \, g \times \frac{b}{2}
\end{aligned}$$

In order to solve the traction force and therefore the traction torque, we substitute unknowns R_f and R_b with F_t:

$$\begin{aligned}
&F_t - (m_v \, g - R_b) \cdot \tan\theta = 0 \\
&R_b = m_v \, g - F_t \cdot \cot\theta
\end{aligned}$$

Further substituting R_b with F_t, we get:

$$\begin{aligned}
&F_t \, (r_w - h) + (m_v \, g - F_t \ \cot\theta) \cdot b = m_v \, g \, \frac{b}{2} \\
&F_t = \frac{\frac{1}{2} \, m_v \, g}{\frac{r_w - h}{a} - \frac{r_w - h}{b}}.
\end{aligned}$$

From the geometry of the plot, we can see

$$\cot\theta = \frac{r_w - h}{a},$$

where a denotes the characterization distance from the front wheel centre to the step. Therefore, the traction torque is:

$$T_t = F_t \times (r_w - h) = \frac{m_v \, g}{2 \, (\frac{1}{a} - \frac{1}{b})}$$

b) Calculate the equivalent gradability
In order to calculate the road slope, the traction torque equates that of overcoming a step with height h:

$$T_t = F_t \times (r_w - h) = m_v \, g \sin\alpha \times r_w.$$

Solving the road slope, we have the exact result for rigid wheels:

$$\sin\alpha = \frac{F_t\,(r_w - h)}{m_v\,g\,r_w} = \frac{1}{2\,r_w} \cdot \frac{ab}{b-a}$$
$$\alpha = \arcsin\left(\frac{ab}{2\,r_w\,(b-a)}\right)$$
$$= \frac{1}{2\,(\frac{r_w}{a} - \frac{r_w}{b})}$$

c) Explore a representation of first approximation
If $a \ll b$, the solution is no longer correlated with the wheelbase b, which gives:

$$\alpha = \arcsin\frac{a}{2\,r_w}$$

Discussion

- Other ways of calculation
From the geometry, it can be seen that

$$a^2 = r_w^2 - (r_w - h)^2 = h \cdot (2\,r_w - h).$$

Thus, the approximated calculation correlates only one geometric parameter $z = \frac{h}{2 \cdot r_w}$:

$$\alpha = arcsin\left(\sqrt{\frac{\frac{h}{r_w}\cdot(2 - \frac{h}{r_w})}{4}}\right)$$
$$= arcsin\left(\sqrt{\frac{\frac{h}{2\,r_w}\cdot(1 - \frac{h}{2\,r_w})}{2}}\right) = \arcsin\left(\sqrt{\frac{z(1-z)}{2}}\right)$$

Correspondingly, the exact solution correlates the relative height $z = h/r_w$ and the relative wheelbase $z_b = b/r_w$:

$$\alpha = \frac{1}{2\,(\frac{r_w}{a} - \frac{r_w}{b})}$$
$$= \frac{1}{2\left(\frac{1}{\sqrt{2\,z(1-z)}} - \frac{1}{z_b}\right)}$$

- Maximum height of the step that can be overcome
Note that the height of the step cannot be larger than the wheel size $z = \frac{h}{2\,r_w} \leq \frac{1}{2}$, otherwise the rotation around the contact point may not be achieved.
- Comparison between the approximated and exact solution:
Assume the relative wheelbase $b/r_w = 6.67$, and different relative step heights as in the Table **??**.

Table 10.1. Different results with different geometric parameters.

z [m]	α_{approx} [rad]	α_{exact} [rad]
0.20	0.2867 (29 %)	0.3090 (32 %)
0.10	0.2137 (22 %)	0.2265 (23 %)
0.05	0.1547 (16 %)	0.1615 (16 %)
0.02	0.0991 (10 %)	0.1020 (10 %)
0.01	0.0704 (7 %)	0.0718 (7 %)

Problem 2.3

Find an equation to evaluate the speed profile of an ICE vehicle under maximum engine torque. Assume a maximum torque curve of the type $T_e = a \cdot \omega_e^2 + b \cdot \omega_e + c$. Include the engine inertia and a traction efficiency η_t.

- Solution

a) Solve theoretically the speed response subject to engine torque
According the vehicle dynamics, the speed dynamics of the vechicle can be described as follows:

$$m_{v,eq}\,\dot{v} = \frac{\gamma}{r_w} \cdot \eta_t T_e - m_v \cdot g \cdot C_r - \frac{1}{2}\,\rho_a \cdot A_f C_d \cdot v^2, \qquad (*)$$

where $m_{v,eq} = m_v + \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2$,$T_e = f(\omega_e)$ in some cases.
Assume T_e is substituted as a function of vehicle speed $v = \frac{r_w}{\gamma} \cdot \omega_e$, and then sorted Equation (**??**) with descending order of power, we have the representation of elementary vehicle speed as follows:

$$(\frac{1}{A\,v^2 + B\,v + C})\;dv = dt \qquad (**)$$

Integrating the elementary speed from the initial state, we have:

$$\int_{v_0}^{v(t)} (\frac{1}{A\,v^2 + B\,v + C})\;dv = \int_{t_0}^{t} dt$$

$$\int_{v_0}^{v(t)} \left[\frac{1}{A\,(v + \frac{B}{2\,A})^2 - \frac{B^2 - 4\,A\,C}{4\,A}}\right]\;dv = t - t_0$$

Solving the equation subject to different values of $\Delta = B^2 - 4\,A\,C$:
(i) If $\Delta \geq 0$
Equation (**??**) gives:

$$\frac{1}{sgn(A)\sqrt{B^2 - 4\,A\,C}}\;\;\ln\left[\frac{2\,A\,v + B - sgn(A)\cdot\sqrt{B^2 - 4\,A\,C}}{2\,A\,v + B + sgn(A)\cdot\sqrt{B^2 - 4\,A\,C}}\right]\Bigg|_{v_0}^{v(t)} = t\Bigg|_{t_0}^{t}$$

Let

$$S_+ = sqrt B^2 - 4\,A\,C$$
$$\hat{S_+} = sgn(A) \cdot \ sqrt B^2 - 4\,A\,C$$
$$K_0 = \frac{2\,A\,v_0 + B - sgn(A) \cdot \sqrt{B^2 - 4\,A\,C}}{2\,A\,v_0 + B + sgn(A) \cdot \sqrt{B^2 - 4\,A\,C}} = \frac{2\,A\,v_0 + B - \hat{S_+}}{2\,A\,v_0 + B + \hat{S_+}},$$

then, the vehicle speed v can be represented as a explicit function of time t, when the vehicle starts are $v|_{t_0=0} = v_0$:

$$K(t) = \frac{2\,A\,v + B - \hat{S_+}}{2\,A\,v + B + \hat{S_+}} = \frac{2\,A\,v_0 + B - \hat{S_+}}{2\,A\,v_0 + B + \hat{S_+}} \cdot e^{\left[\hat{S_+}(t-t_0)\right]} = K_0 \cdot e^{\left[\hat{S_+}(t-t_0)\right]}$$

Equivalently, it can also be represented as

$$\begin{aligned} v &= \frac{\hat{S_+}}{2\,A}\,\frac{1+K(t)}{1-K(t)} - \frac{B}{2\,A} \\ &= \frac{\hat{S_+}[1+K(t)] - B\,[1-K(t)]}{2\,A\,[1-K(t)]} \end{aligned}$$

(ii) If $\Delta < 0$
Equation (**??**) gives:

$$\frac{2}{sgn(A) \cdot \sqrt{4\,A\,C - B^2}} \left[\arctan \frac{2\,A\,v + B}{sgn(A)\,\sqrt{4\,A\,C - B^2}} - \arctan \frac{2\,A\,v_0 + B}{sgn(A)\,\sqrt{4\,A\,C - B^2}}\right] = t - t_0$$

Let

$$S_- = \sqrt{4\,A\,C - B^2}$$
$$\hat{S_-} = sgn(A) \cdot \sqrt{4\,A\,C - B^2},$$

then, the vehicle speed v can be represented as a explicit function of time t, when the vehicle starts are $v|_{t_0=0} = v_0$:

$$\arctan \frac{2\,A\,v + B}{\hat{S_-}} - \arctan \frac{2\,A\,v_0 + B}{\hat{S_-}} = \frac{\hat{S_-}}{2}\,(t - t_0).$$

Equivalently, it can also be represented as

$$v = \frac{\hat{S_-} \ \cdot \tan\left[\frac{\hat{S_-}(t-t_0)}{2} + \arctan \frac{2\,A\,v_0 + B}{\hat{S_-}}\right] - B}{2\,A}$$

b) Specific case of maximum torque input
Given that $T_e = a \cdot \omega_e^2 + b \cdot \omega_e + c$, the corresponding A, B and C in Equation (**??**) can be calculated as follows:

$$A = \left(-\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left(\frac{\gamma}{r_w} \right)^3 \cdot \eta_t \cdot a \right) \cdot \frac{1}{m_{v,eq}}$$
$$B = \left(\frac{\gamma}{r_w} \right)^2 \cdot \eta_t \cdot b \cdot \frac{1}{m_{v,eq}}$$
$$C = \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot c \right) \cdot \frac{1}{m_{v,eq}}$$

Finally, substituting these parameters into Δ and switch to the corresponding equations, we can evaluate the speed profile of an ICE vehicle under maximum engine torque.

Discussion

- Note that γ / r_w generally varies along the acceleration, so a solution must be obtained step by step.
- Specific cases of parameters
  (i) Decided by normal shape of maximum torque curve
  Given the fact that typical maximum torque curve can be emulated by a downward parabolar, which means:

  $$a < 0,\ b \geq 0,\ c > 0.$$

  This parameter set gives both possibilities for Δ to be positive or negative.
  (ii) Decided by different accerleration of vehicle
  When $B = 0$ and $C < 0$ (coasting), then $D < 0$ and one finds back (2.18) for the coasting velocity.

Problem 2.4

Evaluate the 0-100 km/h time precisely using the result of Problem 2.3. Use the following data: engine launch speed = 2500 rpm, engine upshift speed $\omega_{e,max} = 6500\,\text{rpm}$, $\gamma / r_w = \{46.48, 29.13, 20.39, 15.04, 11.39\}$, $a = -4.38 \cdot 10^{-4}\,\text{Nm}\,\text{s}^2$, $b = 0.3514\,\text{Nm}\,\text{s}$, $c = 80\,\text{Nm}$, and the following vehicle data: $m_v = 1240$ kg, $A_f \cdot c_d = 0.65$ m^2, $c_r = 0.009$, $\eta_t = 0.9$. Further assume that a slipping clutch transmits all the torque. The momentum of inertia $\Theta_e = 0.128$ kg·m^2.

- Solution

a) Estimate gear # at the end of acceleration
Since A, B, and C depend on γ, which changes along the speed trajectory, the calculation of $v(t)$ must be separated in segments according to the

gear and the clutch status. The target speed will be reached in the gear # whose γ/r_w is immediately lower than

$$\frac{\omega_{e,max}}{v} = \frac{6500[\text{rpm}] \cdot (2 \cdot \pi/60)}{100\text{km/h}/3.6} = 24.5$$

Thus the target speed is reached in the third gear. Four segments (including takeoff) must be considered.

b) Vehicle mass and inertia
As for the rolling friction, the orignal vehicle mass is used for reaction force calculation:

$$m_v = 1240\,\text{kg}.$$

As for the dynamic force, the equivalent mass with engine inertia is considered for each gear #:

$$m_{v,eq} = m_v + \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2$$

c) ***1^st^ segment*: takeoff,** $v_0 = 0$, $t_0 = 0$
Although the vehicle is at standstill, the engine is operating at the idle speed. Using the result of Problem 2.6, we get $\omega_e = 2500$ rpm; and substituting torque parameters, we further have:

$$T_e = a \cdot \omega_e^2 + b \cdot \omega_e + c = 142.0 \text{ Nm}$$

as a constant torque during slipping-clutch segment.

$$\begin{aligned}
m_{v,eq} &= m_v + \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2 = 1240 + 0.128 \cdot 46.48^2 = 1517 \text{ kg} \\
A &= -\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot \frac{1}{m_{v,eq}} = -\frac{1}{2} \cdot \frac{1.2 \cdot 0.65}{1517} = -2.571 \times 10^{-4} \\
B &= 0 \\
C &= \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot T_e\right) \cdot \frac{1}{m_{v,eq}} = \\
&= \frac{-9.81 \cdot 1240 \cdot 0.009 + 46.48 \cdot 0.9 \cdot 142}{1517} = 3.844 \\
\Delta &= -4 \cdot A \cdot C > 0 \\
\hat{S_+} &= sgn(A) \cdot \sqrt{-4 \cdot A \cdot C} = -1 \cdot \sqrt{4 \cdot 2.571 \times 10^{-4} \cdot 3.844} = -0.06287
\end{aligned}$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of first segment.
To be more specific, the synchronization time between the engine and the vehicle (clutch closed) is when

$$v = \frac{\omega_e[\text{rpm}] \cdot (\pi/30) \cdot r_w}{\gamma} = 5.633 \text{ m/s} = 20.28 \text{ km/h},$$

thus the final state constraint of the first segment appears to be:

$$K(t_0) = \frac{2 \cdot A \cdot v - S}{2 \cdot A \cdot v + S} = \frac{2 \cdot (-2.571) \cdot 10^{-4} \cdot 5.633 + 0.06287}{2 \cdot (-2.571) \cdot 10^{-4} \cdot 5.633 - 0.06287} = -0.9119$$

Because the evolution of $K(t)$ satisfies $K(t) = K_0 \cdot e^{\hat{S_+}(t-t_0)]}$, the synchronize time is thus

$$S_+ = sgn(A) \cdot \sqrt{\Delta} = -0.06287, \; K_0 = -1$$

$$t = \frac{\ln\left(\frac{K(t_0)}{K_0}\right)}{\hat{S_+}} = \frac{\ln(\frac{-0.9119}{-1})}{-0.06287} = 1.467 \text{ s}$$

d) ***2nd segment, 1st gear,*** $v_0 = 5.633$ m/s, $t_0 = 1.467$ s

Here $\gamma / r_w = 46.48$, thus

$$m_{v,eq} = m_v + \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2 = 1240 + 0.128 \cdot 46.48^2 = 1517 \text{ kg}$$

$$A = \left(-\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left(\frac{\gamma}{r_w}\right)^3 \cdot \eta_t \cdot a\right) \cdot \frac{1}{m_{v,eq}}$$

$$= \left(-\frac{1}{2} \cdot 1.2 \cdot 0.65 - 46.48^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4}\right) \cdot \frac{1}{1517} = -0.02635$$

$$B = \left(\frac{\gamma}{r_w}\right)^2 \cdot \eta_t \cdot b \cdot \frac{1}{m_{v,eq}} = \frac{46.48^2 \cdot 0.9 \cdot 0.3514}{1517} = 0.4504$$

$$C = \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot c\right) \cdot \frac{1}{m_{v,eq}}$$

$$= \frac{-9.81 \cdot 1240 \cdot 0.009 + 46.48 \cdot 0.9 \cdot 80}{1517} = 2.134$$

$$\Delta = B^2 - 4 \cdot A \cdot C = 0.4504^2 + 4 \cdot 0.02635 \cdot 2.134 = 0.4278 > 0$$

$$\hat{S_+} = sgn(A) \cdot \sqrt{\Delta} = -0.6541$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of second segment.

$$K(t_0) = \frac{2 \cdot A \cdot v_0 + B - \hat{S_+}}{2 \cdot A \cdot v_0 + B + \hat{S_+}}$$

$$= \frac{-2 \cdot 0.02635 \cdot 5.633 + 0.4504 + 0.6541}{-2 \cdot 0.02635 \cdot 5.633 + 0.4504 - 0.6541}$$

$$= -1.613$$

This phase ends when $\omega_e = \omega_{e,max}$, thus when

$$v = \frac{\omega_{e,max} \cdot (\pi/30) \cdot r_w}{\gamma_1} = \frac{6500 \cdot 2 \cdot \pi}{60 \cdot 46.48} = 14.64 \text{ m/s} = 52.70 \text{ km/h}.$$

and the final state constraint gives:

$$K(t_1) = \frac{2 \cdot A \cdot v + B - \hat{S_+}}{2 \cdot A \cdot v + B + \hat{S_+}} = \frac{-2 \cdot 0.02635 \cdot 14.64 + 0.4504 + 0.6541}{-2 \cdot 0.02635 \cdot 14.64 + 0.4504 - 0.6541} = -0.3414$$

thus at time

$$t_1 = t_0 + \ln\left(\frac{K(t_1)}{K(t_0)}\right) \cdot \frac{1}{\hat{S_+}} = 1.467 + \ln\left(\frac{-0.3414}{-1.613}\right) \cdot \frac{1}{-0.6541} = 1.467 + 2.374 = 3.841 \text{ s}$$

e) ***3rd segment, 2nd gear,*** $v_0 = 14.64$ m/s, $t_0 = 3.841$ s
Here $\gamma/r_w = 29.13$

$$m_{v,eq} = m_v + \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2 = 1240 + 0.128 \cdot 29.13^2 = 1349 \text{ kg}$$

$$A = \left(-\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left(\frac{\gamma}{r_w}\right)^3 \cdot \eta_t \cdot a\right) \cdot \frac{1}{m_{v,eq}}$$

$$= \left(-\frac{1}{2} \cdot 1.2 \cdot 0.65 - 29.13^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4}\right) \frac{1}{1349} = -0.007512$$

$$B = \left(\frac{\gamma}{r_w}\right)^2 \cdot \eta_t \cdot b \cdot \frac{1}{m_{v,eq}} = \frac{29.13^2 \cdot 0.9 \cdot 0.3514}{1349} = 0.1989$$

$$C = \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot c\right) \cdot \frac{1}{m_{v,eq}}$$

$$= \frac{-9.81 \cdot 1240 \cdot 0.009 + 29.13 \cdot 0.9 \cdot 80}{1349} = 1.474$$

$$\Delta = 0.1989^2 + 4 \cdot 0.007512 \cdot 1.474 = 0.08385 > 0$$

$$\hat{S_+} = -sgn(A) \cdot \sqrt{\Delta} = -0.2896$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of third segment.

$$K(t_1) = \frac{2 \cdot A \cdot v_1 + B - \hat{S_+}}{2 \cdot A \cdot v_1 + B + \hat{S_+}}$$

$$= \frac{-2 \cdot 0.007512 \cdot 14.64 + 0.1989 + 0.2896}{-2 \cdot 0.007512 \cdot 14.64 + 0.1989 - 0.2896} = -0.8645$$

This phase ends when $\omega_e = \omega_{e,max}$, thus when

$$v = \frac{\omega_{e,max} \cdot (\pi/30) \cdot r_w}{\gamma_1} = \frac{6500 \cdot 2 \cdot \pi}{60 \cdot 29.13} = 23.37 \text{ m/s} = 84.1 \text{ km/h}$$

and

$$\begin{aligned} K(t_2) &= \frac{2 \cdot A \cdot v_2 + B - \hat{S_+}}{2 \cdot A \cdot v_2 + B + \hat{S_+}} \\ &= \frac{-2 \cdot 0.007512 \cdot 23.37 + 0.1989 + 0.2896}{-2 \cdot 0.007512 \cdot 23.37 + 0.1989 - 0.2896} = -0.3110 \end{aligned}$$

thus at time

$$t_2 = t_1 + \ln\left(\frac{K(t_2)}{K(t_1)}\right) \cdot \frac{1}{\hat{S_+}} = 3.841 + \ln\left(\frac{-0.3110}{-0.8645}\right) \cdot \frac{1}{-0.2896} = 3.841 + 3.530 = 7.371 \text{ s}$$

f) ***4th segment, 3rd gear,*** $v_0 = 23.37$ m/s, $t_0 = 7.371$ s
Here $\gamma/r_w = 20.39 < 24.5 = \frac{\omega_{e,max}}{v_{max}}$

$$\begin{aligned} m_{v,eq} &= m_v + \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2 = 1240 + 0.128 \cdot 20.39^2 = 1293 \text{ kg} \\ A &= \left(-\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left(\frac{\gamma}{r_w}\right)^3 \cdot \eta_t \cdot a\right) \cdot \frac{1}{m_{v,eq}} \\ &= \left(-\frac{1}{2} \cdot 1.2 \cdot 0.65 - 20.39^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4}\right) \frac{1}{1293} = -0.002886 \\ B &= \left(\frac{\gamma}{r_w}\right)^2 \cdot \eta_t \cdot b \cdot \frac{1}{m_{v,eq}} = \frac{20.39^2 \cdot 0.9 \cdot 0.3514}{1293} = 0.1017 \\ C &= \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot c\right) \cdot \frac{1}{m_{v,eq}} \\ &= (-9.81 \cdot 1240 \cdot 0.009 + 20.39 \cdot 0.9 \cdot 80)/1293 = 1.0507 \\ \Delta &= B^2 - 4\,A\,C = 0.1017^2 + 4 \cdot 0.002886 \cdot 1.0507 = 0.2247 > 0 \\ \hat{S_+} &= sgn(A) \cdot \sqrt{\Delta} = -0.1499 \end{aligned}$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of fourth segment.

$$\begin{aligned} K(t_2) &= \frac{2 \cdot A \cdot v_2 + B - \hat{S_+}}{2 \cdot A \cdot v_2 + B + \hat{S_+}} \\ &= \frac{-2 \cdot 0.002886 \cdot 23.37 + 0.1017 + 0.1499}{-2 \cdot 0.002886 \cdot 23.37 + 0.1017 + 0.1499} = -0.6374 \end{aligned}$$

We want to calculate the time to reach $v = 100$ km/h $= 27.78$ m/s that will take place in this segment. Thus, this phase ends when $v = v_{max}$, thus when

$$K(t_3) = \frac{2 \cdot A \cdot v_2 + B - \hat{S_+}}{2 \cdot A \cdot v_2 + B + \hat{S_+}}$$
$$= \frac{-2 \cdot 0.002886 \cdot 27.78 + 0.1017 - 0.1499}{-2 \cdot 0.002886 \cdot 27.78 + 0.1017 + 0.1499} = -0.4376$$

thus at time

$$t_3 = t_2 + \ln\left(\frac{K(t_2)}{K(t_1)}\right) \cdot \frac{1}{\hat{S_+}} = 7.371 + \frac{\ln\left(\frac{-0.4376}{-0.6374}\right)}{-0.1499} = 7.371 + 2,509 = 9.88 \text{ s}$$

Problem 2.5

Consider again Problem 2.4 for an engine with negligible inertia Θ_e. Compare the result with (2.16).

- Solution

a) ***1^st^ segment*: takeoff,** $v_0 = 0$, $t_0 = 0$
Although the vehicle is at standstill, the engine is operating at the idle speed. Using the result of Problem 2.6, we get $\omega_e = 2500$ rpm; and substituting torque parameters, we further have:

$$T_e = a \cdot \omega_e^2 + b \cdot \omega_e + c = 142.0 \text{ Nm}$$

as a constant torque during slipping-clutch segment.

$$m_{v,eq} = m_v = 1240 \text{ kg}$$
$$A = -\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot \frac{1}{m_{v,eq}} = -\frac{1}{2} \cdot \frac{1.2 \cdot 0.65}{1240} = -3.145 \times 10^{-4}$$
$$B = 0$$
$$C = \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot T_e\right) \cdot \frac{1}{m_{v,eq}} =$$
$$= \frac{-9.81 \cdot 1240 \cdot 0.009 + 46.48 \cdot 0.9 \cdot 142}{1240} = 4.702$$
$$\Delta = -4 \cdot A \cdot C = 5.915 \times 10^{-3} > 0$$
$$\hat{S_+} = sgn(A) \cdot \sqrt{-4 \cdot A \cdot C} = -1 \cdot \sqrt{4 \cdot 3.145 \times 10^{-4} \cdot 4.702} = -0.07691$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of first segment.
To be more specific, the synchronization time between the engine and the vehicle (clutch closed) is when

$$v = \frac{\omega_e[\text{rpm}] \cdot (\pi/30) \cdot r_w}{\gamma} = 5.633 \text{ m/s} = 20.28 \text{ km/h},$$

thus the final state constraint of the first segment appears to be:

$$K(t_0) = \frac{2 \cdot A \cdot v - S}{2 \cdot A \cdot v + S} = \frac{2 \cdot (-3.145) \cdot 10^{-4} \cdot 5.633 + 0.07691}{2 \cdot (-3.145) \cdot 10^{-4} \cdot 5.633 - 0.07691} = -0.9119$$

Because the evolution of $K(t)$ satisfies $K(t) = K_0 \cdot e^{\hat{S_+}(t-t_0)]}$, the synchronize time is thus

$$S_+ = sgn(A) \cdot \sqrt{\Delta} = -0.07691, \; K_0 = -1$$

$$t = \frac{\ln\left(\frac{K(t_0)}{K_0}\right)}{\hat{S_+}} = \frac{\ln(\frac{-0.9119}{-1})}{-0.07691} = 1.199 \text{ s}$$

b) ***2^nd^ segment, 1^st^ gear,*** $v_0 = 5.633$ m/s, $t_0 = 1.199$ s

Here $\gamma/r_w = 46.48$, thus

$$\begin{aligned}
m_{v,eq} &= m_v = 1240 \text{ kg} \\
A &= \left(-\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left(\frac{\gamma}{r_w}\right)^3 \cdot \eta_t \cdot a\right) \cdot \frac{1}{m_{v,eq}} \\
&= \left(-\frac{1}{2} \cdot 1.2 \cdot 0.65 - 46.48^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4}\right) \cdot \frac{1}{1240} = -0.03224 \\
B &= \left(\frac{\gamma}{r_w}\right)^2 \cdot \eta_t \cdot b \cdot \frac{1}{m_{v,eq}} = \frac{46.48^2 \cdot 0.9 \cdot 0.3514}{1240} = 0.5510 \\
C &= \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot c\right) \cdot \frac{1}{m_{v,eq}} \\
&= \frac{-9.81 \cdot 1240 \cdot 0.009 + 46.48 \cdot 0.9 \cdot 80}{1240} = 2.611 \\
\Delta &= B^2 - 4 \cdot A \cdot C = 0.5510^2 + 4 \cdot 0.03224 \cdot 2.611 = 0.6403 > 0 \\
\hat{S_+} &= sgn(A) \cdot \sqrt{\Delta} = -0.8002
\end{aligned}$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of second segment.

$$\begin{aligned}
K(t_0) &= \frac{2 \cdot A \cdot v_0 + B - \hat{S_+}}{2 \cdot A \cdot v_0 + B + \hat{S_+}} \\
&= \frac{-2 \cdot 0.03224 \cdot 5.633 + 0.5510 + 0.8002}{-2 \cdot 0.03224 \cdot 5.633 + 0.5510 - 0.8002} \\
&= -1.613
\end{aligned}$$

This phase ends when $\omega_e = \omega_{e,max}$, thus when

$$v = \frac{\omega_{e,max} \cdot (\pi/30) \cdot r_w}{\gamma_1} = \frac{6500 \cdot 2 \cdot \pi}{60 \cdot 46.48} = 14.64 \text{ m/s} = 52.70 \text{ km/h}.$$

and the final state constraint gives:

$$K(t_1) = \frac{2 \cdot A \cdot v + B - \hat{S_+}}{2 \cdot A \cdot v + B + \hat{S_+}} = \frac{-2 \cdot 0.03224 \cdot 14.64 + 0.5510 + 0.8002}{-2 \cdot 0.03224 \cdot 14.64 + 0.5510 - 0.8002} = -0.3413$$

thus at time

$$t_1 = t_0 + \ln\left(\frac{K(t_1)}{K(t_0)}\right) \cdot \frac{1}{\hat{S_+}} = 1.199 + \ln\left(\frac{-0.3413}{-1.613}\right) \cdot \frac{1}{-0.8002} = 1.199 + 1.941 = 3.140 \text{ s}$$

c) ***3rd segment, 2nd gear,*** $v_0 = 14.64$ m/s, $t_0 = 3.140$ s
Here $\gamma / r_w = 29.13$

$$\begin{aligned}
m_{v,eq} &= m_v = 1240 \text{ kg} \\
A &= \left(-\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left(\frac{\gamma}{r_w}\right)^3 \cdot \eta_t \cdot a\right) \cdot \frac{1}{m_{v,eq}} \\
&= \left(-\frac{1}{2} \cdot 1.2 \cdot 0.65 - 29.13^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4}\right) \frac{1}{1240} = -0.008173 \\
B &= \left(\frac{\gamma}{r_w}\right)^2 \cdot \eta_t \cdot b \cdot \frac{1}{m_{v,eq}} = \frac{29.13^2 \cdot 0.9 \cdot 0.3514}{1240} = 0.2164 \\
C &= \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot c\right) \cdot \frac{1}{m_{v,eq}} \\
&= \frac{-9.81 \cdot 1240 \cdot 0.009 + 29.13 \cdot 0.9 \cdot 80}{1240} = 1.603 \\
\Delta &= 0.2164^2 + 4 \cdot 0.008173 \cdot 1.603 = 0.09923 > 0 \\
\hat{S_+} &= -sgn(A) \cdot \sqrt{\Delta} = -0.3150
\end{aligned}$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of third segment.

$$\begin{aligned}
K(t_1) &= \frac{2 \cdot A \cdot v_1 + B - \hat{S_+}}{2 \cdot A \cdot v_1 + B + \hat{S_+}} \\
&= \frac{-2 \cdot 0.008173 \cdot 14.64 + 0.2164 + 0.3150}{-2 \cdot 0.008173 \cdot 14.64 + 0.2164 - 0.3150} = -0.8644
\end{aligned}$$

This phase ends when $\omega_e = \omega_{e,max}$, thus when

$$v = \frac{\omega_{e,max} \cdot (\pi/30) \cdot r_w}{\gamma_1} = \frac{6500 \cdot 2 \cdot \pi}{60 \cdot 29.13} = 23.37 \text{ m/s} = 84.1 \text{ km/h}$$

and

$$K(t_2) = \frac{2 \cdot A \cdot v_2 + B - \hat{S_+}}{2 \cdot A \cdot v_2 + B + \hat{S_+}}$$
$$= \frac{-2 \cdot 0.008173 \cdot 23.37 + 0.2164 + 0.3150}{-2 \cdot 0.008173 \cdot 23.37 + 0.2164 - 0.3150} = -0.3108$$

thus at time

$$t_2 = t_1 + \ln\left(\frac{K(t_2)}{K(t_1)}\right) \cdot \frac{1}{\hat{S_+}} = 3.841 + \ln\left(\frac{-0.3108}{-0.8644}\right) \cdot \frac{1}{-0.3150} = 3.140 + 3.247 = 6.387 \text{ s}$$

d) ***4^th^ segment, 3^rd^ gear,*** $v_0 = 23.37$ m/s, $t_0 = 6.387$ s
Here $\gamma/r_w = 20.39 < 24.5 = \frac{\omega_{e,max}}{v_{max}}$

$$m_{v,eq} = m_v = 1240 \text{ kg}$$
$$A = \left(-\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left(\frac{\gamma}{r_w}\right)^3 \cdot \eta_t \cdot a\right) \cdot \frac{1}{m_{v,eq}}$$
$$= \left(-\frac{1}{2} \cdot 1.2 \cdot 0.65 - 20.39^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4}\right) \frac{1}{1240} = -0.003009$$
$$B = \left(\frac{\gamma}{r_w}\right)^2 \cdot \eta_t \cdot b \cdot \frac{1}{m_{v,eq}} = \frac{20.39^2 \cdot 0.9 \cdot 0.3514}{1240} = 0.1060$$
$$C = \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot c\right) \cdot \frac{1}{m_{v,eq}}$$
$$= (-9.81 \cdot 1240 \cdot 0.009 + 20.39 \cdot 0.9 \cdot 80)/1240 = 1.0956$$
$$\Delta = B^2 - 4\,A\,C = 0.1060^2 + 4 \cdot 0.003009 \cdot 1.0956 = 0.2242 > 0$$
$$\hat{S_+} = sgn(A) \cdot \sqrt{\Delta} = -0.1563$$

Substituting unknowns in the result of Problem 2.6 and solve the equations subject to initial and final conditions of velocity, we find the time of fourth segment.

$$K(t_2) = \frac{2 \cdot A \cdot v_2 + B - \hat{S_+}}{2 \cdot A \cdot v_2 + B + \hat{S_+}}$$
$$= \frac{-2 \cdot 0.003009 \cdot 23.37 + 0.1060 + 0.1563}{-2 \cdot 0.003009 \cdot 23.37 + 0.1060 + 0.1563} = -0.6372$$

We want to calculate the time to reach $v = 100$ km/h $= 27.78$ m/s that will take place in this segment. Thus, this phase ends when $v = v_{max}$, thus when

$$K(t_3) = \frac{2 \cdot A \cdot v_2 + B - \hat{S_+}}{2 \cdot A \cdot v_2 + B + \hat{S_+}}$$
$$= \frac{-2 \cdot 0.003009 \cdot 27.78 + 0.1060 - 0.1563}{-2 \cdot 0.003009 \cdot 27.78 + 0.1060 + 0.1563} = -0.4374$$

thus at time

$$t_3 = t_2 + \ln\left(\frac{K(t_2)}{K(t_1)}\right) \cdot \frac{1}{\hat{S_+}} = 6.387 + \frac{\ln\left(\frac{-0.0.4374}{-0.6372}\right)}{-0.1563} = 6.387 + 2.404 = 8.791 \text{ s}$$

Discussion

In reality, usually the acceleration time is calculated without accounting for the engine inertia, and is usually based on a simplified estimation. According to the given parameters, the maximum power is achieved when derivative of power is set to 0:

$$\frac{d}{d\omega_e}(\omega_e \cdot T_e) = \frac{d}{d\omega_e}(a\,\omega_e^3 + b\,\omega_e^2 + c\,\omega_e) = 0,$$

which gives $\omega_e^* = 631.3$ rad/s $= 6028$ rpm, and $P_{max}|_{\omega_e^*} = 80.35$ kW. And the minimum power:

$$P_{e,min} = \omega_{e,launch} \cdot T_{e,launch} = 2500 \cdot \frac{\pi}{30} \cdot 142 = 37.18 \text{ kW}$$

With these values, there are two basic ways of estimation:

- Using the maximum engine power

$$t = \frac{v^2 \cdot m_v}{P_{e,max}} = \frac{27.8^2 \cdot 1240}{80.35 \times 10^3} = 11.90 \text{ s},$$

  which means an over-estimation of 35%.
- Using the corrected engine power

$$\frac{\tilde{P_{max}}}{2}\bar{P} = \frac{P_{e,max} + P_{e,min}}{2}$$

  The corrected maximum power is therefore:

$$\tilde{P_{max}} = 2 \cdot \bar{P} = 117.5 \text{ kW},$$

  and the corresponding acceleration time is:

$$t = \frac{v^2 \cdot m_v}{\tilde{P_{e,max}}} = \frac{27.78^2 \cdot 1240}{117.5} = 8.14 \text{ s},$$

  which gives an underestimation of 7.4%.

Problem 2.6

Find an equation to calculate the takeoff time (=time to synchronise the speed before and after the clutch) as a function of engine launch speed and torque. Assume that the clutch is slipping but transmitting the whole engine torque.

- Solution

Assume the ratio of gear-ratio to wheel diameter is kept at gear 1 during the take-off time; and the engine torque is kept constant while engine speed ω_e and wheel speed v are therefore decoupled.

a) Check *Delta* of the motion
Given the constant torque as input, the law of motion of the type $\dot{v} = A \cdot v^2 + C$, where

$$A = -\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot \frac{1}{m_{v,eq}} \qquad C = \left(-g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot T_e \right) \cdot \frac{1}{m_{v,eq}}$$

Given the fact that $A < 0, B = 0$, we have $\Delta = -4\,A\,C > 0$, then $\hat{S_+} = sgnA \cdot \sqrt{\Delta} = -\sqrt{\Delta}$.

b) Solve the motion law with final condition
Given that $t_0 = 0, v_0 = 0$, then

$$K_0 = \frac{2\,A\,v_0 + B - \hat{S_+}}{2\,A\,v_0 + B + \hat{S_+}} = -1.$$

and the synchronized speed of the wheel should balance the idel speed of the engine:

$$v_f = \omega_{e,idle} \cdot \frac{r_w}{\gamma}.$$

then

$$K(t_f) = \frac{2\,A\,v_f + B - \hat{S_+}}{2\,A\,v_f + B + \hat{S_+}} = \frac{2\,A\,v_f - \hat{S_+}}{2\,A\,v_f + \hat{S_+}}.$$

Since $K(t_f) = K_0 \cdot e^{\hat{S_+}(t_f - t_0)}$, we have

$$t_f = \frac{\ln \frac{\frac{2\,A\,v_f - \hat{S_+}}{2\,A\,v_f + \hat{S_+}}}{-1}}{\hat{S_+}}$$

Therefore,

$$t_{takeoff} = \frac{\ln(-K)}{\hat{S_+}}$$

Problem 2.7

Evaluate the coasting speed and the roll-out time without acting on the brakes for a vehicle with an initial speed $v_0 = 50$ km/h and $m_v = 1200$ kg, $A_f \cdot c_d = 0.65\,\mathrm{m}^2$, $c_r = 0.009$. Assume the clutch open (no engine friction).

- Solution

Assume the coasting speed is defined when the engine is disengaged and the resistence loss of the vehicle matches exactly the decrease of its kinetic energy:

$$\frac{d}{dt}v_c(t) = -\frac{1}{2\,m_v}\cdot \rho_a A_f \cdot C_d\, v_c^2(t) - g\,C_r$$

and we let:

$$\alpha^2 = \frac{\rho_a \cdot A_f C_d \cdot v_c^2(t)}{2\,m_v} = -A$$
$$\beta^2 = g\,C_r = -C$$

a) Check *Delta* of the motion
Given the zero torque as input(engine disengaged), the law of motion of the type $\dot{v} = A \cdot v^2 + C$, where

$$A = -\frac{1}{2\,m_v}\cdot \rho_a A_f \cdot C_d \qquad C = -g\,C_r$$

Given the fact that $A < 0, B = 0, C < 0$, we have $\Delta = -4\,A\,C < 0$, therefore we have equation (2.18).

b) Calculate coasting speed
The coasting speed as a function of time (2.18) is

$$v(t) = \frac{\beta}{\alpha}\cdot \tan\left\{\arctan\left(\frac{\alpha}{\beta}\cdot v_0\right) - \alpha\cdot\beta\cdot t\right\}$$

where

$$\alpha = \sqrt{\frac{1}{2}\cdot\frac{\rho_a \cdot A_f \cdot c_d}{m_v}} = \sqrt{\frac{1}{2}\cdot\frac{1.2\cdot 0.65}{1200}} = 0.01803$$
$$\beta = \sqrt{g\cdot c_r} = \sqrt{9.81\cdot 0.009} = 0.2971$$

c) Calculate the roll-off time
The braking time at which $v = 0$ is calculated by solving:

$$0 = \frac{\beta}{\alpha}\tan\left[\arctan\left(\frac{\alpha}{\beta}v_0\right) - \alpha\,\beta\cdot t\right]$$
$$t_{rollout} = \frac{1}{\alpha\cdot\beta}\cdot\arctan\left(\frac{\alpha}{\beta}\cdot v_0\right)$$
$$= \frac{1}{0.018\cdot 0.297}\cdot\arctan\left(\frac{0.018}{0.297}\cdot\frac{50}{3.6}\right) = 130.75\text{ s}$$

Discussion
Take care of the radians calculation in the function arctan.

Mechanical Energy Demand in Driving Cycles

Problem 2.8

Evaluate the traction energy and the recuperation energy for the MVEG–95 for the vehicle examples of Fig. 2.8, left and right, assuming perfect recuperation.

- Solution

  Assume the time intervals in traction mode are not subject ot change during artificial cycles like MVEG-95, so the equations used in this problem is only valid if this assumption holds; and the track length of MVEG-95 is 0.114 km.

a) Left graph configuration ($A_f \cdot C_d = 0.7\ \text{m}^2, m_v = 1500\ \text{kg}, C_r = 0.012$)

(i) The traction energy is given by (2.31), assuming no recuperation.

$$\begin{aligned}\bar{E} = E_{diss} + E_{circ} &= [1.9 \times 10^4 \cdot A_f C_d + 8.4 \times 10^2 \cdot m_v\, C_r + 10 \cdot m_v] \cdot x_{tot} \\ &= \big(0.7 \cdot 1.9 \cdot 10^4 + 1500 \cdot 0.012 \cdot 8.4 \cdot 10^2 + 1500 \cdot 10\big) \\ &= 43.42\ {}^{\text{MJ}}\!/_{100\ \text{km}} \ \cdot x_{tot} = 43.42 \cdot 0.114 = 4.950\ \text{MJ}.\end{aligned}$$

(ii) The total energy is given by (2.35), assuming perfect recuperation.

$$\begin{aligned}\bar{E}_{rec} = E_{diss} &= [2.2 \times 10^4 \cdot A_f C_d + 9.8 \times 10^2 \cdot m_v\, C_r] \cdot x_{tot} \\ &= \big(0.7 \cdot 2.2 \cdot 10^4 + 1500 \cdot 0.012 \cdot 9.81 \cdot 10^2\big) = \\ &= 33.04\ {}^{\text{MJ}}\!/_{100\ \text{km}} \ \cdot x_{tot} = 3.767\ \text{MJ}.\end{aligned}$$

(iii) The energy that can be recuperated:

$$\Delta\bar{E} = \bar{E} - \bar{E}_{rec} = 1.183\ \text{MJ} \qquad (24.75\%\ \text{of })\bar{E}.$$

b) Right graph configuration ($A_f \cdot C_d = 0.4\ \text{m}^2, m_v = 750\ \text{kg}, C_r = 0.008$)

(i) The traction energy is given by (2.31), assuming no recuperation.

$$\begin{aligned}\bar{E} = E_{diss} + E_{circ} &= [1.9 \times 10^4 \cdot A_f C_d + 8.4 \times 10^2 \cdot m_v\, C_r + 10 \cdot m_v] \cdot x_{tot} \\ &= \big(0.4 \cdot 1.9 \cdot 10^4 + 1500 \cdot 0.008 \cdot 8.4 \cdot 10^2 + 750 \cdot 10\big) \\ &= 20.14\ {}^{\text{MJ}}\!/_{100\ \text{km}} \ \cdot x_{tot} = 20.14 \cdot 0.114 = 2.296\ \text{MJ}.\end{aligned}$$

(ii) The total energy is given by (2.35), assuming perfect recuperation.

$$\begin{aligned}\bar{E}_{rec} = E_{diss} &= [2.2 \times 10^4 \cdot A_f C_d + 9.8 \times 10^2 \cdot m_v\, C_r] \cdot x_{tot} \\ &= \big(0.4 \cdot 2.2 \cdot 10^4 + 750 \cdot 0.008 \cdot 9.81 \cdot 10^2\big) = \\ &= 14.68\ {}^{\text{MJ}}\!/_{100\ \text{km}} \ \cdot x_{tot} = 1.674\ \text{MJ}.\end{aligned}$$

(iii) The energy that can be recuperated:

$$\Delta\bar{E} = \bar{E} - \bar{E}_{rec} = 0.622\ \text{MJ} \qquad (27.09\%\ \text{of })\bar{E}.$$

The potential of regenerative braking is more important for the smaller vehicle, with smaller front area, vehicle mass and rolling friction coefficient.

Problem 2.9

Calculate the mean force and fuel consumption data shown in Fig. 2.8 left.

- Solution

  Assume left graph configuration ($A_f \cdot C_d = 0.7$ m$^2, m_v = 1500$ kg$, C_r = 0.012$)

a) **Case 1: No recuperation**
Reading constants from **(2.30)**, we get the weights during traction modes of the cycle MVEG-95:

$$\bar{F}_{trac,a} = \frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot 319 = \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 319 = 134.0 \text{ N}$$
$$\bar{F}_{trac,r} = m_v \cdot g \cdot c_r \cdot 0.856 = 1500 \cdot 9.81 \cdot 0.012 \cdot 0.856 = 151.2 \text{ N}$$
$$\bar{F}_{trac,m} = m_v \cdot 0.101 = 1500 \cdot 0.101 = 151.5 \text{ N}$$

The mechanical energy per 100 km that corresponds to 1 N is

$$dm_f = 1 \text{ N} \cdot 10^5 \text{ m} = 10^5 \text{ J} = 10^5/3600 = 27.78 \text{ Wh}.$$

And according to the caption of Figure 2.8, Diesel's LHV is $10^4 W\,h/l$, which gives:

$$1 \text{ N} = 27.78 \text{ Wh/100km} = 2.778 \times 10^{-3} \text{ l/100km}$$

Therefore, for no-recuperation,

$$\overset{*}{V} = \bar{F}_{trac} \times dm_f = 436.6 \times 2.778 \times 10^{-3} = 1.213 \text{ l/100km}$$

b) **Case 2: Perfect recuperation**
Reading constants from **(2.34)**, we get the weights for perfect recuperation over the cycle MVEG-95:

$$\bar{F}_a = \frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot 363 = \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 363 = 152.5 \text{ N}$$
$$\bar{F}_r = m_v \cdot g \cdot c_r \cdot 1 = 1500 \cdot 9.81 \cdot 0.012 \cdot 1 = 176.6 \text{ N}$$

Therefore, for perfect-recuperation,

$$\overset{*}{V} = \bar{F} \times dm_f = 329.1 \times 2.778 \times 10^{-3} = 0.9142 \text{ l/100km}$$

<u>**Discussion**</u>
(i) Evaluation of the differences in mean force

$$\bar{F}_{m,r} = 152.5 + 176.6 - (134.0 + 151.2) = 43.9 \text{ N}$$
$$\bar{F}_{m,b} = 436.6 - 329.1 = 107.5 \text{ N}$$

where $\bar{F}_{m,r}$ denotes the mean force that is used to overcome the driving resistance in the non-traction phases; while $\bar{F}_{m,b}$ denotes the part of mean force that is later dissipated by heat with the brakes.

(ii) Explanatory comments on calculating mean force with/without recuperation

- When no recuperation is done, F_{trac} overcomes aerodynamic drag, rolling friction drag and acceleration requirement when 'trac" mode is on; while during coasting, non of them is of the concern of energy consumption, since the kinetic energy at high vehicle speed apparently cost more energy and cannot be recuperated
- When perfect recuperation is done, F_{trac} overcomes aerodynamic drag, rolling friction drag and acceleration requirement all through the cycle. Especially during coasting, braking or not braking is still of the energy concern since the kinetic energy at high speed can be later recuperated, if the speed profile can be satisfied with the help of "recuperative brake". If it is not the case, mechanical brake has to intervene, so as to dissipate the remaining part of energy and satisfy speed profile.

Problem 2.10

Calculate the data in Fig. 2.9, left and right.

- Solution

a) **Full-sized vehicle** $((A_f \cdot C_d = 0.7\ \mathrm{m}^2, m_v = 1500\ \mathrm{kg}, C_r = 0.012))$
The cycle energy assuming no recuperation is given by (2.31),

$$\begin{aligned}\bar{E} &= [1.9 \times 10^4 \cdot A_f C_d + 8.4 \times 10^2 \cdot m_v\, C_r + 10 \cdot m_v] \cdot x_{tot} \\ &= 0.7 \cdot 1.9 \cdot 10^4 + 1500 \cdot 0.012 \cdot 8.4 \cdot 10^2 + 1500 \cdot 10 \\ &= 43 \cdot 10^3\ {}^{\mathrm{kJ}}\!/_{100\ \mathrm{km}},\end{aligned}$$

thus

$$
\begin{aligned}
S(A_f \cdot c_d) &= \frac{\partial E}{\partial (A_f \cdot C_d)} \cdot \frac{A_f \cdot C_d}{E(A_f \cdot C_d)} \\
&= \frac{1.9 \cdot 10^4 \cdot 0.7}{43 \cdot 10^3} = 0.3046 \\
S(c_r) &= \frac{\partial E}{\partial (C_r)} \cdot \frac{C_r}{E(C_r)} \\
&= \frac{1500 \cdot 8.4 \cdot 10^2 \cdot 0.012}{43 \cdot 10^3} = 0.3463 \\
S(m_v) &= \frac{\partial E}{\partial (m_v)} \cdot \frac{m_v}{E(m_v)} \\
&= \frac{(0.012 \cdot 8.4 \cdot 10^2 + 10) \cdot 1500}{43 \cdot 10^3} = 0.6899
\end{aligned}
$$

b) **Light-weight vehicle** $((A_f \cdot C_d = 0.4 \text{ m}^2, m_v = 750 \text{ kg}, C_r = 0.008))$

$$
\begin{aligned}
\bar{E} &= [1.9 \times 10^4 \cdot A_f C_d + 8.4 \times 10^2 \cdot m_v\, C_r + 10 \cdot m_v] \cdot x_{tot} \\
&= 0.4 \cdot 1.9 \cdot 10^4 + 750 \cdot 0.008 \cdot 8.4 \cdot 10^2 + 750 \cdot 10 \\
&= 20 \cdot 10^3 \ {}^{\text{kJ}}/_{100 \text{ km}}
\end{aligned}
$$

thus

$$
\begin{aligned}
S(A_f \cdot c_d) &= \frac{\partial E}{\partial (A_f \cdot C_d)} \cdot \frac{A_f \cdot C_d}{E(A_f \cdot C_d)} \\
&= \frac{1.9 \cdot 10^4 \cdot 0.4}{20 \cdot 10^3} = 0.3774 \\
S(c_r) &= \frac{\partial E}{\partial (C_r)} \cdot \frac{C_r}{E(C_r)} \\
&= \frac{750 \cdot 8.4 \cdot 10^2 \cdot 0.008}{20 \cdot 10^3} = 0.2502 \\
S(m_v) &= \frac{\partial E}{\partial (m_v)} \cdot \frac{m_v}{E(m_v)} \\
&= \frac{(0.008 \cdot 8.4 \cdot 10^2 + 10) \cdot 750}{20 \cdot 10^3} = 0.6226
\end{aligned}
$$

Discussion

From the result we can conclude, with an advanced vehicle concept (usually light-weight and smaller rolling friction):

- relative dominance of the vehicle mass on the energy consumption is unchanged around a level of 60% to 70%, which makes kinetic energy recuperation an interesting choice.
- Relative influence of the rolling fricition becomes less than that of $A_f \cdot C_d$ (coefficient of aerodynamic force.)

Problem 2.11

Calculate which constant vehicle speed on a flat road is responsible for the same energy demand at the wheels along a MVEG–95 cycle, in the case of no recuperation and of perfect recuperation, respectively. Assume the lightweight vehicle data of Fig. 2.8, right side of Figure: $\{A_f \cdot c_d, m_v, c_r\} = \{0.4\ \mathrm{m}^2, 750\ \mathrm{kg}, 0.008\}$.

- Solution

a) **No recuperation**
The cycle energy is calculated in Problem 2.8 and it is $\bar{E} = 20.14 \cdot 10^3$ kJ/100 km. The mean traction force is

$$\bar{F} = \frac{\bar{E}}{100\ \mathrm{km}} = 201.4\ \mathrm{N}$$

To have the same $\bar{F}$, find v such that

$$\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 + m_v \cdot g \cdot c_r = \bar{F}$$

$$v = \sqrt{2 \cdot \frac{\bar{F}_{trac} - m_v \cdot g \cdot c_r}{\rho_a \cdot A_f \cdot c_d}}$$

$$v = \sqrt{2 \cdot \frac{201.4 - 750 \cdot 9.81 \cdot 0.008}{1.2 \cdot 0.4}} = 24.35\ \mathrm{m/s} = 87.67\ \mathrm{km/h}$$

b) **Perfect recuperation**
Here, $\bar{E} = 14.68 \cdot 10^3$ kJ/100 km, thus

$$\bar{F} = 146.8\ \mathrm{N}$$
$$v = 19.5\ \mathrm{m/s} = 70\ \mathrm{km/h}$$

To have the same $\bar{F}$, find v such that

$$\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 + m_v \cdot g \cdot c_r = \bar{F}$$

$$v = \sqrt{2 \cdot \frac{\bar{F}_{trac} - m_v \cdot g \cdot c_r}{\rho_a \cdot A_f \cdot c_d}}$$

$$v = \sqrt{2 \cdot \frac{146.8 - 750 \cdot 9.81 \cdot 0.008}{1.2 \cdot 0.4}} = 19.14\ \mathrm{m/s} = 68.9\ \mathrm{km/h}$$

Discussion

- Note that the mechanical mean force is not accounted for, because the problem assumes instantaneous energy consumption during flat road constant speed cruising.
- Note that for artificial driving cycles like MVEG-95, the equivalent speed can also be calculated by equating mean force representation with **(2.31)** or **(2.35)**, respectively.

Problem 2.12

Calculate the maximum mass allowed for a recuperation system with $\eta_{rec} =$ 40%. Use the vehicle parameters of Fig. 2.12: $\{A_f \cdot c_d, m_v, c_r\} = \{0.7\ \text{m}^2, 1500\ \text{kg}, 0.012\}$.

- Solution

a) Deriving the energy demand with a real recuperation device
Using the equations **(2.31),(2.35),(2.39),(2.40)**:

$$
\begin{aligned}
E_{diss} + E_{circ} &= [1.9 \times 10^4 \cdot A_f C_d + 8.4 \times 10^2 \cdot m_v\, C_r + 10 \cdot m_v] \\
E_{diss} &= [2.2 \times 10^4 \cdot A_f C_d + 9.8 \times 10^2 \cdot m_v\, C_r] \\
E(\eta_{rec}, m_{rec}) &= \bar{E}_{diss}(m_{rec}) + (1 - \eta_{rec})\, E_{circ}(m_{rec}) \\
\bar{E}_{circ} &= \bar{E} - \bar{E}_{diss}
\end{aligned}
$$

we get:

$$
\begin{aligned}
&\bar{E}(\eta_{rec}, m_{rec}) \\
&= [22000 - 3000(1 - \eta_{rec})]\, A_f \cdot C_d + [980 - 140(1 - \eta_{rec})]\, C_r\, (m_v + m_{rec} + 10\,(1 - \eta_{rec}) \cdot (m_v + m_{re} \\
&= (22000 - 3000) \times 0.6 \times 0.7 + [980 - 140 \times 0.6] \times 0.012\,(1500 + m_{rec}) + 10 \times 0.6\,(1500 + m_{rec})
\end{aligned}
$$

c) Equate the energy demand and get the maximum mass of recuperation device
When we have the maximum weight of recuperation device, the mean energy demand must equal $\bar{E}$, which has been calculated in Problem 2.8 as $43.4 \cdot 10^3$ kJ/100 km. Thus

$$
\begin{aligned}
\tilde{m}_v &= \frac{(43.4 - 14.1) \cdot 10^3}{16.75} = 1750\ \text{kg} \\
&14140 + 16.752 \times (1500 + m_{rec}) = 43.66 \times 10^3 \\
m_{rec} &= 262.2\ \text{kg}.
\end{aligned}
$$

The maximum weight is the one that leads to an energy demand equal to the energy demand without recuperation. Thus, from (2.41)

$$
\begin{aligned}
\bar{E}(\eta_{rec}, m_{rec}) &= 0.7 \cdot (2.2 \cdot 10^4 - 0.6 \cdot 3 \cdot 10^3) + \\
&\qquad + 0.012 \cdot \tilde{m}_v \cdot (9.8 \cdot 10^2 - 0.6 \cdot 1.4 \cdot 10^2) + \\
&\qquad + 0.6 \cdot 10 \cdot \tilde{m}_v = \\
&= 14.1 \cdot 10^3 + (10.75 + 6) \cdot \tilde{m}_v\ (\text{kJ/100 km})
\end{aligned}
$$

where $\tilde{m}_v = m_v + m_{rec}$.

IC-Engine-Based Propulsion Systems

Gear-box Models

Problem 3.1

Improve (3.9) in order to take into account the engine inertia and the transmission efficiency.

- Solution

  Assume the largest gear ratio (the smallest gear in a manual gear box) is often chosen according to the towing requirement at constant vehicle speed v_w.

a) Calculate moment of inertia
   Including the engine inertia

$$m_{v,eq} = m_v + m_{e,eq}$$
$$m_{e,eq} = \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2$$

b) List the power balance
   Equating the traction power with the towing requirement, we have:

$$P_{trac} = \left[m_v + \Theta_e \frac{\gamma_1^2}{r_w^2}\right] g \sin(\alpha_{max}) \cdot v_w = T_{e,max}(\frac{v_w}{r_w} \cdot \gamma_1) \cdot \frac{v_w}{r_w} \cdot \gamma_1 \cdot \eta_t$$

   Thus, after rewriting the equation, we have:

$$\left(\frac{\gamma_1}{r_w}\right)^2 - \frac{T_e \cdot \eta_t}{\Theta_e \cdot a} \cdot \frac{\gamma_1}{r_w} + \frac{m_v}{\Theta_e} = 0,$$

   where the towing acceleration $a = g \sin(\alpha_{max})$.
   This is a quadratic equation in γ. γ_1 can be evaluated by imposing that $a = a_{max}$. With $\Theta_e = 0$ and $\eta_t = 1$, obtain back equation (3.9).
   Discussion
   Since the maximum engine torque depends on the vehicle speed and first gear's ratio, iterations may be necessary to come up with T_e and γ_1 values that both fit.

Problem 3.2

Dimension the first gear of an ICE-based powertrain not based on a given gradability as in (3.9) but in order to obtain a given acceleration at vehicle take-off. Do the calculations according to Problem 3.1, using the following data: $m_v = 1100\,\text{kg}$, payload $m_p = 100\,\text{kg}$, equivalent mass of the wheels $m_{r,w} = 1/30$ of m_v, $c_r = 0.009$, transmission efficiency $\eta_t = 0.9$, $T_e = 142\,\text{Nm}$ at $\omega_{takeoff}$, desired acceleration $a = 4\,\text{m/s}^2$, engine inertia $\Theta_e = 0.128\,\text{kgm}^2$, $r_w = 30\,\text{cm}$. Compare with the approximate solution of (3.9).

- Solution

Theoretical grounds
From the result of Problem 3.1, γ_1 can be dimensioned through the towing requirement at take-off speed.

$$P_{trac} = \left[m_v + \Theta_e \frac{\gamma_1^2}{r_w^2}\right] a_{max} \cdot v_w = T_{e,max}(\frac{v_w}{r_w} \cdot \gamma_1) \cdot \frac{v_w}{r_w} \cdot \gamma_1 \cdot \eta_t$$

$$\left(\frac{\gamma_1}{r_w}\right)^2 - \frac{T_e \cdot \eta_t}{\Theta_e \cdot a_{max}} \cdot \frac{\gamma_1}{r_w} + \frac{m_v}{\Theta_e} = 0 \qquad (*)$$

a) Solving Equation (**??**) with the flat-road desired acceleration,

$$m_{v,eq} = m_v + m_{wheel} + m_{payload} = 1100\,(1 + 1/30) + 100$$

$$\left(\frac{\gamma_1}{r_w}\right)^2 - \frac{142.0 \times 0.9}{0.128 \times 4} \cdot \frac{\gamma_1}{r_w} + \frac{1100\,(1 + 1/30) + 100}{0.128} = 0$$

Let $x = \frac{\gamma_1}{r_w}$, we have

$$x^2 - 249.61\,x + 9661 = 0$$

Solving the quadratic equation, we have:

$$\left.\frac{\gamma_1}{r_w}\right|_{a_{max}} = \frac{249.61 \pm 153.82}{2} = \begin{cases} 201.7 \\ 47.89 \end{cases}$$

We take the smaller root as the solution $\gamma_1 = 47.89$, for the sake of fuel economy, gearbox sizing and drivability.

b) Solving Equation (**??**) assuming a maximum gradability of 27.64^o at take-off speed,

$$m_{v,eq} = m_v + m_{wheel} + m_{payload} = 1100\,(1 + 1/30) + 100$$

$$\left(\frac{\gamma_1}{r_w}\right)^2 - \frac{142.0 \times 0.9}{0.128 \times 9.81 \times \sin(27.64^o)} \cdot \frac{\gamma_1}{r_w} + \frac{1100\,(1 + 1/30) + 100}{0.128} = 0$$

Let $x = \frac{\gamma_1}{r_w}$, we have

$$x^2 - 224.65\,x + 9661 = 0$$

Solving the quadratic equation, we have:

$$\left.\frac{\gamma_1}{r_w}\right|_{a_{max}} = \frac{224.65 \pm 108.74}{2} = \begin{cases} 166.7 \\ 57.96 \end{cases}$$

We take the smaller root as the solution $\gamma_1 = 57.96$, for the sake of fuel economy, gearbox sizing and drivability.

c) Calculate the approximate solutions with Equation (3.9)
Accroding to (3.9), we have:

$$\gamma_1 = \frac{m_v\, r_w \cdot a_{max}}{T_{e,max}(\omega_e)},$$

where a_{max} denotes the desired acceleration or gradability $g \cdot \sin(\alpha_{max})$. Therefore, the approximated solutions are:

$$\gamma_{1,accr} = \frac{[1100\,(1 + 1/30) + 100] \times 0.30 \times 4}{142.0} = 34.83,$$

for desired acceleration of 4 m/s^2

$$\gamma_{1,grad} = \frac{[1100\,(1 + 1/30) + 100] \times 0.30 \times 9.81 \times \sin(27.64^o)}{142.0} = 38.71,$$

for desired gradability of 27.64^o

d) Comparison of Problem 3.2 with approximated equation (3.9)
The results of the four cases are listed in the table: From the results, we

Table 10.2. Comparison of first gear with different ways of calculation.

Results	$\frac{\gamma_1}{r_w}$	$\gamma_1\|_{r_w=0.3}$m
Exact with a_{flat}	47.89	14.367
Approx with a_{flat}	34.83	10.449
Exact with α_{grad}	57.96	17.388
Approx with α_{grad}	38.71	11.613

could conclude that

- Including more types of losses (e.g. rotational moment of inertia/ transmission efficiency) would increase the gear ratio γ_1.
- Increasing the requirement of gradability or acceleration capability would decrease the 1^{st} order term of the quadratic equation (**??**), which indirectly increases the gear ratio γ_1.

Discussion

The multiple roots of quadratic equation adds to complexity of the first gear dimensionization. However, usually only the smaller positive root makes sense for the following reasons:

(i) Since the resistence curve of the vehicle correlates the reciprocal of gear ratio, a too large gear ratio may lead to a resistence curve lying in an area of low fuel efficiency, as is shown in Figure **??**.

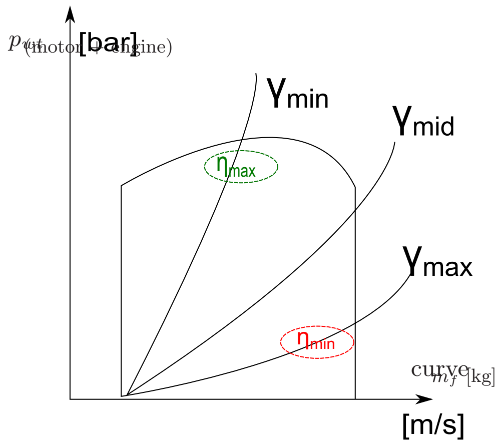

Fig. 10.8. Resistence curves with different gear ratios.

(ii) The 5^{th} out of 6-speed gearbox is usually defined to locate a resistence curve in the best fuel economy area. If too large gear ratio is chosen, the resistance curves of gear #1 and gear #5 could be too far away, which will cause bad sizing and a huge gearbox in the end.

(iii) Since the range of engine speed is set to be around 1000 rpm to 6000 rpm for gasoline engine (for diesel engine even narrower), too large γ_1 means a small range of speed increase in the first gear. Thus the increasing requirement of gear shifts will be detrimental to drivability. For example, for a 30 cm wheel with $\gamma_1 = 201.7$, $1000 - 6000$ rpm range means a speed increase from 1.80 km/h to 11.21 km/h, which is far from acceptable drivability.

Problem 3.3

Dimension the fifth gear in a six-gear transmission for maximum power using (3.10) for a vehicle with the following characteristics: curb $m_v = 1100\,\text{kg}$, performance mass $= 100\,\text{kg}$, $c_r = 0.009$, $A_f \cdot c_d = 0.65\,\text{m}^2$, $r_w = 30\,\text{cm}$, transmission efficiency $= 0.9$, $P_{e,max} = 80.4\,\text{kW}$ at 6032 rpm.

- Solution

  Assume the maximum speed v_{max} is achieved at gear #5, and with the maximum engine power at its corresponding engine speed.

a) Maximum traction power at wheel
Having the engine power and transmission efficiency, we have:

$$P_{trac,max} = P_{e,max} \cdot \eta_t = 72.36 \text{ kW}.$$

b) Solving equation (3.10), and find maximum speed:

$$\begin{aligned} P_{trac,max} &= F_{max} \cdot v_{max} \\ &= v_{max} \cdot \left[m_v \, g \, C_r \, v_{max} + \frac{1}{2} \rho_a \, A_f C_d \, v_{max}^2 \right] \Big|_{gear=5} . \end{aligned}$$

By solving the equation

$$-72.36 \times 10^3 + 105.95 \, v_{max} + 0.39 \, v_{max^3} = 0,$$

we have the only solution of real number

$$v_{max}|_{gear=5} = 55.45 \text{ m/s} = 199.6 \text{ km/h}$$

c) Calculate the gear ratio
With the maximum speed achieved with maximum engine power,

$$\begin{aligned} \frac{\gamma_5}{r_w} &= \frac{\pi}{30} \times \frac{\omega_e|_{P_{max}}}{v_{max}} \\ &= \frac{\pi}{30} \cdot \frac{6032}{55.45} = 11.392 \\ \gamma_5 &= 3.418. \end{aligned}$$

where a common wheel radius of 30 cm is used for calculation.

Problem 3.4

Consider again the system of Problems 3.2–3.3. Calculate the vehicle speed values, v_j at which the engine is at its maximum speed, for all the gears j.

- Solution

Assume only the 2^{nd} to 4^{th} gear can be chosen according to a fixed law, while the 1^{st},5^{th} and 6^{th} are chosen with towing requirement, maximum power and fuel efficiency, respectively.

a) Using geometric law

(i) Calculate the common ratio

$$\begin{aligned} \frac{1}{\kappa} &= R^{1/4} = [\frac{47.9}{11.4}]^{\frac{1}{4}} = 1.432, \\ \frac{\gamma_1}{\gamma_2} &= \frac{\gamma_2}{\gamma_3} = \frac{\gamma_3}{\gamma_4} = \frac{\gamma_4}{\gamma_5} = \frac{1}{\kappa} = 1.432, \end{aligned}$$

(ii) Find corresponding gear ratios

$$\begin{aligned}
\frac{\gamma_2}{r_w} &= \frac{\gamma_1}{r_w} \cdot \kappa = \frac{47.89}{1.432} = 33.443, \\
\frac{\gamma_3}{r_w} &= \frac{\gamma_2}{r_w} \cdot \kappa = \frac{47.89}{1.432^2} = 23.354, \\
\frac{\gamma_4}{r_w} &= \frac{\gamma_3}{r_w} \cdot \kappa = \frac{47.89}{1.432^3} = 16.309, \\
\frac{\gamma_5}{r_w} &= 11.392 \text{ (fixed and confirmed).}
\end{aligned}$$

b) Using arithmetic law

(i) Calculate the common difference

$$\begin{aligned}
R &= \frac{1}{k} \cdot \left(\frac{r_w}{\gamma_i} - \frac{r_w}{\gamma_{i-k}} \right) \\
&= \left(\frac{1}{11.392} - \frac{1}{47.89} \right) \cdot \frac{1}{4} = 0.0167.
\end{aligned}$$

(ii) Find corresponding gear ratios

$$\begin{aligned}
\frac{\gamma_2}{r_w} &= \frac{1}{\frac{r_w}{\gamma_1} + R} = \frac{1}{\frac{1}{47.89} + 0.0167} = 26.609, \\
\frac{\gamma_3}{r_w} &= \frac{1}{\frac{r_w}{\gamma_2} + R} = \frac{1}{\frac{1}{26.609} + 0.0167} = 18.422, \\
\frac{\gamma_4}{r_w} &= \frac{1}{\frac{r_w}{\gamma_3} + R} = \frac{1}{\frac{1}{18.422} + 0.0167} = 14.088, \\
\frac{\gamma_5}{r_w} &= 11.392 \text{ (fixed and confirmed).}
\end{aligned}$$

c) Validate the shifting speeds
The shifting speeds denotes the vehicle speeds at the maximum engine speed $\omega_{e,gs} = 6032$ rpm.

$$v_k = \frac{\pi \cdot \omega_{e,gs}}{30 \cdot \frac{\gamma_k}{r_w}},$$

thus the values in the following table.

Problem 3.5

Calculate the approximate efficiency of a clutch during a vehicle takeoff maneuver.

Table 10.3. the maximum speed in each gear #.

	Geometric	Arithmetic
v_1	13.19 m/s = 47.48 km/h	
v_2	18.89 m/s= 68.00 km/h	23.74 m/s= 85.46 km/h
v_3	27.05 m/s= 97.37 km/h	34.29 m/s=123.4 km/h
v_4	38.73 m/s=139.4 km/h	44.84 m/s=161.3 km/h
v_5	55.45 m/s = 199.6 km/h	

- Solution

  Assume
  - In a first approximation, the transmitted torque could be considered as equal to the engine torque while the clutch is slipping.
  - The engine speed is approximately constant at the launch value ω_e.
  - The final condition of the launching maneuver is $\omega_{gb}(t_f) = \omega_e$.

a) Derive first approximation of launching dynamics
The speed downstream of the clutch is given by the differential equation

$$\Theta_{gb} \cdot \frac{d\omega_{gb}}{dt} = T_e - T_{loss},$$

with $\Theta_{gb} = \Theta_v/\gamma^2$, which can be approximated by neglecting the losses. Thus

$$\omega_{gb}(t) = \frac{T_e}{\Theta_{gb}} \cdot t.$$

b) Solve the dynamics subject to final state constraint $\omega_{gb}(t_f) = \omega_e$.
Given that the launch maneuver ends when $\omega_{gb}(t_f) = \omega_e$. The launch time is therefore

$$t_f = \omega_e \cdot \frac{\Theta_{gb}}{T_e}.$$

The energy provided by the engine during this time is

$$E_e = \int_0^{t_f} T_e \cdot \omega_e dt = T_e \cdot \omega_e \cdot t_f = \Theta_{gb} \cdot \omega_e^2.$$

The energy transferred to the vehicle is the kinetic energy at the end of launch, which is

$$\begin{aligned} E_v &= \frac{1}{2} \cdot \Theta_v \cdot \omega_{w,0}^2, \\ &= \frac{1}{2} \cdot \Theta_{gb} \cdot \omega_e^2. \end{aligned}$$

since $\omega_{w,0} = \omega_e/\gamma$ and $\Theta_{gb} = \Theta_v/\gamma^2$. Therefore, the efficiency is $E_v/E_e = 0.5$.

Discussion

The corresponding synchronization speed is

$$v = \frac{\omega_e}{\frac{\gamma}{r_w}}.$$

Thus the energy lost is $E_e - E_v = \frac{1}{2} \cdot \Theta_{gb} \cdot \omega_e^2$ which coincides with (3.16),

$$E_c = \frac{1}{2} \cdot \Theta_v \cdot \omega_{w,0}^2.$$

Fuel Consumption of IC Engine Powertrains

Problem 3.6

Find the CO_2 emission factor (g/km) as a function of the fuel consumption rate (l/100 km) for gasoline and diesel fuels. Use these average fuels (gasoline, diesel) data: density $\rho = \{0.745, 0.832\}$ kg/l, carbon dioxide to fuel mass fraction $m = \{3.17, 3.16\}$.

- Solution

Assume

Consider the stoichiometric fuel burning reaction, with a fuel of the molar composition CH_a:

$$\mathrm{CH_a} + \left(1 + \frac{a}{4}\right) \mathrm{O_2} \rightleftharpoons \mathrm{CO_2} + \left(\frac{a}{2}\right) \mathrm{H_2O}$$

a) Find the factor a from CO_2 to mass fraction
The mol CO_2/mol emission factor is 1. The kg CO_2/kg emission factor is thus

$$m = \frac{M_{CO_2}}{M_{fuel}} = \frac{12 + 2 \cdot 16}{12 + a} = \frac{44}{12 + a}$$

Table 10.4. Factor a of Diesel and Gasoline.

Variables	Gasoline	Diesel
m	3.17	3.16
a	1.880	1.924

b) Mass of CO_2 per litre of fuel
The kg/l factor is $m \cdot \rho$, where ρ (kg/l) is the fuel density.
where the fuel consumption of some engine is $\dot{m}_f = C$ l/100km,

Table 10.5. CO_2 emission factor of Diesel and Gasoline.

Variables	Gasoline	Diesel
$m \cdot \rho$ [gCO_2/l]	2.362	2.629
$CO_2 factor$ [gCO_2/km]	23.62$\cdot$ C	26.29$\cdot$ C

Problem 3.7

Calculate the fuel consumption and the CO_2 emission rate for the MVEG–95 cycle for a vehicle having the following characteristics: $m_v = 1100\,\text{kg}$, payload $= 100\,\text{kg}$, $c_d \cdot A_f = 0.7$, $c_r = 0.013$, $e_{gb} = 0.98$, $P_{0,gb} = 3\%$, $P_{aux} = 250\,\text{W}$, $v_{launch} = 3\,\text{m/s}$, $e = 0.4$, $P_{e,0} = 1.26\,\text{kW}$, $P_{e,max} = 66\,\text{kW}$, diesel fuel ($H_f = 43.1\,\text{MJ/kg}$, $\rho_f = 832\,\text{g/l}$), idle consumption $\overset{*}{V}_{f,idle}= 150\,\text{g/h}$. The declared CO_2 emission rate for this car is 99 g/km.

- Solution

  Assume the parameters of the driving cycle MVEG-95:

  - Launch event happens every 105 s.
  - Idling time is around 300 s.
  - The track length of the cycle is 11.4 km.
  - The fraction of time during traction mode is $trac = 0.6$.

a) Calculate mean force:
Assuming no recuperation, negligible engine inertia, the mean force of cycle MVEG-95 can be calculated as follows:

$$\begin{aligned}\bar{F} &= F_{trac,r} + F_{trac,a} + F_{trac,m} \\ &= \frac{1}{x_{tot}}\left\{\sum_{trac} m_v\, g\, C_r\, \bar{v}_i \cdot h + \sum_{trac} \frac{1}{2}\rho_a A_f C_d\, \bar{v}_i^{\,3} \cdot h + \sum_{trac} m_v \bar{a}_i \cdot \bar{v}_i \cdot h\right\} \\ &= 1200 \times 9.81 \times 0.013 \times 0.856 + \frac{1}{2} \times 1.2 \times 0.7 \times 319 + 1200 \times 0.101 \\ &= 386.2\ \text{N} = 38.62 \times 10^3\ \text{kJ/100km}\end{aligned}$$

b) Traction power considering different loads
Just like the calculation from **(3.23)** to **(3.32)**, we further assume
- launch event happens every 105 s, this part of energy is accounted for with average power.
- during the calculation of engine load, only traction mode is considered, and thus the power need to be compensated with a coefficienct of $\frac{1}{trac}$

First, we calculate the equivalent engine power considering the traction mode and transmission loss:

$$
\begin{aligned}
\overline{P_{trac,@wheel}} &= \frac{\overline{F}\cdot\overline{v}}{trac} \\
&= \frac{386.2\times 9.5}{0.6} = 6.114\text{kW} \\
\overline{P_{trac,@gb}} &= \frac{\overline{P_{trac,@wheel}} + P_{0,gb}}{\eta_{gb}} \\
&= \frac{(1+3\%)\times(6.114\times 10^3)}{0.98} = 6.426 \text{ kW}
\end{aligned}
$$

Then, we list the main component of engine power:

Traction power

$$\overline{P_{trac,@eng}} = 6.426 \text{ kW}$$

Launch power

$$\overline{P_{launch,@eng}} = \frac{\frac{1}{2}\,\Theta_v\,\omega_{w,0}^2}{t_{launch}} = 51.43 \text{ W}$$

Auxiliary power

$$\overline{P_{aux,@eng}} = 250 \text{ W}$$

So the total engine power is evaluated as:

$$
\begin{aligned}
\overline{P_{e,@eng}} &= \overline{P_{trac,@eng}} + \overline{P_{launch,@eng}} + \overline{P_{aux,@eng}} \\
&= 6.426 + 0.05143 + 0.250 \\
&= 6.727 \text{ kW}
\end{aligned}
$$

c) Calculate average fuel power
Using Willan's approach and compensating back the non-traction time intervals, we have:

$$
\begin{aligned}
\eta_e &= \frac{e\cdot\bar{P}_e}{\bar{P}_e + P_{0,e}} \\
&= \frac{0.4\times 6.727}{6.727+1.26} \\
&= 33.69\%. \\
\overline{P_{f,trac}} &= trac\cdot\frac{\bar{P}_e}{\eta_e} \\
&= 11.98\text{kW}.
\end{aligned}
$$

d) Calculate the fuel consumption in category
The fuel consumption from traction force is:

$$\overset{*}{V}_f = \frac{\overline{P_{f,trac}}}{H_l\cdot\rho_f} = \frac{11.98\times 10^3 \text{ J/s}}{43.1\times 10^6 \text{ J/kg}\times 0.832 \text{ kg/l}} = 3.34\times 10^{-4} \text{ l/s}.$$

In order to switch the unit to the fuel consumption per unit distance, instead of per second, the average cycle speed is used for the approximation:

$$\bar{v} = 9.5 \text{ m/s} = 9.5 \times 10^{-5}[100\text{km/s}].$$

Therefore, the traction fuel consumption with the unit l/100km given an approximated average speed is:

$$\begin{aligned}\overset{*}{V}_f \ \text{l/100km} &= \frac{\overset{*}{V}_f \ \text{l/s}}{v} \\ &= \frac{3.34 \times 10^{-4}}{9.5 \times 10^{-5}} \\ &= 3.517 \text{ l/100km}.\end{aligned}$$

Additionally, during the standstill time intervals, the idling fuel consumption should also be calculated from the given idle consumption:

$$\begin{aligned}\overset{*}{V}_{f,idle} \ \text{l/100km} &= \frac{\overset{*}{V}_{f,idle} \ \text{g/s}}{\rho \ \text{g/l}} \cdot \frac{t_{idle}}{L_{track} \ [100\text{km}]} \\ &= \frac{\frac{150}{3600}}{832} \cdot \frac{300}{0.114} \\ &= 0.1318 \text{ l/100km}.\end{aligned}$$

Consequently, the total fuel consumption of running MVEG-95 is $\overset{*}{V}_{\Sigma} = 3.517 + 0.1318 = 3.649$ l/100km.

e) Switch to the CO_2 emission
Using diesel's emission factor calculated from Problem 3.6,

$$m_{CO_2} = 26.29 \cdot C \text{ gCO}_2\text{/km},$$

where C denotes the fuel consumption in 100 km.

$$m_{CO_2} = 26.29 \times 3.649 = 95.93 \text{ gCO}_2\text{/km}.$$

f) Comparison with the declared value 99 gCO_2/km
The additional CO_2 not taken into account in this method amounts to 3.07 gCO_2/km(*env*.3.1%). The probable reason is the way of averaging velocity profile in traction fuel consumption calculation and approximating idling time in the idle consumption.
Discussion
Take care of the simplified ways of fuel consumption calculation:
- As for the idling consumption, the total time interval of idling and the length of track are used for the approximation.
- As for the launch event compensation, the frequency of launch event is used to calculate the average power, and therefore the equivalent fuel consumption.

- As for the traction fuel consumption, the average cycle cpeed and the length of track are used to switch the unit from fuel power to distance-specifc fuel consumption.

Problem 3.8

Evaluate in a first approximation the contribution of stop-and-start, regenerative braking ($m_{rec} = 20\% \cdot m_v$, $\eta_{rec} = 0.5$) and optimization of power flows in reducing the fuel consumption when the system of Problem 3.7 is hybridized. Assume an 80% charge-discharge efficiency for the reversible storage system.

- Solution

From the result of Problem 3.7,

$$\overset{*}{V}_{idle} \Big/ \overset{*}{V}_{tot} = 4\%$$

Now consider regenerative braking. Redo calculations of Problem 3.7 with $m_v = 1100 \cdot 1.2 = 1320$ kg.

$$\begin{aligned}
\bar{F}_{trac,r} &= 1320 \cdot 9.81 \cdot 0.013 \cdot 0.856 = 144 \text{ N} \\
\bar{F}_{trac,a} &= \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 319 = 134 \text{ N} \\
\bar{F}_{trac,m} &= 1420 \cdot 0.101 = 142 \text{ N} \\
\bar{F}_{trac} &= 144 + 134 + 142 = 420 \text{ N}
\end{aligned}$$

For ideal regenerative braking,

$$\begin{aligned}
\bar{F}_{trac,r} &= 1320 \cdot 9.81 \cdot 0.013 \cdot 1 = 168 \text{ N} \\
\bar{F}_{trac,a} &= \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 363 = 152 \text{ N} \\
\bar{F}_{trac,m} &= 0 \text{ N} \\
\bar{F}_{trac} &= 168 + 152 = 320 \text{ N}
\end{aligned}$$

For $\eta_{rec} = 0.5$, the traction force is

$$\bar{F}_{trac} = 320 + (1 - 0.5) \cdot (420 - 320) = 370 \text{ N}$$

Thus the potential gain due to regenerative braking seems rather limited,

$$\begin{aligned}
\bar{P}_{trac} &= \frac{370 \cdot 9.5}{0.6} = 5.8 \text{ kW} \\
\bar{P}_1 &= \frac{5.8 \cdot 10^3 \cdot 1.03}{0.98} = 6.1 \text{ kW} \\
\bar{P}_{start} &= \frac{1}{2} \cdot \frac{1200 \cdot 3^2}{105} = 51 \text{ kW} \\
\bar{P}_e &= 6.1 \cdot 10^3 + 0.25 + 0.05 = 6.4 \text{ kW} \\
\eta_e &= \frac{e \cdot \bar{P}_e}{\bar{P}_e + P_{0,e}} = \frac{0.4 \cdot 6.4}{6.4 + 1.26} = 0.33 \\
\bar{P}_f &= \frac{0.6 \cdot \bar{P}_e}{\eta_e} = 11.6 \text{ kW} \\
\overset{*}{V}_{trac} &= \frac{11.6 \cdot 10^3}{43.1 \cdot 10^6 \cdot 0.832} = 3.2 \cdot 10^{-4} \text{ l/s} = \\
&= \frac{3.2 \cdot 10^{-4}}{9.5} \cdot 10^5 \text{ l/100 km} = 3.4 \text{ l/100 km}
\end{aligned}$$

In summary we have 3.6 l/100 km w.r.t. 3.4 l/100 km. The benefit due to regenerative braking is equal to the benefit due to idle consumption suppression and they amount to 3%. To evaluate the potential benefit of engine operating point shifting, assume that the engine could be able to work always at its maximum efficiency point, thus at $P_{e,max} = 66$ kW. The efficiency is $\eta_e = 0.4 \cdot 66/(66 + 1.26) = 0.39$. Moreover, during a time t_1, the engine delivers its surplus power to the battery, to be reused later. An energy balance across the traction phase yields $\bar{P}_e \cdot 0.6 = P_{e,max} \cdot t_1 + (P_{e,max} - \bar{P}_e) \cdot t_1 \cdot \eta_{acc}$, where η_{acc} is the efficiency of the accumulation system (to be charged and then discharged). Assuming $\eta_{acc} = 0.8$, from the latter one calculates

$$t_1 = \frac{6.4 \cdot 0.6}{66 + (66 - 6.4) \cdot 0.8} = 0.034,$$

thus

$$\begin{aligned}
\bar{P}_f &= \frac{0.034 \cdot 66 \cdot 10^3}{0.39} = 5.7 \text{ kW} \\
\overset{*}{V}_{trac} &= \frac{5.7 \cdot 10^3}{43.1 \cdot 1^6 \cdot 0.832} = 1.6 \cdot 10{-}4 \text{ l/s} = \\
&= \frac{1.6 \cdot 10^{-4}}{9.5} \cdot 10^5 \text{ l/100 km} = 1.7 \text{ l/100 km}
\end{aligned}$$

Ideal gain due to optimization of power flows = (3.4 − 1.7)/3.6 = 47%.

Electric and Hybrid-Electric Propulsion Systems

Electric Propulsion Systems

Problem 4.1

Design an electric powertrain for a small city car having the following characteristics: curb mass = 840 kg, payload = 2 · 75 kg, tires: 155/65/14T, $c_d \cdot A_f = 1.85\,\text{m}^2$, rolling resistance coefficient = 0.009, to meet the following performance criteria: (i) max speed = 65 km/h, (ii) max grade = 16%, (iii) 100 km range. Assume perfect recuperation, overall efficiency of 0.6, and 85% SoC window. Choose motor size in a class with a maximum speed of 6000 rpm and the number of battery modules having a capacity of 1.2 kWh each.

- Solution

  For $v_{max} = 65$ km/h = 18.1 m/s, the required power is

$$P_{max} = m_v \cdot g \cdot c_r \cdot v_{max} + 0.5 \cdot 1.2 \cdot c_d \cdot A_f \cdot v_{max}^3 = 8 \text{ kW}.$$

The max speed of the motor is $\omega_{m,max} = v_{max} \cdot \gamma / r_w$ where γ is the reduction ratio and r_w is the wheel radius. The wheel radius is obtained from the tire specifications (see Problem 2.1) as

$$\frac{14''}{2} + 0.65 \cdot 0.155 \text{ m} = \frac{14 \cdot 0.0254}{2} + 0.65 \cdot 0.155 = 0.28 \text{ m}.$$

If one fixes $\omega_{m,max} = 6000$ rpm = 628 rad/s, then $\gamma = 628/18.05 \cdot 0.28 = 9.7$.

The max torque is

$$T_{m,max} = r_w/\gamma \cdot m_v \cdot g \cdot \sin(\alpha) = 0.28/9.7 \cdot (840 + 150) \cdot 9.8 \cdot 0.16 = 45 \text{ Nm}.$$

Thus the base speed is $P_{m,max}/T_{m,max} = 8000/45 = 178$ rad/s = 1700 rpm. The base to max speed ratio is 1:3.5, which is a reasonable design choice.

Assuming an efficiency $\eta = 0.6$ and perfect recuperation, the mean traction force for an ECE drive cycle is (see (2.34))

$$\bar{F} = m_v \cdot g \cdot c_r + \frac{1}{2} \cdot 1.2 \cdot A_f \cdot c_d \cdot 100 + 840 \cdot 0.14 = 303 \text{ N}.$$

Thus the energy required is $303/0.6 \cdot 100 \cdot 10^3 = 50.5$ MJ = 14 kWh. Add an unused 15% range and obtain 16.1 kWh. Using 6V/200 Ah (1.2 kWh) modules, 14 modules would be needed for a stored energy of 16.8 kWh.

Problem 4.2

Find an equation for the AER D_{ev} of a full electric vehicle as a function of its vehicle parameters, battery capacity and powertrain efficiency. Then evaluate the D_{ev} for a bus with the following characteristics: $\eta_{rec} = 100\%$, $\eta_{sys} = 0.45$ (including unused SoC), $c_r = 0.006$, $A_f \cdot c_d = 6.8 \cdot 0.62$, $Q_{bat} = 89\,\text{Ah}$, $U_{bat} = 600\,\text{V}$, $m_v = 14.6\,\text{t}$, without payload and with a load of 60 passengers, respectively. Assume a MVEG-type speed profile.

- Solution

Equation (2.30) is used for the energy at the wheels. To have energy demand in Wh/km, divide the outcome of (2.30) by the factor $100 \cdot 3.6$. Now, if the battery capacity is expressed in Ah,

$$D_{ev} = \frac{Q_{bat} \cdot U_{bat}}{\frac{E_{rec,MVEG-95}}{100 \cdot 3.6 \cdot \eta_{sys}}}.$$

For the numerical case without payload,

$$\bar{E} = 6.8 \cdot 0.62 \cdot 2.2 \cdot 10^4 + 14620 \cdot 0.006 \cdot 9.81 \cdot 100 = 1.8 \cdot 10^5 \text{ kJ/100 km} = \\ = 500 \text{ Wh/km},$$

$$D_{ev} = \frac{89 \cdot 600}{\frac{500}{0.45}} = 48 \text{ km}.$$

For a payload of $60 \cdot 75$ kg = 4500 kg,

$$\bar{E} = 6.8 \cdot 0.62 \cdot 2.2 \cdot 10^4 + 19120 \cdot 0.006 \cdot 9.81 \cdot 100 = 2.05 \cdot 10^5 \text{ kJ/ km} = \\ = 570 \text{ Wh/km},$$

$$D_{ev} = \frac{89 \cdot 600}{\frac{570}{0.45}} = 42 \text{ km}.$$

Problem 4.3

The 2011 Nissan Leaf electric vehicle has been rated by the EPA as achieving 99 mpg equivalent or 34 kWh/100 miles. Justify this rating.

- Solution

Just consider that the energy content of one U.S. liquid gallon of gasoline is 33.41 kWh. Then

$$33.4 \, \frac{kWh}{gal} \cdot \frac{1}{99} \, \frac{gal}{miles} \cdot 100 = 34 \, \frac{kWh}{100 \, miles}.$$

Hybrid-Electric Propulsion Systems

Problem 4.4

Classify the five different parallel hybrid architectures, (1) single-shaft with single clutch between engine and electric machine (E-c-M-T-V), (2) single-shaft with single clutch between engine–electric machine and transmission (E-M-c-T-V) or (M-E-c-T-V), (3) two-clutches single-shaft (E-c-M-c-T-V), (4) double-shaft (E-c-T-M-V), (5) double-drive, with respect to the following features:

- regenerative braking: optimized (without unnecessary losses) / not optimized
- ZEV mode: optimized (without unnecessary losses) / not optimized
- stop-and-start: optimized (independent from vehicle motion) / not optimized / not possible
- battery recharge at vehicle stop: possible / not possible
- gear synchronization: optimized (no additional inertia on the primary shaft) / not optimized
- compensation of the torque “holes” during gear changes: possible / not possible
- active dampening of engine idle speed oscillations: possible / not possible

- Solution

  Architecture (1):

- compensation of the torque ”holes” during gear changes not possible
- reg. braking ideal
- ZEV mode ideal
- stop/start compromised
- battery recharge at vehicle stop impossible
- gear synchronization compromised (but compensation through the electric machine itself possible)
- active dampening impossible

Architecture (2), e.g., an Honda IMA-type system (E-M-c-T-V), or a belt starter-alternator case (M-E-c-T-V):

- reg. braking compromised
- ZEV mode compromised
- stop/start ideal
- active dampening possible
- battery recharge at vehicle stop possible
- gear synchronization ideal

Architecture (3):

- reg. braking ideal
- ZEV mode ideal
- stop/start ideal
- active dampening possible
- battery recharge at vehicle stop possible
- gear synchronization ideal

Architecture (4):

- reg. braking compromised
- ZEV mode compromised
- stop/start compromised

- active dampening impossible
- battery recharge at vehicle stop impossible
- gear synchronization ideal
- compensation of the torque "holes" during gear changes possible

Architecture (5):

- reg. braking ideal
- ZEV mode not ideal
- stop/start impossible (would need an additional starter machine)
- active dampening impossible (see stop/start)
- battery recharge at vehicle stop impossible
- gear synchronization ideal
- compensation of the torque "holes" during gear changes possible

Find a summary of these features in the table below.

	E-c-M-T-V	E-M-c-T-V	E-c-M-c-T-V	E-c-T-M-V	E-c-T-V-M
RB	✓	×	✓	✓	✓
ZEV	✓	×	✓	✓	✓
S/S	×	✓	✓	×	×
rech. at stop	×	✓	✓	×	×
gear sync.	×	✓	✓	✓	✓
comp. holes	×	×	×	✓	✓
act. dmp.	×	✓	✓	×	×

Problem 4.5

Determine the overall degrees of freedom u in modeling (i) a parallel hybrid, (ii) a series hybrid, (iii) a combined hybrid, with the quasistatic approach. For (ii) and (iii) use both the generator causality depicted in Figs. 4.11 – 4.13 and the alternative causality introduced in Sect. 4.4.

- Solution

Parallel hybrid There are 6 blocks $\{V, T, E, M, P, B\}$ and 7 relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_e, T_m, \gamma) = 0$
3. $f_{T,2}(v, \omega_e, \gamma) = 0$
4. $f_{T,3}(v, \omega_m, \gamma) = 0$
5. $f_E(u_e, T_e, \omega_e) = 0$ (u_e is the engine control vector)
6. $f_M(T_m, \omega_m, P_m) = 0$
7. $f_P(P_m, P_b) = 0$

between the 10 variables $v, F_t, T_e, T_m, \omega_e, u_e, \omega_m, \gamma, P_m, P_b$. Consider γ as fixed. Thus there are two independent variables. In the quasistatic approach, v is known, thus the remaining degree of freedom is, e.g., the torque split ratio at the torque coupler u (needed to solve the $f_{T,1}$ equation).

Series hybrid There are 7 blocks and relationships

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_m) = 0$
3. $f_{T,2}(v, \omega_m) = 0$
4. $f_M(T_m, \omega_m, P_m) = 0$
5. $f_P(P_m, P_g, P_b) = 0$
6. $f_G(P_g, \omega_g, T_g) = 0$
7. $f_E(u_e, T_e, \omega_e) = 0$

between the 10 variables $v, F_t, T_m, \omega_m, P_m, P_b, P_g, T_g = T_e, \omega_g = \omega_e, u_e$. Thus there are three independent variables. In the quasistatic approach, v is known, thus the remaining degrees of freedom are the power split ratio u (needed to solve the f_P equation) and the generator speed ω_g (needed to solve the f_G equation). In the alternative causality of the generator block, generator speed and torque T_g are used to solve the f_G equation.

Combined hybrid There are 8 blocks and 11 relationships

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_f) = 0$
3. $f_{T,2}(v, \omega_f) = 0$
4. $f_{PSD,1}(\omega_f, \omega_g, \omega_e) = 0$
5. $f_{PSD,2}(\omega_f, \omega_g, \omega_m) = 0$
6. $f_{PSD,3}(T_f, T_g, T_e) = 0$
7. $f_{PSD,4}(T_f, T_g, T_m) = 0$
8. $f_M(T_m, \omega_m, P_m) = 0$
9. $f_P(P_m, P_g, P_b) = 0$
10. $f_G(P_g, \omega_g, T_g) = 0$
11. $f_E(u_e, T_e, \omega_e) = 0$

between the 14 variables $v, F_t, \omega_f, T_f, T_m, \omega_m, P_m, T_g, \omega_g, P_g, P_b, T_e, \omega_e, u_e$. Thus there are three independent variables. The degrees of freedom are the same as for the series hybrid case.

Problem 4.6

Perform the same analysis as in Problem 4.5 with the dynamic approach. Calculate the number n_v of variables in the flowcharts of Figs. 4.11 – 4.13. Then calculate the number n_e of the equations available using the simple models presented in this chapter. Finally evaluate the manipulated variables that are necessary to realize the degrees of freedom (DOF) determined in Problem 4.5.

- Solution

Parallel hybrid There are 6 blocks and $n_e = 10$ relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_e, T_m, \gamma) = 0$
3. $f_{T,2}(v, \omega_e, \gamma) = 0$
4. $f_{T,3}(v, \omega_m, \gamma) = 0$
5. $f_{M,1}(T_m, I_m) = 0$
6. $f_{M,2}(\omega_m, U_m, u_m) = 0$ (u_m is the motor control vector)
7. $f_{P,1}(U_m, U_b) = 0$
8. $f_{P,2}(I_m, I_b) = 0$
9. $f_E(u_e, T_e, \omega_e) = 0$
10. $f_B(I_b, U_b) = 0$

between the $n_v = 10$ variables represented in the figure. If γ is fixed, the control inputs u_e, u_m determine the vehicle speed and the torque split ratio.

Series hybrid There are 6 blocks and $n_e = 12$ relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_m) = 0$
3. $f_{T,3}(v, \omega_m) = 0$
4. $f_{M,1}(T_m, I_m) = 0$
5. $f_{M,2}(\omega_m, U_m, u_m) = 0$
6. $f_{P,1}(U_m, U_b, U_g) = 0$
7. $f_{P,2}(I_m, I_b) = 0$
8. $f_{P,3}(I_m, I_g) = 0$
9. $f_{G,1}(T_g, I_g) = 0$
10. $f_{G,2}(\omega_g, U_g, u_g) = 0$ (u_g is the generator control vector)
11. $f_E(u_e, T_e, \omega_e) = 0$
12. $f_B(I_b, U_b) = 0$

between the $n_v = 12$ variables represented in the figure. The control inputs u_e, u_m, and u_g determine the vehicle speed, the power split ratio, and the generator speed.

Combined hybrid There are 8 blocks and $n_e = 16$ relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_f) = 0$
3. $f_{T,2}(v, \omega_f) = 0$
4. $f_{PSD,1}(\omega_f, \omega_g, \omega_e) = 0$
5. $f_{PSD,2}(\omega_f, \omega_g, \omega_m) = 0$
6. $f_{PSD,3}(T_f, T_g, T_e) = 0$
7. $f_{PSD,4}(T_f, T_g, T_m) = 0$
8. $f_{M,1}(T_m, I_m) = 0$
9. $f_{M,2}(\omega_m, U_m, u_m) = 0$
10. $f_{P,1}(U_m, U_b, U_g) = 0$
11. $f_{P,2}(I_m, I_b) = 0$
12. $f_{P,3}(I_m, I_g) = 0$

13. $f_{G,1}(T_g, I_g) = 0$
14. $f_{G,2}(\omega_g, U_g, u_g) = 0$
15. $f_E(u_e, T_e, \omega_e) = 0$ (u_e is the engine control vector)
16. $f_B(I_b, U_b) = 0$

between the $n_v = 16$ variables represented in the figure. The control inputs u_e, u_m, and u_g determine the vehicle speed, the power split ratio, and the generator speed.

***Problem* ??**

Perform the same analysis as in Problems 4.5 – 4.6 for an electric powertrain powered by a battery and a supercapacitor.

- Solution

Quasistatic approach There are 6 blocks $\{V, T, M, P, B, SC\}$ and 5 relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_m) = 0$
3. $f_{T,2}(v, \omega_m) = 0$
4. $f_M(T_m, \omega_m, P_m) = 0$
5. $f_P(P_m, P_b, P_{sc}) = 0$

between the 7 variables $v, F_t, T_m, \omega_m, P_m, P_b, P_{sc}$. Thus there are two independent variables. In the quasistatic approach, v is known, thus the remaining degree of freedom is, e.g., the power split ratio at the DC link u (needed to solve the f_P equation).

Dynamic approach There are 6 blocks and $n_e = 10$ relationships

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_m) = 0$
3. $f_{T,2}(v, \omega_m) = 0$
4. $f_{M,1}(T_m, I_m) = 0$
5. $f_{M,2}(\omega_m, U_m, u_m) = 0$
6. $f_{P,1}(U_m, U_b, U_{sc}) = 0$
7. $f_{P,2}(I_m, I_b) = 0$
8. $f_{P,3}(I_m, I_{sc}) = 0$
9. $f_B(I_b, U_b) = 0$
10. $f_{SC}(I_{sc}, U_{sc}) = 0$

between the $n_v = 10$ variables. If there is only one control input u_m the power split ratio cannot be chosen. Thus a second controllable component is needed, typically a DC–DC converter on either the supercapacitor or the battery side.

Problem 4.8

For a plug-in hybrid, the fuel consumption according to UN/ECE regulation [91] is

$$C = \frac{D_e \cdot C_1 + D_{av} \cdot C_2}{D_e + D_{av}},$$

where C_1 is the fuel consumption in charge-depleting mode, C_2 is the consumption in charge-sustaining mode, D_e is the electric range, and D_{av} is 25 km, the assumed average distance between two battery recharges. Estimate the fuel consumption of the electric system of Problem 4.1 equipped with a range extender having a max power of 5 kW and an efficiency of 0.4.

- Solution

$D_e = 100$ km, $C_1 = 0$, $D_{av} = 25$ km. To evaluate the fuel consumption in charge-sustaining mode, divide the cycle into two phases, with (i) APU on, and (ii) APU off. The mean force is the same. During phase (i),

$$E_{bat} = -F_r \cdot e_b \cdot x_{on},$$

where F_r is the mean traction force to recharge the battery, $e_b = \sqrt{e}$ is the battery efficiency, and x_{on} is the distance covered during the phase (i). During phase (ii)

$$E_{bat} = \frac{F}{e} \cdot (x_{tot} - x_{on}).$$

By equalizing these two energy terms (charge sustaining),

$$x_{on} = \frac{x_{tot} \cdot F}{F + F_r \cdot e \cdot \sqrt{e}}.$$

The APU mean power during phase (i) is

$$P_{apu} = \left(F_r + \frac{F}{\sqrt{e}}\right) \cdot \frac{x_{on}}{t_{tot}} = \frac{F + F_r \cdot \sqrt{e}}{F + F_r \cdot e \cdot \sqrt{e}} \cdot \frac{\bar{v}}{\sqrt{e}}$$

and the average fuel power is

$$P_f = \frac{P_{apu}}{e_{apu} \cdot \frac{x_{on}}{x_{tot}}}$$

Numerically,

$$F_r = \frac{P_{apu,max}}{\bar{v}} - \frac{F}{\sqrt{e}} = \frac{5 \cdot 10^3}{9.5} - \frac{303}{\sqrt{0.6}} = 135 \text{ Nm}$$

$$P_{apu} = \frac{303 + 135 \cdot \sqrt{0.6}}{303 + 135 \cdot 0.6 \cdot \sqrt{0.6}} \cdot \frac{9.5}{\sqrt{0.6}} = 4.1 \text{ kW}$$

$$P_f = \frac{4.1 \cdot 10^3}{0.4} = 10.4 \text{ kW}$$

$$V_f = \frac{10.4 \cdot 10^3}{43.5 \cdot 10^6 \cdot 0.75} = 3.2 \cdot 10^{-4} \text{ l/s} = \frac{3.2 \cdot 10^{-4}}{9.5} \cdot 1 \cdot 10^5 = 3.4 \text{ l/100 km} = C_2$$

Thus the combined fuel consumption is

$$C = \frac{D_e \cdot C_1 + D_{av} \cdot C_2}{D_e + D_{av}} = \frac{100 \cdot 0 + 25 \cdot 3.4}{125} = 0.68 \text{ l/100 km}$$

Motor and Motor Controller

Problem 4.9

Consider a separately-excited DC motor having the following characteristics: $R_a = 0.05\,\Omega$, battery voltage $= 50\,\mathrm{V}$ (neglect battery resistance), rated power $= 4\,\mathrm{kW}$, nominal torque constant $\kappa_a = \kappa_i = 0.25\,\mathrm{Wb}$, aimed at propelling a small city vehicle. Calculate the motor voltage and current limits, then the flux weakening region limit (maximum torque and base speed). Calculate the step-down chopper duty-cycle α for the following operating points: (i) $\omega_m = 100\,\mathrm{rad/s}$ and $T_m = 15\,\mathrm{Nm}$; (ii) $\omega_m = 300\,\mathrm{rad/s}$ and $T_m = 8\,\mathrm{Nm}$.

- Solution

The maximum voltage is $U_{max} = 50\,\mathrm{V}$. The maximum admissible current is calculated by forcing $U_a = U_{max}$ and $\omega_m \cdot T_m = P_{max}$. The following quadratic equation is obtained,

$$\begin{aligned} U_{max} \cdot I_{max} &= R_a \cdot I_{max}^2 + P_{max}, \qquad \text{or} \\ 0.05\ \Omega \cdot I_{max}^2 - 50\ \mathrm{V} \cdot I_{max} + 4000\ \mathrm{W} &= 0, \end{aligned}$$

whose solution is $I_{max} = 88\,\mathrm{A}$. Thus the maximum torque is $88 \cdot 0.25 = 22\,\mathrm{Nm}$. The flux weakening region limit occurs when $U_a = U_{max}$ thus

$$\begin{aligned} \frac{R_a \cdot T_m}{\kappa_a} + \kappa_a \cdot \omega_m &= U_{max}, \qquad \text{or} \\ 0.2 \cdot T_m + 0.25 \cdot \omega_m &= 50, \end{aligned}$$

extending from $T_m = 250\,\mathrm{Nm}$ on the torque axis to $\omega_m = 200\,\mathrm{rad/s}$ on the speed axis. The base speed is $\omega_b = 4000/22 = 182\,\mathrm{rad/s}$. For the first operating point,

$$\begin{aligned} I_a &= \frac{T_m}{\kappa_a} = \frac{15}{0.25} = 60\ \mathrm{A} \\ U_a &= R_a \cdot I_a + \kappa_a \cdot \omega_m = 0.05 \cdot 60 + 0.25 \cdot 100 = 3 + 25 = 28\ \mathrm{V}. \end{aligned}$$

Both current and voltage limits are respected. The chopper duty-cycle is $\alpha = 28/50 = 56\%$. The second operating point belongs to the flux weakening region. In fact, if one were to calculate the current and voltage with the above equations, $I_a = 8/0.25 = 32\,\mathrm{A}$ would be obtained, but $U_a = 0.05 \cdot 32 + 0.25 \cdot 300 = 77\,\mathrm{V}$ that is beyond the admissible voltage. Thus κ_a must be reduced. To find κ_a such that $U_a = 50\,\mathrm{V}$ ($\alpha = 100\%$), the following equation is used

$$U_{max} = \frac{R_a \cdot T_m}{\kappa_a} + \kappa_a \cdot \omega_m,$$

which leads to $\kappa_a = 0.16\,\text{Wb}$. An approximated value is obtained by neglecting the resistance as $\kappa_a = U_{max}/\omega_m = 50/300 = 0.17\,\text{Wb}$.

Problem 4.10

For the DC motor of Problem 4.9, evaluate the approximation of mirroring the efficiency from the first to the fourth quadrant, for the two operating points (i) $\omega_m = 50\,\text{rad/s}$ and $T_m = 22\,\text{Nm}$; (ii) $\omega_m = 300\,\text{rad/s}$, $T_m = 8\,\text{Nm}$. Assume further that $P_{l,c} = 0$.

- Solution

From (4.14), for $T_m > 0$

$$\frac{1}{\eta_m} = 1 + \frac{R_a \cdot T_m}{\kappa_a^2 \cdot \omega_m},$$
$$P_l = \frac{R_a \cdot T_m^2}{\kappa_a^2}.$$

For $T_m < 0$

$$\eta_m = 1 + \frac{R_a \cdot T_m}{\kappa_a^2 \cdot \omega_m} = 0.88.$$

For the point (i)

$$\eta_m(50, 22) = \frac{1}{1 + \frac{0.05 \cdot 22}{0.25^2 \cdot 50}} = 0.74,$$
$$\eta_m(50, -22) = 1 - \frac{0.05 \cdot 22}{0.25^2 \cdot 50} = 0.65,$$
$$P_l = 0.05 \cdot \left(\frac{22}{0.25}\right)^2 = 387\ \text{W}.$$

In the field weakening region, for the point (ii),

$$\eta_m(300, 8) = \frac{1}{1 + \frac{0.05 \cdot 8}{0.25^2 \cdot 300}} = 0.98,$$
$$\eta_m(300, -8) = 1 - \frac{0.05 \cdot 8}{0.25^2 \cdot 300} = 0.98,$$
$$P_l = 0.05 \cdot \left(\frac{8}{0.25}\right)^2 = 51\ \text{W}.$$

For the given values, the approximation of mirroring the efficiency is better as the losses decrease.

Problem 4.11

Using the PMSM model 4.40–4.43 calculate a static control law, i.e., a selection of reference values I_d, I_q as a function of torque and speed, such that the stator current intensity is minimized. Do the calculation in the case (i) $I_s \leq I_{max}$, $U_s \leq U_{max} = mU_m$ (maximum torque region) and (ii) when the voltage constraint is active (flux weakening region). Neglect the stator resistance R_s and consider a machine with $p = 1$. The stator current and voltage intensities are defined as

$$I_s^2 = I_q^2 + I_d^2, \quad U_s^2 = U_q^2 + U_d^2.$$

Evaluate the base speed.

- Solution

If R_s is neglected, the static counterparts of (4.26)-(4.28) are

$$\begin{aligned} U_q &= \omega_m \cdot (\varphi_m + L_s \cdot I_d), \\ U_d &= -\omega_m \cdot L_s \cdot I_q, \\ T'_m &= \frac{2}{3} T_m = \varphi_m \cdot I_q. \end{aligned}$$

To obtain the desired torque, set $I_q = T'_m/\varphi_m$. To minimize I_s without constraints, $I_d = 0$. Under these conditions, $I_s = I_q = T'_m/\varphi_m$ and

$$U_s^2 = \omega_m^2 \cdot \left(\varphi_m^2 + \left(\frac{L_s \cdot T'_m}{\varphi_m} \right)^2 \right).$$

Such a situation is valid in (i), i.e., if $I_s \leq I_m$, i.e., if $T'_m \leq \varphi_m \cdot I_{max}$, and if

$$U_s^2 = \omega^2 \cdot \left(\varphi_m^2 + \left(\frac{L_s \cdot T'_m}{\varphi_m} \right)^2 \right) \leq U_{max}^2.$$

The *base speed* is obtained from the intersection of the latter limits, i.e., for

$$\omega_b = \frac{U_{max}}{\sqrt{\varphi_m^2 + (L_s \cdot I_{max})^2}}.$$

If a negative I_d is allowed, points above the base speed are obtained. In case (ii), $U_s = U_{max}$ and $T'_m = \varphi_m \cdot I_q$ and I_d is calculated (a second-order equation is obtained) as

$$I_d = \sqrt{\left(\frac{U_{max}}{L_s \omega_m} \right)^2 - \left(\frac{T'_m}{\varphi_m} \right)^2} - \frac{\varphi_m}{L_s}.$$

Problem 4.12

Evaluate the torque limit curve for a PMSM both (i) in the maximum torque region and (ii) in the flux weakening region, see Problem 4.11, assuming $R_s = 0$. Evaluate the transition curve between these two regions. Assume that $\varphi_m > L_s I_{max}$ (why is that important?).

- Solution

The torque limit in the maximum torque region is simply $T'_{max} = \varphi_m I_{max}$. In the flux weakening region, the torque limit is generally lower than $\varphi_m I_{max}$ and is calculated using a graphical construction. The maximum current ("I") curve is a circle in the I_d–I_q plane, with center at the origin and radius I_{max}. The maximum voltage ("U") curve is a circle with center $[-\varphi_m/L_s, 0]$ and radius $U_{max}/(\omega_m L_s)$. Under the assumption that $\varphi_m > L_s I_{max}$, the center of U-curve is found outside the I-curve.

Thus the largest value of I_q that fulfills both constraints is where the two curves I and U intersect. In this case, the coordinates of the intersection are

$$I_d = \frac{\left(\frac{U_s}{\omega_m}\right)^2 - \varphi^2 - L_s^2 \cdot I_{max}^2}{2 \cdot \varphi_m \cdot L_s},$$

$$I_q = \sqrt{I_{max}^2 - \frac{1}{4 \cdot \varphi_m^2 \cdot L_s^2}\left(\left(\frac{U_{max}}{\omega_m}\right)^2 - \varphi_m^2 - (L_s \cdot I_{max})^2\right)^2},$$

from whence $T'_{max}(\omega_m) = \varphi_m \cdot I_q(\omega_m)$.

The maximum speed at which the maximum torque is null is

$$\omega_{max} = \frac{U_{max}}{\varphi_m - L_s \cdot I_{max}}.$$

The transition curve is the locus of the torque points that can be still achieved with $I_d = 0$. It is given by the intersection of U-curve with the y-axis, resulting in

$$L_s^2 \cdot I_q^2 = \left(\frac{U_{max}}{\omega_m}\right)^2 - \varphi_m^2,$$

that is the transition curve sought with $T'_m = \varphi_m I_q$. In particular, for $I_q = I_{max}$, obtain the base speed as

$$\omega_b^2 = \frac{U_{max}^2}{\varphi_m^2 + L_s^2 \cdot I_{max}^2}$$

Problem 4.13

Using the same assumptions as in Problem 4.11, evaluate the maximim power curve as a function of speed.

- Solution

The general expression for the maximum power is

$$P_{max} = \omega_m \cdot T_{max}.$$

Let us calculate the maximum of P_{max}.

$$\frac{dP_{max}}{d\omega_m} = 0 \Rightarrow T_{max} + \omega_m \cdot \frac{dT_{max}}{d\omega_m} = 0.$$

But $T_{max} = \varphi_m \cdot I_q$ and $I_q^2 = f(X)$ as given by Problem 4.11, where $X = U_{max}/\omega_m$. Consequently, the maximum condition is given by

$$2 \cdot I_q \cdot \frac{dI_q}{d\omega_m} = -\frac{df}{dX} \cdot \frac{X}{\omega} \Rightarrow \varphi_m \cdot I_q - \varphi_m \cdot \frac{df}{dX} \cdot \frac{X}{2 \cdot I_q} = 0 \Rightarrow 2 \cdot f = \frac{df}{dX} \cdot X.$$

After having calculated the derivative df/dX, obtain a 2nd-order equation in the variable X, whose solution is $X^2 = \varphi_m^2 - L_s^2 I_{max}^2$, from whence

$$\omega_P = \frac{U_{max}}{\sqrt{\varphi_m^2 - L_s^2 \cdot I_{max}^2}}.$$

For this speed,

$$f = 4 \cdot L_s^2 \cdot I_{max}^2 (\varphi^2 - L_s^2 \cdot I_{max}^2) = 4 \cdot L_s^2 \cdot I_{max}^2 (\frac{U_{max}}{\omega_m})^2,$$

$$I_q^2 = f/(4 \cdot \varphi^2 \cdot L_s^2) \Rightarrow I_q = \frac{U_{max} \cdot I_{max}}{\varphi_m \cdot \omega_m},$$

and finally $T_m = \frac{U_{max} \cdot I_{max}}{\omega_m}$ or $P_{max} = U_{max} \cdot I_{max}$.

Problem 4.14

Equation 4.42 is only valid when $L_d = L_q = L_s$. In the general case in which $L_d \neq L_q$, the correct equation is

$$T_m = \frac{3}{2} \cdot p \cdot I_q \cdot (\varphi_m - p \cdot (L_q - L_d) \cdot I_d).$$

Consider again Problem 4.11 and derive a static control law I_d, I_q that minimizes the current intensity (MTPA), assuming that the constraints over current and voltage are not active, for a motor where $p = 4$, $R_s = 0.07\,\Omega$, $L_q = 5.4 \cdot 10^{-3}\,$H, $L_d = 1.9 \cdot 10^{-3}\,$H, $\varphi_m = 0.185\,$Wb and $T = 50\,$Nm. Then compare the result with that obtained for $\Delta L = L_q - L_d = 0$.

- Solution

To evaluate I_d, a procedure similar to that of Problem 4.11 is used. Now minimize $I_d^2 + I_q^2$, subject to the condition

$$\varphi_m \cdot I_q - \Delta L \cdot I_d \cdot I_q = T'_m.$$

This is a parameter optimization problem in the two parameters I_d, I_q. Build the Hamiltonian

$$H = I_d^2 + I_q^2 + \mu \cdot (\varphi_m \cdot I_q - \Delta L \cdot I_d \cdot I_q - T'_m).$$

Pontryagin's Minimum Principle reads

$$\begin{aligned}
\frac{dH}{dI_d} &= 2 \cdot I_d - \mu \cdot \Delta L \cdot I_q = 0,\\
\frac{dH}{dI_q} &= 2 \cdot I_q + \mu \cdot \varphi_m - \mu \cdot \Delta L \cdot I_d = 0,
\end{aligned}$$

from whence

$$\Delta L \cdot I_d^2 - \varphi_m \cdot I_d - \Delta L \cdot I_q^2 = 0,$$

that is, a quadratic equation is found. Now combine this equation with the torque equation to have I_d and I_q as a function of T'_m,

$$\begin{aligned}
\Delta L \cdot (T'_m)^2 &= (\Delta L \cdot I_d^2 - \varphi_m \cdot I_d) \cdot (\varphi_m - \Delta L \cdot I_d)^2 =\\
&= \Delta L^3 \cdot I_d^4 - 3 \cdot \varphi_m \cdot \Delta L^2 \cdot I_d^3 + 3 \cdot \varphi_m^2 \cdot \Delta L \cdot I_d^2 - \varphi_m^3 \cdot I_d =\\
&= I_d \cdot (\Delta L \cdot I_d - \varphi_m)^3
\end{aligned}$$

For the numerical case, $\Delta L = 3.5 \cdot 10^{-3}\,\mathrm{H}$, $T'_m = 50/(3/2 \cdot 4) = 8.33\,\mathrm{Nm}$, thus

$$\begin{aligned}
0 = &\left(3.5 \cdot 10^{-3}\right)^3 \cdot I_d^4 - 3 \cdot 0.185 \cdot \left(3.5 \cdot 10^{-3}\right)^2 \cdot I_d^3 + \\
&+ 3 \cdot 0.185^2 \cdot 3.5 \cdot 10^{-3} \cdot I_d^2 - 0.185^3 \cdot I_d - \\
&+ 3.5 \cdot 10^{-3} \cdot 8.33^2 \qquad \Rightarrow \quad I_d = -17 \text{ A},\\
0 = &\, 3.5 \cdot 10^{-3} \cdot 17^2 + 0.185 \cdot 17 - 3.5 \cdot 10^{-3} \cdot I_q^2 \qquad \Rightarrow \quad I_q = \;\; 34 \text{ A}.
\end{aligned}$$

Verify that

$$T_m = 4 \cdot \frac{3}{2} \cdot 34 \cdot (0.185 + 3.5 \cdot 10^{-3} \cdot 17) = 50 \text{ Nm}.$$

With $\Delta L = 0$, one would have obtained $I_d = 0$ and

$$I_q = 50/4/(3/2)/0.185 = 45 \text{ A}.$$

Problem 4.15

Calculate the torque characteristic curve $T_m(\omega_m, U_s)$ of a PMSM having the following characteristics: $R_s = 0.2\,\Omega$, $L = 0.003\,\mathrm{H}$, $\frac{3}{2}\varphi_m = 0.89\,\mathrm{Wb}$, $p = 1$, for a voltage intensity (see definition in Problem 4.11) $U_s = 30\,\mathrm{V}$. Derive an affine approximation of the DC-motor type, $T_{m,lin}(\omega_m, U_s)$. Evaluate the torque error $\varepsilon(\omega_m) \triangleq U_s^2(T_m) - U_s^2(T_{m,lin})$ and calculate its maximum value.

- Solution

If $L_d = L_q$, the MTPA (maximum torque per ampere) control is (see Problem 4.11) $I_d = 0$, $I_q = T'_m/\varphi_m$. Consequently, the voltage is

$$\begin{aligned} U_q = &\quad \omega_m \cdot \varphi_m + R_s \cdot I_q = \omega_m \cdot \varphi_m + \frac{R_s \cdot T'_m}{\varphi_m}, \\ U_d = &-\omega_m \cdot L_s \cdot I_q = -\omega_m \cdot \frac{L_s \cdot T'_m}{\varphi_m}. \end{aligned}$$

The torque characteristic curve $T'_m = T'_m(\omega_m, U_s)$ for a given $U_s = \sqrt{U_d^2 + U_q^2}$ is given by

$$U_s^2 = \omega_m^2 \cdot \varphi_m^2 + \frac{R_s^2 \cdot (T'_m)^2}{\varphi_m^2} + 2 \cdot \omega_m \cdot R_s \cdot T'_m + \omega_m^2 \cdot \frac{L_s^2 \cdot (T'_m)^2}{\varphi_m^2} \qquad \Rightarrow$$

$$(\varphi_m \cdot U_s)^2 - \omega_m^2 \cdot \varphi_m^4 = (T'_m)^2 \cdot \left((\omega_m \cdot L_s)^2 + R_s^2\right) + 2 \cdot R_s \cdot \omega_m \cdot \varphi_m^2 \cdot T'_m.$$

For $\omega_m = 0$, the breakaway torque is

$$T'_{br} = \frac{U_s \cdot \varphi_m}{R_s}.$$

The zero torque speed is

$$\omega_0 = \frac{U_s}{\varphi_m}.$$

The affine approximation of the characteristic curve is

$$T'_{lin}(\omega_m, U_s) = T'_{br} + \left.\frac{\partial T'_m}{\partial \omega_m}\right|_{\omega_m = 0} \cdot \omega_m$$

where $T'_m = T'_m(\omega, U_s)$ is derived from the equation above. From a comparison with the DC-motor characteristic curve, one can make the equivalence $\kappa_a = \varphi_m$, and

$$T'_{lin}(\omega_m, U_s) = \frac{\varphi_m}{R_s} \cdot U_s - \frac{\varphi_m^2}{R_s} \cdot \omega_m.$$

At $\omega_m = \omega_0$ the approximated torque is

$$T'_0 = \frac{\varphi_m}{R_s} \cdot U_s - \frac{\varphi_m}{R_s} \cdot U_s = 0.$$

Thus, at $\omega 0$ the error is zero with respect to the nonlinear characteristic curve. To generally evaluate this error, calculate $U_s(T_m)$ from the nonlinear curve and $U_s(T_{m,lin})$ from the affine curve. One obtains

$$U_s^2(T_m) - U_s^2(T_{m,lin}) = \varepsilon(\omega_m).$$

This term can be calculated using the results above, such that

$$\begin{aligned}
\varepsilon(\omega_m) &= \left(R_s^2 + \omega_m^2 \cdot L_s^2\right) \cdot \frac{(T'_m)^2}{\varphi_m^2} + 2 \cdot \omega_m \cdot R_s \cdot T'_m - \\
&\quad + \left(1 + \frac{\omega_m^2 \cdot L_s^2}{R_s^2}\right) \cdot (U_s - \omega_m \cdot \varphi_m)^2 - 2 \cdot \omega_m \cdot \varphi_m \cdot (U_s - \omega_m \cdot \varphi_m) \\
&= \frac{R_s^2 \cdot (T'_m)^2}{\varphi_m^2} + \frac{\omega_m^2 \cdot L_s^2 \cdot (T'_m)^2}{\varphi_m^2} + \omega_m^2 \cdot \varphi_m^2 + 2 \cdot \omega_m \cdot R_s \cdot T'_m - U_s^2 - \\
&\quad + \left(\frac{\omega_m \cdot L_s}{R_s}\right)^2 \cdot (U_s - \omega_m \cdot \varphi_m)^2 \\
&= -\left(\frac{\omega_m \cdot L_s}{R_s}\right)^2 \cdot (U_s - \omega_m \cdot \varphi_m)^2,
\end{aligned}$$

which is zero for $\omega_m = 0$ and $\omega_m = \omega_0$. Define

$$E(\omega_m) = \omega_m^2 \cdot (U_s - \omega_m \cdot \varphi_m)^2.$$

The maximum value for $E(\omega_m)$ is obtained by differentiating w.r.t. ω_m:

$$\begin{aligned}
\frac{dE}{d\omega_m} &= 2 \cdot \omega_m \cdot (U_s - \omega_m \cdot \varphi_m)^2 + 2 \cdot \omega_m^2 \cdot (U_s - \omega_m \cdot \varphi_m) \cdot (-\varphi_m) = 0 \\
&\Rightarrow \quad U_s - \omega_m \cdot \varphi_m = \omega_m \cdot \varphi_m \\
&\Rightarrow \quad \omega_{max,E} = \frac{U_s}{2 \cdot \varphi_m} = \frac{\omega_0}{2}.
\end{aligned}$$

The maximum error is

$$\varepsilon(\omega_{max,E}) = \left(U_s^2 \cdot \frac{L_s}{4 \cdot \varphi_m \cdot R_s}\right)^2.$$

In relative terms

$$\frac{\varepsilon(\omega_{max,E})}{U_s^2} = \left(\frac{U_s \cdot L_s}{4 \cdot \varphi_m \cdot R_s}\right)^2.$$

With the numerical data $\varepsilon(\omega_{max,E}) = 14.4\,\mathrm{V}^2$ or in relative terms $14.4/30^2 = 1.6\%$.

Problem 4.16

A simple thermal model of an electric machine reads

$$C_{t,m} \cdot \frac{d}{dt}\vartheta_m(t) = P_l(t) - \frac{\vartheta_m(t) - \vartheta_a}{R_{th}},$$

where $C_{t,m}$ is an equivalent thermal capacity, R_{th} is an equivalent thermal resistance, $\vartheta_m(t)$ is the relevant motor temperature, and ϑ_a is the external temperature. Derive the current limitation to I_{max} from thermal considerations using the models of Sect. 4.3.3 (DC motor). How would the result change if other losses of the type $\beta \cdot \omega_m$ (iron losses, mechanical losses) were taken into account?

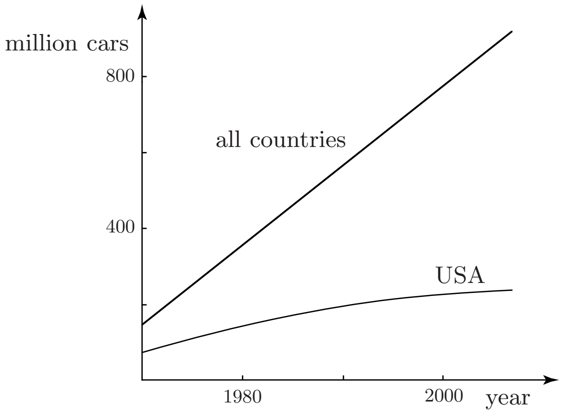

Fig. 10.9. *Schematic* representation of the development of the number of passenger cars operated worldwide.

- Solution

Behind the limitation $I_a = I_{max}$ leading to $T_m = T_{max}$ (for $\omega_m < \omega_b$) there is a temperature limitation $\vartheta_m = \vartheta_{max}$. Consider the DC motor model. Here the only loss is due to ohmic losses

$$P_l = R_a \cdot I_a^2.$$

The motor (windings) temperature varies according to this power dissipated into heat and according to heat exchange to the ambient, so

$$C_{t,m} \cdot \frac{d\vartheta_m}{dt} = P_l - \frac{\vartheta_m - \vartheta_a}{R_{th}} = R_a \cdot I_a^2 - \frac{\vartheta_m - \vartheta_a}{R_{th}}. \tag{10.24}$$

To guarantee that $\vartheta_m < \vartheta_{max}$, it should be

$$R_a \cdot I_a^2 < \alpha \cdot (\vartheta_{max} - \vartheta_a),$$

or

$$I_a < \sqrt{\frac{\vartheta_{max} - \vartheta_a}{R_{th} R_a}} = \text{const.} = I_{max}.$$

If other losses of the type $\beta \cdot \omega_m$ are considered (the factor β could in turn be dependent on ω_m), the condition on temperature reads

$$R_a \cdot I_a^2 + \beta \cdot \omega_m < \frac{\vartheta_m - \vartheta_a}{R_{th}},$$

from whence

$$I_a < \sqrt{\frac{\vartheta_{max} - \vartheta_a - \beta \cdot \omega_m}{R_{th} R_a}} = I_{max}(\omega_m).$$

Thus the maximum torque for $\omega_m < \omega_b$ would decrease with ω_m.

Problem 4.17

Derive a (simplified) rule to express peak torque limits of an electric machine as a function of application time. Use the result of Problem 4.16.

- Solution

Relax the condition on stationary temperature. The solution to the ODE (**??**) is then

$$\vartheta_m(t) = \vartheta_a + (\vartheta_{stat} - \vartheta_a) \cdot (1 - e^{-\frac{t}{\tau}}),$$

where $\vartheta_{stat} = \vartheta_a + R_{th}R_a \cdot I_a^2$, and $\tau = C_{t,m}R_{th}$. Impose that $\vartheta(t) = \vartheta_{max}$ and obtain I_{max} as a function of time,

$$\begin{aligned}
\vartheta_{max} - \vartheta_a &= R_{th}R_a \cdot I_{max}^2(t) \cdot \left(1 - e^{-\frac{t}{\tau}}\right) \\
\Rightarrow \quad I_{max}(t) &= \sqrt{\frac{(\vartheta_{max} - \vartheta_a)}{R_{th}R_a \cdot \left(1 - e^{-\frac{t}{\tau}}\right)}} \\
\Rightarrow \quad T_{max}(t) &= \kappa_a \cdot I_{max}(t)
\end{aligned}$$

For $t \to \infty$, one finds

$$T_{max} = \kappa_a \cdot \sqrt{\frac{\vartheta_{max} - \vartheta_{amb}}{R_{th}R_a}},$$

which is the result of Problem 4.16.

Problem 4.18

One PMSM has the following design parameter: external diameter $d_1 = 0.145\,\text{m}$, weight $m_1 = 14\,\text{kg}$, length $l_1 = 0.06\,\text{m}$. At 5500 rpm it delivers a maximum torque $T_1 = 12\,\text{Nm}$. Predict the power P_2 and the weight m_2 for a similarly designed machine with a diameter $d_2 = 0.2\,\text{m}$ and a length $l_2 = 0.2\,\text{m}$. Compare the cases in which the design is made (i) on the basis of constant tangential stress and peripheral speed, or (ii) of constant speed.

- Solution

Case (i). The power of machine 1 is

$$P_1 = \frac{5500 \cdot \pi}{30} \cdot 12 = 6.9 \text{ kW}.$$

The mean rotor speed is

$$c_m = \frac{\omega_1 \cdot d_1}{2} = \frac{5500 \cdot \pi}{30} \cdot \frac{0.145}{2} = 41.8 \text{ m/s}.$$

The mean *pressure* is

$$p_{me} = \frac{T_1}{2 \cdot V_1} = \frac{12}{2 \cdot \left(\frac{\pi \cdot 0.145^2}{4} \cdot 0.06\right)} = 6 \text{ kPa}.$$

Indeed, $P_1 = \pi \cdot d_1 \cdot l_1 \cdot p_{me} \cdot c_m = \pi \cdot 0.145 \cdot 0.06 \cdot 6 \cdot 10^3 \cdot 41.8 = 6.9$ kW. For machine 2,

$$P_2 = \pi \cdot 0.2 \cdot 0.2 \cdot 6 \cdot 10^3 \cdot 41.8 = 31.5 \text{ kW},$$
$$m_2 = m_1 \cdot \left(\frac{d_2}{d_1}\right)^2 \cdot \frac{l_2}{l_1} = 14 \cdot \left(\frac{0.2}{0.145}\right)^2 \cdot \frac{0.2}{0.06} = 89 \text{ kg},$$

assuming constant density.

Case (ii). What changes now is that

$$P = \omega \cdot p_{me} \cdot \pi \cdot \frac{d^2}{2} \cdot l.$$

For machine 2,

$$P_2 = \frac{\pi \cdot 5500}{30} \cdot 6 \cdot 10^3 \cdot \pi \cdot \frac{0.2^2}{2} \cdot 0.2 = 43 \text{ kW},$$

while m_2 is unchanged. The specific power is the same for machine one and two,

$$\frac{6.9 \cdot 10^3}{14} = \frac{43 \cdot 10^3}{89} = 0.49 \text{ kW/kg},$$

Problem 4.19

Evaluate the specific power of a motor and inverter assembly, knowing that $\left(\frac{P}{m}\right)_{\text{motor}} = 1.2\,\text{kW/kg}$ and $\left(\frac{P}{m}\right)_{\text{inverter}} = 11\,\text{kW/kg}$.

- Solution

The specific power of the motor system is simply

$$\left(\frac{P}{m}\right) = \frac{1}{\frac{1}{\left(\frac{P}{m}\right)_{\text{motor}}} + \frac{1}{\left(\frac{P}{m}\right)_{\text{inverter}}}} = \frac{1}{\frac{1}{1.2} + \frac{1}{11}} = 1.08 \text{ kW/kg}.$$

Range extenders

Problem 4.20

Consider an APU for a series hybrid. Given the engine model

$$\begin{aligned}
P_f &= \frac{P_e + P_0}{e}, \\
\frac{1}{e} &= 5.07 - 0.0117 \cdot \omega_e + 1.50 \cdot 10^{-5} \cdot \omega_e^2 = a_1 + b_1 \cdot \omega_e + c_1 \cdot \omega_e^2 \\
\frac{P_0}{e} &= -1.22 \cdot 10^3 + 31.7 \cdot \omega_e + 0.421 \cdot \omega_e^2 = a_2 + b_2 \cdot \omega_e + c_2 \cdot \omega_e^2 \\
T_{max} &= 96.9 + 1.35 \cdot \omega_e - 0.0031 \cdot \omega_e^2 = h \cdot \omega_e^2 + g \cdot \omega_e + f,
\end{aligned}$$

and a generator model with constant efficiency $\eta_g = 0.92$, derive an OOL structure $\hat{\omega}(P_g)$. Then calculate the operating points for (i) $P_g = 10\,\text{kW}$, (ii) $P_g = 40\,\text{kW}$, and (iii) $P_g = 60\,\text{kW}$.

- Solution

The problem is finding ω_g for each $P_g = \eta_g \cdot P_e$ such that P_f is minimized. By differentiating P_f with respect to $\omega_e = \omega_g$ one obtains

$$\frac{dP_f}{d\omega_g} = b_2 + 2 \cdot c_2 \cdot \omega_g + b_1 \cdot P_e + 2 \cdot c_1 \cdot \omega_g \cdot P_e = 0 \; ,$$

thus

$$\hat{\omega} = -\frac{b_1 \cdot P_e + b_2}{2 \cdot (c_1 \cdot P_e + c_2)} \; . \tag{10.25}$$

For the case (i), $\hat{\omega} = 81.7\,\text{rad/s}$, which is below the minimum APU speed, therefore $\hat{\omega} = 1000$ rpm $= 104.7\,\text{rad/s}$.

For the case (ii), $\hat{\omega} = 222\,\text{rad/s}$. The torque is $T_g = 196\,\text{Nm}$, which is below the maximum torque at the speed $\hat{\omega}$. Thus the operating point is admissible.

For the case (iii), $\hat{\omega}$ would be equal to $261\,\text{rad/s}$ and the torque would be $250\,\text{Nm}$, while the maximum torque at that speed is $239\,\text{Nm}$. Thus a different calculation should be used: find $\hat{\omega}$ such that

$$\left(96.9 + 1.35 \cdot \hat{\omega} - 0.0031 \cdot \hat{\omega}^2\right) \cdot \hat{\omega} = \frac{60 \cdot 10^3}{0.92} \; .$$

The solution is $\hat{\omega} = 279\,\text{rad/s}$.

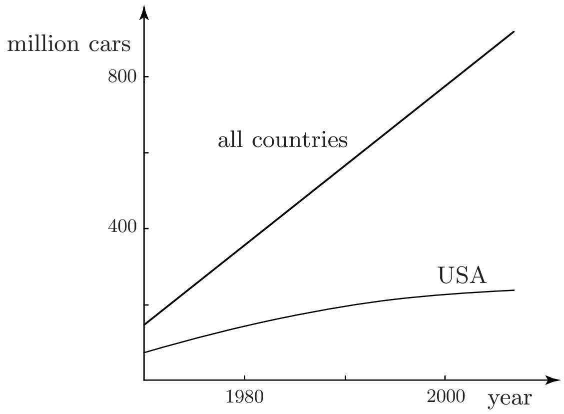

Fig. 10.10. *Schematic* representation of the development of the number of passenger cars operated worldwide.

Problem 4.21

For the APU model of Problem 4.20, find a piecewise affine approximation $P_f = a + b \cdot P_g$. Evaluate the error with respect to the nonlinear model, for (i) $P_g = 10\,\text{kW}$, (ii) $P_g = 40\,\text{kW}$, and (iii) $P_g = 60\,\text{kW}$.

- Solution

Three discontinuity points are identified: $P_e = 0$, $P_e = P_1$ such that $\hat{\omega} = 1000\,\text{rpm}$, $P_e = P_2$ such that the engine torque limitation is active, and $P_e = P_{max}$. The value P_1 is calculated as the root of the equation

$$\begin{aligned}\hat{\omega}(P_1) = 1000\,rpm \quad &\Rightarrow \quad b_1 \cdot P_1 + b_2 = -2 \cdot \frac{1000 \cdot \pi}{30} \cdot (c_1 \cdot P_1 + c_2) \\ &\Rightarrow \quad P_1 = 14 \text{ kW} \,.\end{aligned}$$

The value P_2 is calculated as the root of the equation

$$P_2 = \hat{\omega}(P_2) \cdot T_{max}(\hat{\omega}(P_2)) \quad \Rightarrow \quad P_2 = 62 \text{ kW} \,.$$

The value of P_{max} is given by finding the stationary point of

$$P(\omega_e) = \left(h \cdot \omega_e^2 + g \cdot \omega_e + f\right) \cdot \omega_e \quad \Rightarrow \quad \frac{dP}{d\omega_e} = 3 \cdot h \cdot \omega_e^2 + 2 \cdot g \cdot \omega_e + f = 0 \,,$$

which gives $\hat{\omega} = \omega_e = 324$ rad/s and $P_{max} = 68.3$ kW.
The value of P_f at $P_e = 0$ is

$$P_f(0^+) = c_2 \cdot \left(\frac{1000 \cdot \pi}{30}\right)^2 + b_2 \cdot \frac{1000 \cdot \pi}{30} + a_2 = 6.7 \text{ kW} \,.$$

The value of P_f at $P_e = P_1$ is

$$\begin{aligned}&\left(c_1 \cdot 14 \cdot 10^3 + c_2\right) \cdot \left(\frac{1000 \cdot \pi}{30}\right)^2 + \\ &\quad + \left(b_1 \cdot 14 \cdot 10^3 + b_2\right) \cdot \frac{1000 \cdot \pi}{30} + a_1 \cdot 14 \cdot 10^3 + a_2 = 62.9 \text{ kW} \,.\end{aligned}$$

The value of P_f at $P_e = P_2$ is calculated after having calculated the

$$\omega_e = -\frac{b_1 \cdot P_2 + b_2}{2 \cdot (c_1 \cdot P_2 + c_2)} = 256 \text{ rad/s},$$

with $b = -694$ and $c = 1.35$ (and $a = 3.13 \cdot 10^5$). Then

$$P_f = c \cdot \omega_e^2 + b \cdot \omega_e + a = 224.4 \text{ kW}.$$

The value of P_f at $P_e = P_{max}$ is calculated with $a = 3.45 \cdot 10^5$, $b = -767$, $c = 1.45$. The value is $P_f = 248.7$ kW. The piecewise affine model is therefore

$$
\begin{aligned}
&P_f = 0, \quad P_e = 0 \quad (P_g = 0) \\
&P_f = 6.7 \cdot 10^3 + \frac{62.9 - 6.7}{14} \cdot P_e, && \text{for} \quad 0 < \frac{P_e}{10^3} < 14 \\
&P_f = 62.9 \cdot 10^3 + \frac{224.4 - 62.9}{62 - 14} \cdot \left(P_e - 14 \cdot 10^3\right), && \text{for } 14 < \frac{P_e}{10^3} < 62 \\
&P_f = 224.4 \cdot 10^3 + \frac{248.7 - 224.2}{68.3 - 62} \cdot \left(P_e - 62 \cdot 10^3\right), && \text{for } 62 < \frac{P_e}{10^3} < 68.3
\end{aligned}
$$

The engine power can be replaced by $P_e = P_g/\eta_g$ to get the affine relationship between fuel power and APU power.

For the operating point (i), $P_g = 10$ kW and $P_e = 10.9$ kW. The exact P_f is 50.3 kW. The approximated value is

$$6.7 \cdot 10^3 + \frac{62.9 - 6.7}{14} \cdot 10.9 \cdot 10^3 = 50.5 \text{ kW},$$

with a 2% error.

For the operating point (ii), $P_g = 40$ kW and $P_e = 43$ kW, the exact $P_f = 166.4$ kW. The approximated value is

$$62.9 \cdot 10^3 + \frac{224.4 - 62.9}{62 - 14} \cdot \left(43 \cdot 10^3 - 14 \cdot 10^3\right) = 162.2 \text{ kW},$$

with a 2.5% error.

For the operating point (iii), $P_g = 60$ kW and $P_e = 65.2$ kW, the exact $P_f = 234.7$ kW. The approximated value is

$$224.4 \cdot 10^3 + \frac{248.7 - 224.2}{68.3 - 62} \cdot \left(65.2 \cdot 10^3 - 62 \cdot 10^3\right) = 236.8 \text{ kW},$$

with a 1% error.

***Problem* ??**

Propose an algorithm to calculate the OOL of an engine for a combined hybrid from the data `w4x` (APU speed breakpoint vector), `T4x` (APU torque breakpoint vector), `Tmax(w4x)` (APU maximum torque), and `mfuel(w4x,T4x)` (engine consumption map).

- Solution

```
FOR P = 0 TO max(w4x * Tmax)
 WHILE w = w4x AND P/w < Tmax(w)
   Pf(w) = Hl*mfuel(w,T);
   h(w) = P/Pf(w);
 END
 wopt(P) = arg min(h(w));
END
```

Problem 4.23

Find an optimal design for a range extender working at a stationary operating point, i.e., select optimal values for the displacement volume V_d and the speed ω_e of the engine, neglecting stop-and-start effects and making the following approximations. The IC engine is modeled with a Willans line

$$p_{me} = e \cdot p_{mf} - (p_{me0} + p_{me2} \cdot \omega_e^2)$$

with $e = 0.4$, $p_{me0} = 1.5 \cdot 10^5\,\mathrm{Pa}$, $p_{me2} = 1.4\,\mathrm{Pa}{\cdot}s^2$. The generator is a DC machine with constant armature resistance $R_a = 0.2\,\Omega$ and $\kappa_a = 0.5$. The battery is modeled as an internal voltage source $U_{oc} = 180\,\mathrm{V}$ with an internal resistance $R_b = 0.3\,\Omega$. The overall system efficiency

$$\eta_{ov} = \frac{U_{oc} \cdot I_a}{\overset{*}{m}_f \cdot H_l}$$

should be optimal. The nominal power P_e of the range extender should be greater than 30 kW and the brake mean effective pressure of the engine smaller than 9 bar. The design parameters can be chosen between the following boundaries: $V_d \in [0.5, 2]\,\mathrm{l}$, $\omega_e \in [U_{oc}/\kappa_a, 600]\,\mathrm{rad/s}$.

- Solution

The generator current is

$$I_a = \frac{T_e}{\kappa_a} = \frac{\kappa_a \cdot \omega_e - U_{oc}}{R},$$

where $R = R_a + R_b$. The fuel consumption rate is

$$\overset{*}{m}_f \cdot H_f = \frac{\omega_e \cdot T_e}{e} + \frac{\omega_e}{e} \cdot \left(p_{me0} + p_{me2} \cdot \omega_e^2\right) \cdot \frac{V_d}{4 \cdot \pi}$$

Thus the overall efficiency is

$$\eta_{ov} = \frac{e \cdot U_{oc} \cdot (\kappa_a \cdot \omega_e - U_{oc})}{\kappa_a \cdot \omega_e \cdot (\kappa_a \cdot \omega_e - U_{oc}) + \omega_e \cdot (p_{me0} + p_{me2} \cdot \omega_e^2) \cdot \frac{V_d}{4 \cdot \pi} \cdot R} = \eta_{ov}(\omega_e, V_d).$$

The power is also expressed as a function of ω_e and V_d as

$$P_e = I_a \cdot U_a = I_a \cdot \kappa_a \cdot \omega_e = \frac{(\kappa_a \cdot \omega_e - U_{oc}) \cdot \kappa_a \cdot \omega_e}{R}$$

while the brake m.e.p. is

$$p_{me} = \frac{4 \cdot \pi \cdot T_e}{V_d} = \frac{4 \cdot \pi \cdot \kappa_a \cdot (\kappa_a \cdot \omega_e - U_{oc})}{V_d \cdot R}.$$

It can be seen by inspection that the efficiency decreases for increasing values of V_d. The lower V_d is obtained at the intersection of the two conditions on P_e and p_{me}. The former gives

$$\kappa_a^2 \cdot \omega_e^2 - \kappa_a \cdot U_{oc} \cdot \omega_e - R \cdot P_g = 0 \quad \Rightarrow \quad \omega_e = 484 \text{ rad/s}.$$

The latter condition gives

$$V_d = \frac{4 \cdot \pi}{R \cdot p_{me}} \cdot (\kappa_a^2 \cdot \omega_e - \kappa_a \cdot U_{oc}) = \frac{4 \cdot \pi}{0.5 \cdot 9 \cdot 10^5} \cdot (0.5^2 \cdot 484 - 0.5 \cdot 180) = 0.87 \text{ l}.$$

At this speed, the current is $(0.5 \cdot 484 - 180)/0.5 = 124$ A, the torque is $0.5 \cdot 124 = 62$ Nm, the engine power is thus $484 \cdot 62 = 30$ kW, the fuel power is

$$\frac{30 \cdot 10^3}{0.4} + \frac{484}{0.4} \cdot (1.5 \cdot 10^5 + 1.4 \cdot 484^2) \cdot \frac{0.87 \cdot 10^{-3}}{4 \cdot \pi} = 115 \text{ kW},$$

(engine efficiency 26%), the armature voltage is $0.5 \cdot 484 - 0.2 \cdot 124 = 217$ V, the battery power is $217 \cdot 124 = 26.9$ kW (generator efficiency $26.9/30 = 90\%$), the battery internal power is $180 \cdot 124 = 22.3$ kW (battery internal efficiency 83%), the overall efficiency is 19%, as it can be verified using the expression calculated for $\eta_{ov}(\omega_e, V_d)$.

Problem ??

In Problem 4.23, set the engine displacement volume to $V_d = 1.6$ l and find the optimal operating speed that maximizes the overall efficiency while respecting the two constraints on P_e and p_{me}.

- Solution

The overall efficiency as a function of ω is given by

$$\eta_{ov} = \frac{e \cdot U_{oc} \cdot (\kappa_a \cdot \omega_e - U_{oc})}{\kappa_a \cdot \omega_e \cdot (\kappa_a \cdot \omega_e - U_{oc}) + \omega_e \cdot (p_{me0} + p_{me2} \cdot \omega_e^2) \cdot \frac{V_d}{4 \cdot \pi} \cdot R} = \eta_{ov}(\omega_e).$$

To find the maximum efficiency, differentiate with respect to ω_e,

$$\begin{aligned} &\kappa_a \cdot \big(\kappa_a \cdot \omega_e \cdot (\kappa_a \cdot \omega_e - U_{oc}) + \omega_e \cdot (p_{me0} + p_{me2} \cdot \omega_e^2) \cdot a\big) = \\ &\quad (\kappa_a \cdot \omega_e - U_{oc}) \cdot (3 \cdot a \cdot p_{me2} \cdot \omega_e^2 + 2 \cdot \kappa_a^2 \cdot \omega_e - \kappa_a \cdot U_{oc} + a \cdot p_{me0}), \end{aligned}$$

where $a = V_d \cdot R/4/\pi$ is a constant. By manipulating the expression above, obtain the third-order equation

$$\begin{aligned} &2 \cdot \kappa_a \cdot a \cdot p_{me2} \cdot \omega_e^3 + (\kappa^3 - 3 \cdot a \cdot p_{me2} \cdot U_{oc}) \cdot \omega_e^2 - \\ &\quad + 2 \cdot \kappa_a^2 \cdot U_{oc} \cdot \omega_e + (\kappa_a \cdot U_{oc}^2 - a \cdot U_{oc} \cdot p_{me0}) = 0, \end{aligned}$$

whose solution is $\omega_e = 503$ rad/s for $V_d = 1.6 \cdot 10^{-3}$, $a = 6.37 \cdot^{-5}$.

At this speed, the current is $(0.5 \cdot 503 - 180)/0.5 = 143$ A, the torque is $0.5 \cdot 143 = 71.5$ Nm, the p_{me} is $4 \cdot \pi \cdot 71.5/1.6 \cdot 10^{-3} = 5.6$ bar (thus the constraint is not violated), the engine power is thus $503 \cdot 71.5 = 36$ kW, the fuel power is

$$\frac{36 \cdot 10^3}{0.4} + \frac{503}{0.4} \cdot (1.5 \cdot 10^5 + 1.4 \cdot 503^2) \cdot \frac{1.6 \cdot 10^{-3}}{4 \cdot \pi} = 171 \text{ kW}$$

(engine efficiency 21%), the armature voltage is $0.5 \cdot 503 - 0.2 \cdot 143 = 223$ V, the battery power is $223 \cdot 143 = 31.8$ kW and thus also the second constraint is not violated (generator efficiency $31.8/36 = 88\%$), the battery internal power is $180 \cdot 143 = 25.7$ kW (battery internal efficiency 81%), the overall efficiency is 15%, as it can be verified using the expression calculated for $\eta_{ov}(\omega_e)$. Thus the choice of a non-optimal value for the displacement volume leads to a substantial loss in the overall efficiency.

Batteries

Problem 4.25

For a battery pack having the following characteristics: $Q_{cell} = 5\,\text{Ah}$, $U_{cell} = 3.14 + 1.10 \cdot \xi$ (V), $R_{cell} = 0.005 - 0.0016 \cdot \xi$ (Ω) under discharge and $R_{cell} = 0.0020 \cdot \xi^2 - 0.0020 \cdot \xi + 0.0041$ (Ω) under charge, $N = 96$, calculate the electrochemical power P_{ech} for an electric power demand of 15 kW in charge and discharge, respectively, and for 20% and 90% state of charge.

- Solution

$$P_{ech} = U_{oc} \cdot I_b \quad \text{with} \quad I_b = \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\frac{U_{oc}^2}{4 \cdot R_i^2} - \frac{P_b}{R_i}}.$$

For (i) $P_b = 15\,\text{kW}$ and $q = 0.2$,

$$\begin{aligned}
U_{oc} &= (3.14 + 1.10 \cdot 0.2) \cdot 96 = 323 \text{ V}, \\
R_i &= R_d = (0.005 - 0.0016 \cdot 0.2) \cdot 96 = 0.45\,\Omega, \\
I_b &= \frac{323}{2 \cdot 0.45} - \sqrt{\frac{323^2}{4 \cdot 0.45^2} - \frac{15 \cdot 10^3}{0.45}} = 50 \text{ A}, \\
P_{ech} &= 323 \cdot 50 = 16.1 \text{ kW} \qquad \left(\text{efficiency} = \frac{15}{16.1} = 93\%\right).
\end{aligned}$$

For (ii) $P_b = 15\,\text{kW}$ and $q = 0.9$,

$$\begin{aligned}
U_{oc} &= (3.14 + 1.10 \cdot 0.9) \cdot 96 = 396 \text{ V}, \\
R_i &= R_d = (0.005 - 0.0016 \cdot 0.9) \cdot 96 = 0.34\Omega, \\
I_b &= \frac{396}{2 \cdot 0.34} - \sqrt{\frac{396^2}{4 \cdot 0.34^2} - \frac{15 \cdot 10^3}{0.34}} = 39 \text{ A}, \\
P_{ech} &= 396 \cdot 39 = 15.4 \text{ kW} \qquad \left(\text{efficiency} = \frac{15}{15.4} = 97\%\right).
\end{aligned}$$

For (iii) $P_b = \text{-}15\,\text{kW}$ and $q = 0.2$,

$$
\begin{aligned}
U_{oc} &= 323 \text{ V},\\
R_i = R_c &= (0.0020 \cdot 0.04 - 0.0020 \cdot 0.2 + 0.0041) \cdot 96 = 0.36\Omega,\\
I_b &= \frac{323}{2 \cdot 0.36} - \sqrt{\frac{323^2}{4 \cdot 0.36^2} + \frac{15 \cdot 10^3}{0.36}} = -44 \text{ A},\\
P_{ech} &= 323 \cdot (-44) = -14.3 \text{ kW} \qquad \left(\text{efficiency} = \frac{14.3}{15} = 95\%\right).
\end{aligned}
$$

For (iv) P_b = -15 kW and $q = 0.9$,

$$
\begin{aligned}
U_{oc} &= 396 \text{ V},\\
R_i = R_c &= (0.0020 \cdot 0.81 - 0.0020 \cdot 0.9 + 0.0041) \cdot 96 = 0.38\Omega,\\
I_b &= \frac{396}{2 \cdot 0.38} - \sqrt{\frac{396^2}{4 \cdot 0.38^2} + \frac{15 \cdot 10^3}{0.38}} = -37 \text{ A},\\
P_{ech} &= 396 \cdot 37 = -14.7 \text{ kW} \qquad \left(\text{efficiency} = \frac{14.7}{15} = 98\%\right).
\end{aligned}
$$

Problem 4.26

Find a quadratic approximation for the relationship between battery power P_b and electrochemical power P_{ech}. Compare the results with those of Problem 4.25.

- Solution

The relationship between power and current is

$$
I_b = \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\frac{U_{oc}^2}{2 \cdot R_i^2} - \frac{P_b}{R_i}} \qquad \text{or} \qquad I_b = c - \sqrt{c^2 - a \cdot P_b}.
$$

By expanding this function as a Taylor series, one obtains

$$
\begin{aligned}
I_b(0) &= 0,\\
\left.\frac{dI_b}{dP_b}\right|_{P_b=0} &= \left.\frac{a}{2 \cdot \sqrt{c^2 - a \cdot P_b}}\right|_{P_b=0} = \frac{1}{U_{oc}},\\
\left.\frac{d^2 I_b}{dP_b^2}\right|_{P_b=0} &= \left.\frac{a^2}{4 \cdot (c^2 - a \cdot P_b)^{3/2}}\right|_{P_b=0} = \frac{2 \cdot R_i}{U_{oc}^3}.
\end{aligned}
$$

Thus

$$
\hat{I}_b = \frac{P_b}{U_{oc}} + 2 \cdot \frac{R_i}{U_{oc}^3} \cdot P_b^2.
$$

For the cases of Problem 4.25 and the approximation we get

P_b (kW)	q	U_{oc} (V)	R_i (Ω)	I_b (A)	P_{ech} (kW)
15	0.2	323	0.45	50	16.1
15	0.9	396	0.34	39	15.4
15	0.2	323	0.36	-44	-14.3
15	0.9	369	0.38	-37	-14.7

$\hat{I}_b$ (A)	$\hat{P}_{ech}$ (kW)	error (%)
52.4	16.9	5
40.3	16.0	4
-41.6	-13.4	6
-35.1	-13.9	5

Problem 4.27

One couple of electrodes for a lithium cell has the following characteristics: Negative electrode (graphite): capacity $q_{rev,n} = 340\,\text{mAh/g}$, potential $U_n = 0.25\,\text{V}$, density $\rho_n = 2.2\,\text{g/cm}^3$, thickness $s_n \leq 80\,\mu\text{m}$. Positive electrode ($LiCoO_2$): capacity $q_{rev,p} = 140\,\text{mAh/g}$, potential $U_p = 3.85\,\text{V}$, density $\rho_p = 4\text{g/cm}^3$, thickness $s_p \leq 80\,\mu\text{m}$. Separator, collector: surface density $0.047\,\text{g/cm}^2$. Calculate the cell voltage and the cell specific energy.

- Solution

$$U_{cell} = U_p - U_n = 3.85 - 0.25 = 3.6 \text{ V}.$$

If $Q = Q_n = Q_p$, then

$$q_{rev} = \frac{q_{rev,n} \cdot q_{rev,p}}{q_{rev,n} + q_{rev,p}} = \frac{1}{\frac{1}{340} + \frac{1}{140}} = 99 \text{ mAh/g} \qquad \text{(theoretical)},$$

$$\left(\frac{E}{m}\right)_{rev} = q_{rev} \cdot U_{cell} = 99 \cdot 3.6 = 356 \text{ Wh/kg} \qquad \text{(theoretical)}.$$

To add the masses of the collector and separator, the surface density L_{act} of the active mass must be calculated. To do that, consider the maximum electrode thickness. Since the surface is the same for the anode and the cathode,

$$Q_n = Q_p \quad \rightarrow \quad s_n \cdot q_n \cdot \rho_n \cdot S = s_p \cdot q_p \cdot \rho_p \cdot S$$

$$\frac{s_n}{s_p} = \frac{q_p \cdot \rho_p}{q_n \cdot \rho_n} = \frac{140 \cdot 4}{340 \cdot 2.2} = 0.75, \qquad \text{thus } s_n < s_p$$

Assume $s_p = 80\ \mu\text{m}$, then $s_n = 60\ \mu\text{m}$.

$$\frac{Q_n}{S_n} = \frac{Q_p}{S_p} = \frac{Q}{S} = 80 \cdot 10^{-4} \cdot 140 \cdot 4 \text{ mAh/cm}^2 = 4.48 \text{ mAh/cm}^2,$$

$$L_{act} = \frac{Q}{S} \cdot \frac{1}{q_{rev}} = \frac{4.48}{99} \text{ g/cm}^2 = 0.045 \text{ g/cm}^2 \qquad \text{(active mass)}.$$

Add the mass of the separator to obtain

$$L = L_{act} + L_{sep} = 0.045 + 0.047 = 0.092 \text{ g/cm}^2.$$

The practical specific capacity q_{cell} is thus

$$q_{cell} = \frac{Q}{S} \cdot \frac{1}{L} = \frac{4.48}{0.092} = 49 \text{ mAh/g} \qquad \text{(practical)},$$

or approximatly 50% of the theoretical capacity. The practical energy density e is

$$\left(\frac{E}{m}\right)_{cell} = 3.6 \cdot 49 = 176 \text{ Wh/kg} \qquad \text{(practical)}.$$

The pack specific energy will be even lower.

Problem 4.28

Develop an equation for the battery apparent capacity as a function of the current for constant current discharge using the battery modeling equations of Section 4.5.2. Then evaluate the apparent capacity of a battery with nominal capacity $Q_0 = 72\,\text{Ah}$, $\kappa_4 = -0.005$, $\kappa_2 = 1.22$, at $C/10$, C_1, and C_{10} discharge.

- Solution

$$U_b = U_{oc} - R_i \cdot I_b = (\kappa_1 + \kappa_2 \cdot \xi) - (\kappa_3 + \kappa_4 \cdot \xi) \cdot I_b, \text{ with } \kappa_4 < 0$$
$$\dot{\xi} = -\frac{I_b}{Q_0^*},$$

thus

$$\dot{U}_b = -\kappa_2 \cdot \frac{I_b}{Q_0^*} + \kappa_4 \cdot \frac{I_b^2}{Q_0^*} = -c,$$
$$U_b(t) = U_b(0) - c \cdot t, \text{ where } U_b(0) = \kappa_1 + \kappa_2.$$

The discharge ends when $U_b(t_f) = U_{cut}$, thus for

$$t_f = \frac{U_b(0) - U_{cut}}{c}.$$

The capacity or Ah rate is

$$Q_0 = I_b \cdot t_f = Q_0^* \cdot \frac{U_b(0) - U_{cut}}{\kappa_2 - \kappa_4 \cdot I_b},$$

which is dependent on I_b. To calculate $Q_0(I_b)/Q_0(I_b^*)$, define $K_c = 1 - \kappa_4/\kappa_2 \cdot I_b^*$. One finds that

$$\frac{Q_0}{Q_0^*} = \frac{K_c}{1 + (K_c - 1) \cdot \frac{I_b}{I_b^*}},$$

that is (4.58) with $n = 2$.

The nominal capacity must be retrieved for very slow currents, ideally when $I_b^* = 0$, as

$$Q_0^* = Q_0^* \cdot \frac{U_b(0) - U_{cut}}{\kappa_2} = Q_0^* \cdot \frac{\kappa_1 + \kappa_2 - U_{cut}}{\kappa_2},$$

from whence it must be $U_{cut} = \kappa_1$. The dependency $Q_0(I_b)$ can be rewritten as

$$Q_0(I_b) = \frac{Q_0^*}{1 + e \cdot I_b}, \qquad \text{where} \quad e = \frac{|\kappa_4|}{\kappa_2}.$$

In the numerical case, $e = 0.005/1.22 = 0.0041$ The $C/10$ current is $72/10 = 7.2$ A. For this current, the capacity Q_0 is

$$Q_0 = \frac{72}{1 + 0.0041 \cdot 7.2} = 70 \text{ Ah} \qquad (97\% \text{ of the nominal capacity})$$

and the discharge time t_f is

$$t_f = \frac{70}{7.2} = 9.7 \text{ h}.$$

For a C_1 current $= 72$ A,

$$Q_0 = \frac{72}{1 + 0.0041 \cdot 72} = 56 \text{ Ah} \qquad (77\% \text{ of the nominal capacity}),$$
$$t_f = \frac{56}{72} = 0.78 \text{ h} = 46 \text{ min}.$$

For a C_{10} current $= 720$ A,

$$Q_0 = \frac{72}{1 + 0.0041 \cdot 720} = 18 \text{ Ah} \qquad (25\% \text{ of the nominal capacity}),$$
$$t_f = \frac{18}{720} = 0.025 \text{ h} = 1.5 \text{ min}.$$

Problem 4.29

Verify that the round-trip efficiency of a battery under constant current discharge-charge and for varying parameters U_{oc}, R_i as described in Section 4.5.2 has the same form as (4.82) but with U_{oc} and R_i calculated for $\xi = 0.5$.

- Solution

The energy discharged E_d is

$$E_d = I_b \cdot \int_0^{t_f} U_{oc}(\xi) - R_i(\xi) \cdot I_b \quad dt =$$
$$= I_b \cdot \int_0^{t_f} (\kappa_1 + \kappa_2 \cdot \xi) - (\kappa_3 + \kappa_4 \cdot \xi) \cdot I_b \quad dt.$$

Since $\xi(t) = 1 - I_b/Q_0 \cdot t$ under constant current discharge,

$$E_d = I_b \cdot \left[\kappa_1 \cdot t_f + \kappa_2 \cdot \left(t_f - \frac{I_b \cdot t_f^2}{2 \cdot Q_0}\right) - \kappa_3 \cdot t_f \cdot I_b - \kappa_4 \cdot I_b \cdot \left(t_f - \frac{I_b \cdot t_f^2}{2 \cdot Q_0}\right)\right].$$

Since $t_f = Q_0/I_b$,

$$E_d = I_b \cdot \left[\kappa_1 \cdot t_f + \kappa_2 \cdot \left(t_f - \frac{t_f}{2}\right) - \kappa_3 \cdot t_f \cdot I_b - \kappa_4 \cdot I_b \cdot \left(t_f - \frac{t_f}{2}\right)\right] =$$
$$= I_b \cdot t_f \cdot \left[\kappa_1 + \frac{\kappa_2}{2} - \kappa_3 \cdot I_b - \kappa_4 \cdot \frac{I_b}{2}\right],$$

which is equal to

$$E_d = I_b \cdot t_f \cdot \left(U_{oc,\frac{1}{2}} - R_{i,\frac{1}{2}} \cdot I_b\right)$$

Similarly for E_c,

$$E_c = |I_b| \cdot t_f \cdot \left(U_{oc,\frac{1}{2}} + R_{i,\frac{1}{2}} \cdot |I_b|\right),$$

and thus

$$\eta_b = \frac{U_{oc,\frac{1}{2}} - R_{i\frac{1}{2}} \cdot I_b}{U_{oc,\frac{1}{2}} + R_{i,\frac{1}{2}} \cdot |I_b|}.$$

Supercapacitors

Problem 4.31

Derive Equation (4.117).

- Solution

P_{sc} is considered as a constant. By differentiating (4.116) one obtains

$$2 \cdot U_{sc} \cdot \frac{d}{dt}U_{sc} + \frac{I_{sc}}{C_{sc}} \cdot U_{sc} - \frac{Q_{sc}}{C_{sc}} \cdot \frac{d}{dt}U_{sc} = 0,$$

where the second of (4.115) has been used. By solving (4.116) for Q_{sc}, one obtains

$$Q_{sc} = \frac{C_{sc}}{U_{sc}} \cdot (P_{sc} \cdot R_{sc} + U_{sc}^2),$$

thus

$$2 \cdot U_{sc} \cdot \frac{d}{dt}U_{sc} + \frac{P_{sc}}{C_{sc}} - \frac{d}{dt}U_{sc} \cdot \left(P_{sc} \cdot \frac{R_{sc}}{U_{sc}} + U_{sc}\right).$$

Since $2 \cdot U_{sc} \cdot dU_{sc}/dt = d/dt(U_{sc}^2)$, one obtains

$$\frac{d}{dt}U_{sc}^2 + \frac{P_{sc}}{C_{sc}} - \frac{P_{sc} \cdot R_{sc} \cdot \frac{d}{dt}U_{sc}^2}{2 \cdot U_{sc}^2} - \frac{\frac{d}{dt}U_{sc}^2}{2} = 0 \qquad \Rightarrow \qquad \frac{d}{dt}\frac{U_{sc}^2}{2} + \frac{P_{sc}}{C_{sc}} - \frac{P_{sc} \cdot R_{sc} \cdot \frac{d}{dt}U^2}{2 \cdot U_{sc}^2} = 0,$$

from whence (4.117) follows. For $R_{sc} = 0$ it is found that $dE_{sc}/dt = d/dt(C_{sc} \cdot U_{sc}^2/2) = -P_{sc}$ (P_{sc} positive during discharge).

Problem 4.32

Derive an analytical solution for the discharge of a supercapacitor with maximum power. Then verify the solution with the data of Figure 4.42.

- Solution

At max power

$$P_{sc} = \frac{Q_{sc}^2}{4 \cdot R_{sc} \cdot C_{sc}^2},$$

then from (4.118)

$$U_{sc} = \frac{Q_{sc}}{2 \cdot C_{sc}},$$

from (4.115)

$$R_{sc} \cdot I_{sc} = \frac{Q_{sc}}{C_{sc}} - \frac{Q_{sc}}{2 \cdot C_{sc}} = \frac{Q_{sc}}{2 \cdot C_{sc}}$$

and

$$\frac{dQ_{sc}}{dt} = -\frac{Q_{sc}}{2 \cdot R_{sc} \cdot C_{sc}}.$$

Thus

$$\begin{aligned}
Q_{sc}(t) &= Q_0 \cdot e^{-\frac{t}{2 \cdot C_{sc} \cdot R_{sc}}}, \\
I_{sc}(t) &= \frac{Q_0}{2 \cdot C_{sc} \cdot R_{sc}} \cdot e^{-\frac{t}{2 \cdot C_{sc} \cdot R_{sc}}}, \\
U_{sc}(t) &= \frac{Q_0}{2 \cdot C_{sc}} \cdot e^{-\frac{t}{2 \cdot C_{sc} \cdot R_{sc}}}, \\
P_{sc}(t) &= \frac{Q_0^2}{4 \cdot R_{sc} \cdot C_{sc}^2} \cdot e^{-\frac{t}{C_{sc} \cdot R_{sc}}}.
\end{aligned}$$

For $t = 2$ s, $C_{sc} = 12.5$ F, $R_{sc} = 0.08\Omega$, and $Q_0 = 800$ C, find

$$\begin{aligned}
Q_{sc}(t) &= 800 \cdot e^{-\frac{2}{2 \cdot 12.5 \cdot 0.08}} = 294 \text{ C}, \\
I_{sc}(t) &= \frac{800}{2 \cdot 12.5 \cdot 0.08} \cdot e^{-\frac{2}{2 \cdot 12.5 \cdot 0.08}} = 147 \text{ A}, \\
U_{sc}(t) &= \frac{800}{2 \cdot 12.5} \cdot e^{-\frac{2}{2 \cdot 12.5 \cdot 0.08}} = 12 \text{ V}, \\
P_{sc}(t) &= \frac{800^2}{4 \cdot 0.08 \cdot 12.5^2} \cdot e^{-\frac{2}{12.5 \cdot 0.08}} = 1.7 \text{ kW}.
\end{aligned}$$

Problem 4.33

Yet another definition of supercapacitor efficiency that is sometimes found (e.g., in [222]) is

$$\eta_{sc,d} = 1 - 2\frac{\tau}{t_f},$$

during discharge at constant current, and

$$\eta_{sc,c} = \frac{1}{1 + 2\frac{\tau}{t_f}},$$

during charge at constant current, where $\tau = C_{sc} \cdot R_{sc}$ and t_f has the same meaning as in the text. Explain this definition.

- Solution

The energy amount that can be obtained from a fully-charged supercapacitor with constant-current discharge in the ideal case of negligible resistance is obtained from (4.119) as

$$E_{d,id} = \frac{Q_0^2}{2 \cdot C_{sc}},$$

which coincides with the maximum stored energy defined in (4.129). By defining the efficiency as

$$\eta_{sc,d} = \frac{E_d}{E_{d,id}},$$

find the value in the problem statement.

For charge, define the efficiency as

$$\eta_{sc,c} = \frac{|E_{c,id}|}{|E_c|},$$

and find the value in the problem statement with $|E_{c,id}| = E_{d,id}$.

Electric Power Links

Problem 4.34

Consider again Problem 4.9 and account for a battery internal resistance of $0.025\,\Omega$.

- Solution

Combine DC motor equations and battery equations with $U_b = U_m$ and $I_b = I_m$. The relationship linking the motor torque, speed, and the DC–DC converter duty cycle is

$$\alpha \cdot U_{oc} - \frac{(R_a + \alpha^2 \cdot R_b) \cdot T_m}{\kappa_a} - \kappa_i \cdot \omega_m = 0.$$

The flux weakening region limit is obtained for $\alpha = 1$ as $0.3 \cdot T_m + 0.25 \cdot \omega_m = 50\,\text{V}$ (axis intercepts at 200 rad/s and 167 Nm). The current limit is obtained by setting $\alpha = 1$ and $\omega_m \cdot T_m = P_{m,max}$. The result is the same as in Problem 4.9 but now R_a should be replaced by $R_a + R_b = 0.075\,\Omega$. The new solution is $I_{m,max} = 93\,\text{A}$ (increase).

In the case (i) only the calculation of α changes, since $U_a = 28\,\text{V}$ and $I_a = 60\,\text{A}$ are still admissible values. Knowing that

$$U_b = U_{oc} - R_b \cdot I_b,$$
$$\alpha = \frac{U_a}{U_b} = \frac{I_b}{I_a},$$

one obtains

$$U_a = U_{oc} \cdot \alpha - R_b \cdot I_a \cdot \alpha^2,$$

from whence $\alpha = 57\%$ (increase) and correspondingly $U_b = 49\,\text{V}$, $I_b = 34\,\text{A}$ (increase). Alternatively, α can be directly calculated from the general equation above.

For the case (ii) again R_a is replaced by the sum of the two resistances. The duty cycle is still fixed at $\alpha = 100\%$ and the flux is now $\kappa_i = \kappa_a = 0.14\,\text{Wb}$. Thus the flux weakening increases.

Problem 4.35

For an electric drive including a battery, a step-down DC–DC converter (chopper), and a DC motor, calculate the duty cycle that maximizes the regenerated power during braking. Calculate the corresponding motor current, using the data presented in Problem 4.9.

- Solution

The limitation is imposed by the step-down converter for which $R = U_a/U_b = 1 - \alpha$. Evaluate R as a function of ω_m and I_a:

$$R \cdot U_{oc} - R_b \cdot R^2 \cdot I_a - R_a \cdot I_a - \kappa_i \cdot \omega = 0.$$

For $R = 0$,

$$I_a = -\kappa_i \cdot \frac{\omega}{R_a},$$

but $I_b = R \cdot I_a = 0$ (no recuperation). For $R = \kappa_i \cdot \omega_m / U_{oc}$, $I_a = 0$ and $I_b = 0$ (again no recuperation). Thus there must be a value $R > 0$ that maximizes the recuperated power. The power is

$$P_a = P_b = I_a \cdot (R_a \cdot I_a + \kappa_i \cdot \omega_m) = R_a \cdot I_a^2 + \kappa_i \cdot \omega_m \cdot I_a,$$
$$\frac{dP_a}{dI_a} = 2 \cdot R_a \cdot I_a + \kappa_i \cdot \omega_m = 0 \quad \Rightarrow \quad I_a = -\kappa_i \cdot \frac{\omega_m}{2 \cdot R_a} = I_{a,min}.$$

For this value of current

$$R \cdot U_{oc} + \frac{R_b}{2 \cdot R_a} \cdot \kappa_i \cdot \omega_m \cdot R^2 - \frac{\kappa_i \cdot \omega}{2} = 0 \quad \Rightarrow \quad R_{min}.$$

And inserting the data of Problem 4.9,

$$I_{a,min} = -\frac{\kappa_i \cdot \omega_m}{2 \cdot R_a} = -\frac{0.25 \cdot 100}{2 \cdot 0.05} = -250 \text{ A},$$

$$R^2 \cdot (R_b \cdot I_{a,min}) - u \cdot U_{oc} + R_a \cdot I_{a,min} + \kappa_i \cdot \omega = 0 \quad \Rightarrow \quad R_{min} = 24\%.$$

Problem 4.36

Consider an electric drive including a battery, a boost DC–DC converter and a motor. Derive a relationship between the maximum torque curve of the motor and the converter ratio, assuming that for $\omega_m > \omega_b$, $P_{max} \approx U_{max} \cdot I_{max}$. Conceive a strategy to perform quasistatic simulations in this case.

- Solution

A simplified expression for the motor maximum torque is

$$T_{max} = \min\{k \cdot I_{max}, P_{max}/\omega_m\}$$

where the constant k is given by κ_a in DC motors and by $3/2 \cdot p \cdot \varphi_m$ in PMSMs. The maximum power is $P_{max} = I_{max} \cdot U_{max}$ where U_{max} is the voltage at the DC side of the motor. One obtains

$$P_{max} = U_{max} \cdot I_{max} = I_{max} \cdot R \cdot (U_{oc} - R_b \cdot R \cdot I_{max}) = f(R)$$

In backward modeling, the required torque T_m has to be saturated by the maximum value T_{max}. However, this value depends on R. Physically, there is one value of R that realizes the desired speed and torque (see a similar situation in Problem 4.34). For simulation purposes, one could map the conversion ratio as a function of motor speed and torque and then feed the maximum torque map. Alternatively, T_{max} could be mapped as a function of speed and voltage U_{max} and the latter calculated as $U_b \cdot I_b / I_{max}$.

Problem 4.37

Consider a semi-active power link with a battery, a supercapacitor, an electric motor, and a DC–DC converter on the supercapacitor branch. Derive an analytical relationship between the control factor u (4.132) and the DC–DC converter voltage ratio R. Calculate the values of R to obtain a pure battery supply ($u = 0$) or a pure supercapacitor supply ($u = 1$). Describe the supercapacitor on a quasistatic basis, i.e.,

$$C_{sc} \cdot \frac{U_{s0} - U_{sc}}{\tau} = -I_{sc},$$

where τ is the time step.

- Solution

The battery equation is $U_b = U_{oc} - R_b \cdot I_b$. The DC link equations are $U_m = R \cdot U_{sc} = U_b$. Additionally, $I_b + I_{sc}/R = I_m$. The supercapacitor equation is given in the problem formulation. There are five equations in the six variables U_b, I_b, U_{sc}, I_{sc}, U_m, I_m. However, in dynamic modeling I_m is given from the downstream powertrain. Thus all the other quantities can be calculated as a function of R. The result is

$$U_m = U_b = \frac{\frac{U_{oc}}{R_b} + \frac{C}{\tau}\frac{U_{s0}}{R} - I_m}{\frac{1}{R_b} + \frac{C}{\tau R^2}},$$

from whence U_{sc}, I_{sc}, and I_b are calculated as well. The supercapacitor power P_{sc} is

$$P_{sc} = U_{sc} I_{sc} = \frac{U_m}{R}\frac{C}{\tau}(U_{s0} - U_{sc})$$

and the control ratio u is

$$u = \frac{P_{sc}}{P_m} = \frac{1}{I_m}\left(\frac{CU_{s0}}{\tau R} - \frac{C}{\tau R^2}\frac{\frac{U_{oc}}{R_b} + \frac{C}{\tau}\frac{U_{s0}}{R} - I_m}{\frac{1}{R_b} + \frac{C}{\tau R^2}}\right) = u(R, I_m).$$

To find $u = 0$, the DC–DC converter must be regulated such as

$$R = \frac{U_{oc} - R_b I_m}{U_{s0}}.$$

To obtain $u = 1$, the DC–DC converter must be regulated such as R is the solution of the quadratic equation

$$\tau I_m R^2 - CU_{s0}R + CU_{oc} = 0.$$

The condition to have a solution is

$$(CU_{s0})^2 - 4\tau I_m C U_{oc} > 0.$$

Therefore, pure supercapacitor operation is allowed for

$$I_m < \frac{CU_{s0}^2}{4\tau U_{oc}}.$$

Torque Couplers

Problem 4.38

Consider a through-the-road parallel hybrid. The torque coupling has the following characteristics: transmission ratio between the rear-axle motor and the wheels $\gamma_m = 11$, transmission ratios between the front-axle engine and the wheels $\gamma_e = \{15.02, 8.09, 5.33, 3.93, 3.13, 2.59\}$, wheel radius $r_{wh} = 31.7\,\text{cm}$. Moreover, $T_{m,max} = -T_{m,min} = 140\,\text{Nm}$, $P_{m,max} = -P_{m,min} = 42\,\text{kW}$, engine speed limited to $\omega_{e,max} = 4500\,\text{rpm}$, engine maximum torque $T_{e,max} =$

$1.34 \cdot 10^{-5} \cdot \omega_e^3 - 0.0149 \cdot \omega_e^2 + 4.945 \cdot \omega_e - 243\,(\mathrm{Nm})$, engine minimum torque $T_{e,min} = -20\,\mathrm{Nm}$. For a driving situation with $V = 30\,\mathrm{m/s}$ and $T_{wh} = 675\,\mathrm{Nm}$, determine the limits imposed by the motor operation on the engine operation.

- Solution

The engine speed for various gears is

$$\omega_e = \gamma_e \cdot \frac{V}{r_{wh}} = \{947, 510, 336, 248, 197, 163\}\ \mathrm{rad/s}.$$

Since $\omega_{e,max} = 471\,\mathrm{rad/s}$ only the third to sixth gears are admissible. The engine maximum torque $T_{e,max}$ for the four admissible speeds is

$$T_{e,max} = \{242, 270, 255, 225\}\ \mathrm{Nm}.$$

However, the torque coupling equation reads

$$T_e \cdot \gamma_e + T_m \cdot \gamma_m = T_{wh}.$$

Since $T_m \geq T_{m,min}$, consequently,

$$T_e \leq \frac{T_{wh} - \gamma_m \cdot T_{m,min}}{\gamma_e} = \{259, 351, 441, 533\}.$$

In all cases, the motor imposed limits overshadow the engine physical limits during generating operation. For motoring operation,

$$T_e \geq \frac{T_{wh} - \gamma_m \cdot T_{m,max}}{\gamma_e} = \{1.6, 2.1, 2.7, 3.2\}.$$

that prevents, e.g., purely electric operation ($T_e = 0$).

Power Split Devices

Problem 4.39

Consider a power-split combined hybrid powertrain with a planetary gear set linking the engine, the generator, and the output shafts with the following Willis relation

$$\omega_g = 3.6 \cdot \omega_e - 2.6 \cdot \omega_f.$$

The second electric machine is mounted directly on the output shaft without any reduction gear. The generator has the following characteristics (both in motor and in generator modes): maximum torque = 160 Nm, maximum power = 25 kW, maximum speed = 1200 rad/s. The motor has the following characteristics (both in motoring and in generating mode): maximum torque = 400 Nm, maximum power = 25 kW, and maximum speed = 700 rad/s. The

S.I. engine has the following characteristics: maximum torque curve = {0 87 110 107} Nm @ {400 1300 3700 5000} rpm. The maximum battery power is 35 kW. Consider a driving situation in which the speed of the output shaft is 343 rad/s and the required torque is 129 Nm. Supposing that the decision variables of the energy management strategy are engine speed and torque, evaluate the admissible range of these variables.

- Solution

The degrees of freedom are selected as the engine torque and speed. Thus the admissible range is drawn on the engine speed-torque plane. The engine limits themselves are drawn as straight lines (curve $\mathcal{A}$)

$$\begin{aligned}
&T_e = 0, && \text{for} && \omega_e \leq 42\\
&T_e = \frac{87}{94} \cdot (\omega_e - 42), && \text{for} && 42 < \omega_e \leq 136\\
&T_e = 87 + \frac{23}{251} \cdot (\omega_e - 136), && \text{for} && 136 < \omega_e \leq 387\\
&T_e = 110 - \frac{3}{136} \cdot (\omega_e - 387), && \text{for} && 387 < \omega_e \leq 524
\end{aligned}$$

The motor speed is fixed, i.e., $\omega_m = \omega_f = 343$ rad/s. The motor base speed is $25000/400 = 62$ rad/s. Thus the max torque is $25000/343 = 73$ Nm. The relationship between engine torque, motor torque, and output torque is

$$T_e \cdot \frac{2.6}{3.6} = T_f - T_m.$$

Thus the engine torque corresponding to the maximum motor torque (curve $\mathcal{B}$) is

$$T_e = \frac{129 - 73}{0.72} = 78 \text{ Nm}.$$

Only engine torque values greater than 78 Nm are admissible, since they do not saturate the motor limits.

The relationship between generator torque and engine torque is $T_g = T_e/3.6$. The engine torque corresponding to the maximum generator torque is $160 \cdot 3.6 = 576$ Nm, thus far beyond the engine limits. The generator base speed is $25000/160 = 156$ rad/s (in both rotating directions). The engine speed corresponding to the generator base speed is

$$\frac{\omega_g + 2.6 \cdot \omega_f}{3.6} = \pm 156 + 2.6 \cdot 343 = 291 \text{ and } 204 \text{ rad/s}.$$

Thus outside of this range the max power limit of the generator could limit the engine operation. The max power limit of the generator is $T_g = 25000/\omega_g$, thus in engine variables (curve $\mathcal{C}$) is

$$\frac{T_e}{3.6} = \frac{25000}{3.6 \cdot \omega_e - 892}.$$

The intersection of curve $\mathcal{C}$ with the curve $T_e = 110$ Nm is at 475 rad/s. To be more precise, the intersection should be made with the fourth branch of the curve $\mathcal{A}$, leading to a quadratic equation.

Neglecting in a first approximation the motor and generator losses, the battery power is $P_b = P_m - P_g = P_f - P_e$. The output power is $P_f = 129 \cdot 343 = 44247$ W. Thus the limit $P_b = 35$ kW in engine variables becomes $\omega_e \cdot T_e = 44247$ W $- 35000$ W $= 9247$ W (curve $\mathcal{D}$). The intersection with curve $\mathcal{B}$ is at 118 rad/s. The curve A at this engine speed gives 70 Nm, which is below curve $\mathcal{B}$. Thus the battery constraint is not active at these driving conditions. All the other limits (generator minimum torque, motor minimum torque) are not active as well.

The engine admissible range is thus between curve $\mathcal{B}$ ($T_e = 78$ Nm), curve $\mathcal{A}$ (between $\omega_e = 126$ rad/s and 475 rad/s), and curve $\mathcal{C}$ (between $\omega_e = 475$ rad/s and 524 rad/s).

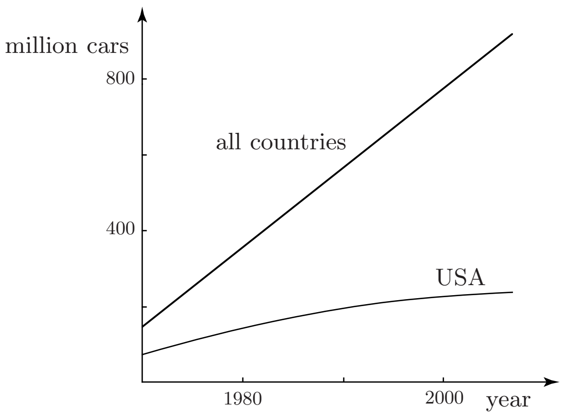

Fig. 10.11. *Schematic* representation of the development of the number of passenger cars operated worldwide.

Problem 4.40

Derive the coupling matrix for the four torque levels of a PSD from the elements of the kinematic matrix in the case of quasistatic modeling. Do the same in the case of forward modeling, i.e., derive (4.166).

- Solution

In backward modeling the kinematic matrix generally reads

$$\omega_e = A \cdot \omega_f + B \cdot \omega_g,$$
$$\omega_m = C \cdot \omega_f + D \cdot \omega_g.$$

Moreover, the power balance of the PSD reads

$$T_e \cdot \omega_e + T_m \cdot \omega_m = T_g \cdot \omega_g + T_f \cdot \omega_f.$$

In backward modeling one wants to calculate T_e and T_m as a function of T_f and T_g. To do so, use the kinematic relationships

$$T_e \cdot A \cdot \omega_f + T_e \cdot B \cdot \omega_g + T_m \cdot C \cdot \omega_f + T_m \cdot D \cdot \omega_g = T_g \cdot \omega_g + T_f \cdot \omega_f.$$

By equating the factors of ω_g and, respectively, ω_f, one obtains

$$\begin{aligned} T_e \cdot A + T_m \cdot C &= T_f, \\ T_e \cdot B + T_m \cdot D &= T_g, \end{aligned}$$

thus,

$$\begin{aligned} T_e &= \frac{1}{A \cdot D - B \cdot C} \cdot (D \cdot T_f - C \cdot T_g), \text{ and} \\ T_m &= \frac{1}{A \cdot D - B \cdot C} \cdot (-B \cdot T_f + A \cdot T_g). \end{aligned}$$

In the case of forward modeling, the input variables are ω_f and ω_g, while the output variables are T_g and T_f. As a result, Equation 4.141 is obtained.

Problem 4.41

For the compound power split device architecture shown in Fig. 4.56, derive the kinematic matrix **M** and the values of the two kinematic nodes.

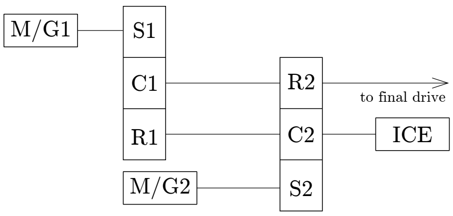

Fig. 10.12. Compound power-split configuration for Problem 4.41

- Solution

Let the first electric machine be the motor and the second the generator. The general relationship of a PGS is

$$\omega_r + z \cdot \omega_s = (1 + z) \cdot \omega_c.$$

For the first PGS,

$$\omega_e + z_1 \cdot \omega_m = (1 + z_1) \cdot \omega_f.$$

For the second PGS,

$$\omega_f + z_2 \cdot \omega_g = (1 + z_2) \cdot \omega_e.$$

Thus the kinematic matrix is $A = 1/(1+z_2)$, $B = z_2/(1+z_2)$, $C = (z_1 + z_2 + z_1 z_2)/z_1(1+z_2)$, $D = -1/z_1(1+z_2)$. The two nodes are calculated (see Problem 4.42) as $K_1 = 1/(1+z_1)$ and $K_2 = 1 + z_2$).

Problem 4.42

Derive equation (4.167) for K_v as a function of K and equation (4.168) for r as a function of K, including the definitions of K_1 and K_2.

- Solution

We use the definition of M as in Problem 4.40. By defining $K = \omega_f/\omega_e$, we have

$$\omega_g/\omega_e = A \cdot K + B \qquad \text{and} \qquad \omega_m/\omega_e = C \cdot K + D.$$

Thus,

$$K_v = \frac{\omega_m}{\omega_g} = \frac{C \cdot K + D}{A \cdot K + B} = \frac{D}{\frac{B \cdot \left(1 - \frac{K}{K_1}\right)}{1 - \frac{K}{K_2}}}$$

from whence the definition of $K_1 = -D/C$ and $K_2 = -B/A$. The power split ratio is

$$r = \frac{\omega_g}{\omega_e} \cdot \frac{T_g}{T_e} = \frac{\omega_m}{\omega_e} \cdot \frac{T_m}{T_e}.$$

By using the torque matrix calculated in Problem 10,

$$r = (A \cdot K + B) \cdot \frac{D \cdot \frac{T_f}{T_e} + C}{B \cdot C - A \cdot D} = (C \cdot K + D) \cdot \frac{B \cdot \frac{T_f}{T_e + A}}{B \cdot C - A \cdot D}.$$

Equating the last two equations and defining $T_f/T_e = X$, derive that

$$(D \cdot X + C) \cdot (A \cdot K + B) = (C \cdot K + D) \cdot (B \cdot X + A)$$

from whence

$$X \cdot (D \cdot (A \cdot K + B) - B \cdot (C \cdot K + D)) = A \cdot (C \cdot K + D) - C \cdot (A \cdot K + B)$$
$$\Rightarrow X \cdot (A \cdot D \cdot K - B \cdot C \cdot K) = A \cdot D - B \cdot C,$$

thus $X = 1/K$. Hence,

$$r = \frac{1}{K} \cdot \frac{(C \cdot K + D) \cdot (B + A \cdot K)}{B \cdot C - A \cdot D} = \frac{1}{K} \cdot D \cdot B \cdot \frac{(1 + \frac{K}{K_1}) \cdot (1 + \frac{K}{K_2})}{B \cdot C - A \cdot D}.$$

As $B \cdot C - A \cdot D$ can be written as

$$D \cdot B \cdot \left(\frac{C}{D} - \frac{A}{B}\right) = D \cdot B \cdot \left(\frac{1}{K_2} - \frac{1}{K_1}\right),$$

equation (4.143) is obtained.

Problem 4.19

Derive (4.163)-(4.164).

- Solution

Speed balance:

$$\omega_e = a \cdot \omega_g + (1-a) \cdot \omega_f,$$

where $a = z/(1+z)$, $1-a = 1/(1+z)$.

Steady-state torque balance:

$$T_e = T_g + T_r,$$
$$T_r = T_f - T_m.$$

Dynamic torque balance (note the sign of inertia torque in the right-hand side):

$$T_e - \Theta_c \cdot \frac{d\omega_e}{dt} = T_g + \Theta_s \cdot \frac{d\omega_g}{dt} + T_r + \Theta_r \cdot \frac{d\omega_f}{dt}.$$

Power balance:

$$\begin{aligned}
\left(T_e - \Theta_c \cdot \frac{d\omega_e}{dt}\right) \cdot \omega_e &= \left(T_g + \Theta_s \cdot \frac{d\omega_g}{dt}\right) \cdot \omega_g + \left(T_r + \Theta_r \cdot \frac{d\omega_f}{dt}\right) \cdot \omega_f = \\
&= \left(T_e - \Theta_c \cdot \frac{d\omega_e}{dt}\right) \cdot a \cdot \omega_g + \left(T_e - \Theta_c \cdot \frac{d\omega_e}{dt}\right) \cdot (1-a) \cdot \omega_f = \\
&= \left(T_g + \Theta_s \cdot \frac{d\omega_g}{dt}\right) \cdot \omega_g + \left(T_r + \Theta_r \cdot \frac{d\omega_f}{dt}\right) \cdot \omega_f = \\
&= \left(T_e - \Theta_c \cdot a \cdot \frac{d\omega_g}{dt} - \Theta_c \cdot (1-a) \cdot \frac{d\omega_f}{dt}\right) \cdot a \cdot \omega_g + \\
&\quad + \left(T_e - \Theta_c \cdot a \cdot \frac{d\omega_g}{dt} - \Theta_c \cdot (1-a) \cdot \frac{d\omega_f}{dt}\right) \cdot (1-a) \cdot \omega_f
\end{aligned}$$

from whence, by equalizing the terms multiplying ω_g and those multiplying ω_f, one obtains

$$\begin{aligned}
a \cdot T_e - \Theta_c \cdot a^2 \cdot \frac{d\omega_g}{dt} - \Theta_c \cdot (1-a) \cdot a \cdot \frac{d\omega_f}{dt} &= T_g + \Theta_s \cdot \frac{d\omega_g}{dt} \\
(1-a) \cdot T_e - \Theta_c \cdot a \cdot (1-a) \cdot \frac{d\omega_g}{dt} - \Theta_c \cdot (1-a)^2 \cdot \frac{d\omega_f}{dt} &= T_r + \Theta_r \cdot \frac{d\omega_f}{dt}
\end{aligned}$$

which are the equations sought, since

$$a \cdot (1-a) = \frac{z}{(1+z)^2}, \quad a^2 = \frac{z^2}{(1+z)^2}, \quad \text{and } (1-a)^2 = \frac{1}{(1+z)^2}.$$

Problem ??

Extend equations (4.149)-(4.150) to the case where there are losses in the planetary gearset.

- Solution

Assume that e_s and e_r are the efficiencies of the contacts carrier-sun and carrier-ring, respectively, the power balance neglecting the inertia terms is simply written as

$$\begin{aligned} P_e &= \frac{1}{e_s} \cdot P_g + \frac{1}{e_r} \cdot P_r, & \\ T_e \cdot \omega_e &= \frac{T_g}{e_s} \cdot \omega_g + \frac{T_r}{e_r} \cdot \omega_f, & \text{from whence} \\ T_g &= e_s \cdot a \cdot T_e, & \text{and} \\ T_r &= e_r \cdot (1-a) \cdot T_e, & \text{where } a = \frac{z}{1+z}. \end{aligned}$$

The lost power P_{lost} is

$$\begin{aligned} P_{lost} &= T_e \cdot \omega_e - T_g \cdot \omega_g - T_r \cdot \omega_f = \\ &= a \cdot T_e \cdot \omega_g \cdot (1-e_s) + (1-a) \cdot T_e \cdot \omega_f \cdot (1-e_r). \end{aligned}$$

Non-electric Hybrid Propulsion Systems

Hybrid-inertial Powertrains

Problem 5.1

Derive a Ragone curve similar to (5.1) and (5.2) for a flywheel battery. Then evaluate the maximum energy and power. Use a simplified expression for the loss power of the type $P_l = R \cdot \omega_f^2$.

- Solution

  Assume constant power output of the flywheel and assume the flywheel starts at speed ω_0.
  newline

a) Dynamics of the flywheel
   For constant power P_f, the dynamic equation:

   $$\Theta_f \cdot \omega_f \cdot \dot{\omega_f} = -P_f - R \cdot \omega_f^2,$$

   which can be rewritten as the following integral:

$$\begin{aligned}
\frac{\Theta_f \cdot d\omega_f}{P_f + R \cdot \omega_f^2} &= -dt \\
\frac{\frac{\Theta_f}{2R} \cdot d(R\omega_f^2 + P_f)}{(R\omega_f^2 + P_f)} &= -dt \\
\frac{\Theta_f}{2R} \ln\left[P_f + R\omega_f^2(\infty)\right]\Big|_{t_0}^{t_\infty} &= -t\Big|_0^\infty \\
\frac{\Theta_f}{2R} \ln\left[\frac{P_f + R\omega_f^2(\infty)}{P_f + R\omega_f^2(0)}\right] &= -(t_\infty - t_0)
\end{aligned}$$

By setting integral limits from $t = 0, \omega_f = \omega_0$ to $t = t_\infty, \omega_f = 0$ yields

$$\begin{aligned}
\frac{\Theta_f}{2R} \ln\left[\frac{P_f + R\omega_f^2(\infty)}{P_f}\right] &= -(t_\infty) \\
t_\infty = \tau \cdot \ln(1 + R \cdot \omega_0^2 / P_f).
\end{aligned}$$

where $\tau \triangleq \Theta_f/(2R)$.

b) Energy delivery

$$\begin{aligned}
E_f &= P_f \cdot t_\infty \\
&= P_f \cdot \left\{ -\frac{\Theta_f}{2R} \cdot \ln\left[\frac{1}{1 + R\frac{\omega_0^2}{P_f}}\right]\right\} \\
&= \frac{\Theta_f}{2R} \cdot P_f \ln\left[1 + R\frac{\omega_0^2}{P_f}\right].
\end{aligned}$$

c) Initial energy available
The initial energy stored in the flywheel is

$$E_0 = 1/2 \cdot \Theta_f \cdot \omega_0^2.$$

Thus the efficiency can be calculated as

$$\eta_f = \frac{E_f}{E_0} = \frac{P_f}{R \cdot \omega_0^2} \cdot \ln\left[1 + \frac{R \cdot \omega_0^2}{P_f}\right].$$

Discussion

If we plot out the Ragone curve $E_f = E_f(P_f)$, it appears to be a monotonously increasing curve, contrarily to battery and supercapacitors. That is, the larger is the power, the larger is the energy that can be extracted from the flywheel and thus its efficiency.

Problem 5.2

Dimension a flywheel for the following application (F1 KERS): $P_{max} = 60\,\mathrm{kW}$, $E_{max} = 500\,\mathrm{kJ}$, $\omega_{max} = 60000\,\mathrm{rpm}$. Assume $\beta = 0.5$, $\rho = 1658\,\mathrm{kg/m^3}$ (carbon fiber), $\rho_a^{0.8}\eta_a^{0.2} = 0.48$ (sealed flywheel).

- Solution

  Assume β denotes the the geometric ratio of the wheel width over the wheel diameter; $d = 0.2\,[\mathrm{m}]$, and q denotes the ratio between inner and outer diameters of the flywheel rings.

a) Calculate the moment of inertia
The maximum speed is $\omega_{max} = 60000\,\mathrm{rpm} = 60000 \cdot 2\pi/60 = 6283\,\mathrm{rad/s}$.
From the maximum energy, the moment of inertia can be calculated with

$$\Theta_f = \frac{2 \cdot E_{max}}{\omega_{max}} = 2 \cdot 500 \cdot 10^3/(6283)^2 = 0.0233 \ \mathrm{kg} \cdot \ \mathrm{m}^2.$$

b) Find the flywheel diameter
By setting $P_{l,a} = P_{max}$ and neglecting bearing losses, the diameter is found from (5.5) by setting $P_{l,a} = P_{max}$, i.e.,

$$\begin{aligned} 0.04 \cdot \rho_a^0.8 \cdot \eta_a^0.2 \cdot u^2.8(t) \cdot (\beta + 0.33) = P_{l,a}(t) = P_{meax} \\ 0.04 \cdot 0.48 \cdot d^{4.6} \cdot (6283/2)^{2.8} \cdot 0.83 = 60 \cdot 10^3, \end{aligned}$$

from whence $d \approx 0.2\,\mathrm{m}$. Thus $b = \beta \cdot d = 0.5 \cdot 0.2 = 0.1\,\mathrm{m}$.

c) Dimension the flywheel mass with material property
Using (5.7), find the geometric ratio q from the equation:

$$\begin{aligned} \Theta_f = \frac{\pi}{2} \cdot \rho \cdot b\frac{d^4}{16}(1 - q^4) \\ 0.026 = 3.14 \cdot 1658/32 \cdot (1 - q^4) \cdot 0.1 \cdot 0.2^4, \end{aligned}$$

from whence $q = 0.4069$.
The mass is obtained from (5.8) as

$$\begin{aligned} m_f &= \rho \cdot \int_{r_{in}}^{r_{out}} 1 \cdot (2\pi r)dr \\ m_f &= \pi\rho b\frac{d^2}{4} \cdot (1 - q^2) \\ m_f &= 0.1 \cdot 0.2^2 \cdot 3.14 \cdot 1658 \cdot (1 - 0.4069^2)/4 = 4.346 \ \mathrm{kg}. \end{aligned}$$

Problem 5.3

Evaluate the charging efficiency of the flywheel of Problem 5.2 for a braking at maximum power for 2 s. Evaluate the round-trip efficiency.

- Solution

a) Charging efficiency

(i) estimate the coefficient of loss power using maximum achievable power
During charging, the flywheel dynamics reads $\Theta_f \cdot \omega_f \cdot \dot{\omega_f} = P_f - R \cdot \omega_f^2$.
Using the data of Problem 5.2, an approximation for R could be

$$R = P_{max}/\omega_{max}^2 = 60 \cdot 10^3/(60000 \cdot 2\pi/60)^2 = 0.0015.$$

Thus, the charging speed trajectory ω_f satisfies:

$$\theta_f \ cdot \omega_f \cdot \dot{\omega_f} = P_f - R \cdot \omega_f^2$$
$$\text{which gives, } \tau \ln \left[\frac{P_f - R \cdot \omega_f^2(\infty)}{P_f - R \cdot \omega_f^2(0)} \right] = -(t_\infty - t_0)$$

By definition, $\tau \triangleq \Theta_f/(2R) = 8.332\,\mathrm{s}$. Therefore, beginning at rest, the speed profile is as follows, with the maximum braking power charging the flywheel for 2 s:

$$\omega_f = \frac{P_f}{R} \left[1 - e^{-\frac{t}{\tau}} \right],$$
$$\omega_f|_{t=2\,\mathrm{s}} = \sqrt{\frac{60 \times 10^3}{1.52 \times 10^{-3}} \left[1 - e^{-\frac{2}{8.332}} \right]} = 2902\,\mathrm{rad/s}.$$

(ii) Charging efficiency
Charging efficiency is the ratio of available energy storage to available energy input from regenerative braking:

$$\begin{aligned} E_{in} =& P_f \cdot t \\ E_{out} =& \frac{1}{2}\,\Theta_f\,\omega_f^2 = \frac{1}{2}\,\Theta_f\,\frac{P_f}{R} \left[1 - e^{-\frac{t}{\tau}} \right] \\ \eta_c =& \frac{E_{out}}{E_{in}} = \frac{\Theta_f}{2\,Rt} \left[1 - e^{-\frac{t}{T}} \right]. \\ =& \frac{8.332\,\mathrm{s}}{2\,\mathrm{s}} \left[1 - e^{-\frac{2\,\mathrm{s}}{8.332\,\mathrm{s}}} \right] = 88.90\%. \end{aligned}$$

b) Calculate discharge and thus roundtrip efficiency
The initial speed for discharging equals the final speed of charging:

$$\omega_{0,disch} = \omega_{f,ch} = 2902\,\mathrm{rad/s}.$$

Thus, the total discharging energy equals:

$$E_{f,disch} = \frac{\Theta}{2\,R}\,P_f \,\ln \left[1 + \frac{R \cdot \omega_{0,disch}^2}{P_f} \right].$$

With the energy storage in the flywheel, the discharging efficiency can be calculated:

$$\eta_{disch} = \frac{E_{f,disch}}{E_{out}} = \left\{ \frac{\ln\left[1 + \frac{R\cdot\omega_{0,disch}^2}{P_f}\right]}{1 - e^{-\frac{t_{ch}}{\tau}}} \right\} = 90.64\%$$

Finally, the round-trip efficiency is therefore $\eta = \eta_{ch} \cdot \eta_{disch} = 80.58\%$.

Discussion

Generally, the maximum energy of flywheel is used to estimate the moment of inertia and mass, while the maximum power is used to estimate the coefficient of loss power.

Problem 5.4

Evaluate the CVT ratio $\nu(t)$ during a deceleration of a vehicle equipped with a flywheel-based KERS and the opening time of the clutch. Use the flywheel data of Problem 5.3. The flywheel is connected to the input stage of the CVT through a fixed-reduction gear with ratio 8.33. Final drive and wheel ratio $\left(\frac{\gamma_{fd}}{r_w}\right) = 13$. Initial conditions: $v(0) = 80\,\text{km/h}$, $\omega_f(0) = 10000\,\text{rpm}$, $m_v = 600\,\text{kg}$, braking time 2 s (assume a constant braking power), CVT range $\nu_{max}/\nu_{min} = 6$.

- Solution

a) Calculate the range of CVT ratio
As for the initial speed,

$$\nu(0) = \frac{\omega_f(0)}{\gamma_f \cdot \nu(0) \cdot \frac{\gamma_{fd}}{r_w}} = \frac{10000 \times 2\,\pi/60}{8.33 \cdot 80/3.6 \cdot 13} = 0.4352.$$

Therefore, given the CVT range, the maximum CVT ratio is as follows:

$$\nu(t) = \nu(0) \times \frac{\nu_{max}}{\nu_{min}} = 6 \times 0.4352 = 2.611.$$

b) List equations from the vehicle side
Given that the vehicle speed trajectory during decceleration is

$$v^2(t) = v^2(0) - 2\,P_b \cdot t_b/m_v$$

, where P_b is the constant braking power which brings the vehicle approximately towards standstill. Thus, the average braking power is

$$P_b = \frac{m_v \cdot \omega^2}{t_b} = 1/2 \cdot 600 \cdot (80/3.6)^2/2 = 74.07\ \text{kW}.$$

The corresponding final speed at the clutch opening time equals:

$$v^2(t_c) = \left(\frac{80}{3.6}\right)^2 - \frac{2 \times 74.07 \times 10^3 \cdot t_c}{600}.$$

c) List equations from the flywheel side
According to the result of Problem 5.3:

$$\frac{P_f}{R} - \omega_f^2(t) = \left[\frac{P_f}{R} - \omega_f^2(0)\right] \cdot e^{-\frac{t}{\tau}}.$$

Using intemediate results from Problem 5.3, the following values are known:

$$P_f = 60\,\text{kW},\ R = 0.01520\,\text{kg} \cdots m^2,\ \tau = 8.332\,\text{s}.$$

Equations from the flywheel side can be established by substituting values in.

d) Solve equations with the CVT ratio constraint

$$\begin{cases} \nu(t_c) = 2.611 = \frac{\omega_f(t_c)}{\nu(t_c)\cdot\gamma_f\cdot\frac{\gamma_{fd}}{r_w}} \\ \omega_f^2(t_c) = -\left[P_f/r - \omega_f^2(0)\right] \cdot e^{-\frac{t}{\tau}+P_f/R} \\ v^2(t_c) = v^2(0) - 2\,P_b \cdot t_c/m_v \end{cases}$$

Substituting one speed with the other, we get:

$$\left(2.611 \times \frac{80}{3.6} \times 8.33 \times 13\right)^2 = \frac{\omega_f^2(t_c)}{v^2(t_c)}. \tag{*}$$

Eliminate the second unknown variable and make equations of t_c only:

$$\frac{1}{3.948 \times 10^7}\left\{P_f/R - \left[Pf/R - \omega_f^2(0)\right] \cdot e^{-\frac{t_c}{\tau}}\right\} = v^2(0) - 2 \times P_b \times t_c/m_v. \tag{10.26}$$

The solution is therefore:

$$\begin{cases} t_c = 1.9999\,\text{s} \approx 2\,\text{s} \\ \omega_f(t_c) = 1431\,\text{rad/s} = 13660\,\text{rpm} \\ v(t_c) = 0.1648\,\text{m/s}. \end{cases}$$

Discussion
After the clutch opening time $t > t_c$, the clutch is open and the vehicle can further decelerate util rest, with the help of tyre friction power.

Hybrid-hydraulic Powertrains

Problem 5.5

Derive a Ragone curve similar to (5.1) and (5.2) for a hydraulic accumulator. Show that for high power this definition is equivalent to that adopted in the text.

- Solution

a) Energy balance
Establish the accumulator's energy balance for Ragone Curve:

$$\begin{cases} \dot{E} = m_g \cdot c_v \cdot \frac{d}{dt}\,\theta_g(t) = -p_g(t) \cdot \frac{d}{dt}\,V_g(t) - h\,A_w(\theta_g(t) - \theta_w) \\ \frac{d}{dt}\,V_g(t) = Q_h(t) \\ P_g(t) = \frac{m_g\,R_g\,\theta_g(t)}{V_g(t)}. \end{cases}$$

b) Power output
As from the textbook, the power of a hydraulic accumulator is output through the change of pressurized flow:

$$P_h = p_g(t) \cdot Q_h(t).$$

Therefore, given that $E = m_g\,c_v\,\theta_g(t)$, rewrite the energy balance as follows:

$$\dot{E} = -P_h - \frac{h\,A_w}{m_g\,c_v} \cdot E + h\,A_w\,\theta_w.$$

Let $\tau = \frac{m_g\,c_v}{h\,A_w}$, and $E^* = E + \tau\,(P_h - h\,A_w\,\theta_w)$, then we have:

$$\frac{dE^*}{E^*} = -\frac{dt}{t}.$$

c) Solve ODE with initial and final conditions.

Assume constant power output.

Assume starting of the cycle is defined as point "B", while the end of power output is reached when gas temperature equals liquid temperature.

$$E|_{t=0} = m_g c_{v,g} \theta_B E|_{t=\infty} = m_g c_{v,g} \theta_w = E_h$$

Solving the ODE subject to these conditions, we get:

$$t_\infty = \tau \cdot \ln\left[\frac{P_h + (E_0 - E_w)}{P_h}\right]$$

So the total energy transmitted is:

$$E_{ha} = P_h \cdot t_\infty = P_h \cdot \tau \cdot \ln\left[1 + \frac{E_0 - E_w}{P_h \cdot \tau}\right]$$

d) Prove the equivalence of validity
According to the reference cycle used for power output:

$$W_{AB} = m_g \cdot c_{v,g}\,(\theta_B - \theta_A) = E_0 - E_w$$

. When $P_h \to \infty$, using L'Hospital rule in limit calculation, we get:

$$E_{ha} = k \cdot (E_0 - E_w) = k \cdot W_{AB},$$

which proves the validity of definition (5.36), as an equivalence of the Ragone Curve function
Discussion

- the energy balance ODE assuming the mode of constant power output is a source of Ragone Curve Function;
- the Ragone Curve shows the energy is positively correlated with the power, so the energy output is hight if the constant power level is higher.

Problem 5.6

Derive (5.48) and (5.50).

- Solution

Fuel-Cell Propulsion Systems

Fuel Cells

Problem 6.1

For high pressures, the thermodynamic properties of gas have to be calculated using the Redlich–Kwong equation of state instead of the ideal gas law. The Redlich–Kwong equation reads

$$p = \frac{\tilde{R} \cdot \vartheta}{\tilde{V} - b} - \frac{a}{\sqrt{\vartheta} \cdot \tilde{V} \cdot (\tilde{V} + b)}, \tag{10.27}$$

where p is pressure, $\tilde{R}$ is the universal gas constant, ϑ is temperature, $\tilde{V}$ is the molar volume. The constants a and b are defined as

$$a = \frac{0.4275 \cdot \tilde{R}^2 \cdot \vartheta_c^{2/5}}{p_c}, \quad b = \frac{0.08664 \cdot \tilde{R} \cdot \vartheta_c}{p_c}, \tag{10.28}$$

where ϑ_c is the temperature at the critical point, and p_c is the pressure at the critical point. Using this equation of state, evaluate the gaseous density of hydrogen at 350 bar, 700 bar, when the gas temperature is 300 K.

- Solution

  Assume the temperature of H_2 is 300K.

a) Gas states and constants
From thermodynamic tables, $\tilde{R} = 8.314\text{J/mol}\cdot\text{K}$, $\theta_c = 32.97\,\text{K}$, $p_c = 1.293$ MPa. Then, we find the constants:

$$\begin{aligned}
a &= \frac{0.4275 \times \tilde{R}^2 \times \theta_c^0.4}{p_c} \\
&= \frac{0.4275 \times 8.314^2 \times 32.97^0.4}{1.293 \times 10^6} \\
&= 9.251 \times 10^{-5}; \\
b &= \frac{0.08664 \times \tilde{R} \times \theta_c}{p_c} \\
&= \frac{0.08664 \times 8.314 \times 32.97}{1.293 \times 10^6} \\
&= 1.837 \times 10^{-5}.
\end{aligned}$$

b) when $p_1 = 350 \times 10^5$ Pa, $\theta_1 = 300$ K.

$$\begin{aligned}
P_1 &= \frac{\tilde{R}\,\theta_1}{\tilde{V_1} - b} - \frac{a}{\sqrt{\theta_1}\cdot\tilde{V_1}\cdot(\tilde{V_1} + b)} \\
350 \times 10^5 &= \frac{8.314 \times 300}{\tilde{V_1} - 1.837 \times 10^{-5}} - \frac{9.251 \times 10^{-5}}{sqrt300 \times \tilde{V_1}(\tilde{V_1} + 1.837 \times 10^{-5})}
\end{aligned}$$

Solve the 3^{rd} order algebraic equation, we get:

$$\begin{cases} \tilde{V_1} = 8.963 \times 10^{-5}\text{m}^3/\text{mol}; \\ \rho_{1,H_2} = \frac{M_h}{\tilde{V_1}} = 22.31\text{kg}/m^2. \end{cases}$$

c) when $p_2 = 700 \times 10^5$ Pa, $\theta_2 = 300$ K.

$$\begin{aligned}
P_2 &= \frac{\tilde{R}\,\theta_2}{\tilde{V_2} - b} - \frac{a}{\sqrt{\theta_2}\cdot\tilde{V_2}\cdot(\tilde{V_2} + b)} \\
700 \times 10^5 &= \frac{8.314 \times 300}{\tilde{V_2} - 1.837 \times 10^{-5}} - \frac{9.251 \times 10^{-5}}{sqrt300 \times \tilde{V_2}(\tilde{V_2} + 1.837 \times 10^{-5})}
\end{aligned}$$

Solve the 3^{rd} order algebraic equation, we get:

$$\begin{cases} \tilde{V_2} = 5.400 \times 10^{-5}\text{m}^3/\text{mol}; \\ \rho_{2,H_2} = \frac{M_h}{\tilde{V_2}} = 37.04\text{kg}/m^3. \end{cases}$$

Discussion

- when $p_1 = 350 \times 10^5$ Pa, $\theta_1 = 300$ K.
  $\rho_{1,idealgas} = \frac{P_1}{R_g\,T_1} = 28.07\text{kg/m}^3$
  Its compressibility factor $Z_1 = \frac{p_1\,\tilde{V}_1}{\tilde{R}\,\theta_1} = \frac{\rho_{1,idealgas}}{\rho_{1,realgas}} = 1.258.$
- when $p_2 = 700 \times 10^5$ Pa, $\theta_2 = 300$ K.
  $\rho_{2,idealgas} = \frac{P_2}{R_g\,T_2} = 56.14\text{kg/m}^3$
  Its compressibility factor $Z_2 = \frac{p_2\,\tilde{V}_2}{\tilde{R}\,\theta_2} = \frac{\rho_{2,idealgas}}{\rho_{2,realgas}} = 1.516.$
- over-estimation of high pressure cases As can be seen from the comparative result, the higher the pressure is, the larger would the over-estimation be.

Problem 6.2

A good approximation of the compressibility factor of hydrogen between pressures p and p_0 is

$$Z = 1 + 0.00063 \cdot \left(\frac{p}{p_0}\right) \tag{10.29}$$

(verify it with the results of Problem 6.1). With this assumption evaluate the energy required to compress 1 kg of hydrogen (from 1 bar) to 350 bar and 700 bar, respectively, at 300 K, under the further assumptions of (i) isothermal compression, (ii) adiabatic compression. Evaluate the result as a percentage of the energy content of hydrogen.

- Solution

a) Derivation of compression work in isothermal and isentropic case:

   (i) Isothermal case
   Given the definition of compressibility,

$$P\,v = Z\,R\theta,$$

   where $Z = 1 + 0.00063 \times \left(\frac{p}{p_0}\right)$. After integrating $dW = v\,dp$ from p_0 to p, we get:

$$\begin{aligned} W_c &= \int_{p_0}^{p} \frac{Z\,R\,\theta}{p}\,dp \\ &= R\,\theta \int_{p_0}^{p} \left(\frac{1}{p}\frac{0.00063}{+}p_0\right)\,dp \\ &= R\,\theta\left[\ln\frac{p}{p_0} + 0.00063\,\frac{p-p_0}{p_0}\right] \end{aligned}$$

   Note that in most cases $p_0 \ll p$.

(ii) Isentropic case
For isentropic processes, $pv^{\gamma} = C$,

$$W_c = \frac{\bar{Z}_{p,p_0} R\theta}{\gamma - 1}\left[\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} - 1\right].$$

where $\bar{Z}$ denotes the average compressibility factor of the initial and final state.

b) $p = 350$ bar, $\theta = 300$ K.

(i) Isothermal case

$$W_c = \frac{8.314}{2 \times 10^{-3}} \times 300 \times [\ln(350) + 0.00063 \times (350 - 1)] = 7.580\text{MJ/kg}.$$

(ii) Isentropic case

$$W_c = \frac{\frac{1+1.258}{2} \times \frac{8.314}{2\times10^{-3}\times300}}{\gamma - 1} \times \left(350^{\frac{\gamma-1}{\gamma}} - 1\right) = 15.25\text{MJ/kg}.$$

(iii) Comparison with LHV
Since for H_2, $LHV = 120$ MJ/kg, the ratio of isothermal compression work to LHV is 6.317%, while that of the isentropic compression work is 12.71%.

c) $p = 700$ bar, $\theta = 300$ K.

(i) Isothermal case

$$W_c = \frac{8.314}{2 \times 10^{-3}} \times 300 \times [\ln(700) + 0.00063 \times (700 - 1)] = 8.719\text{MJ/kg}.$$

(ii) Isentropic case

$$W_c = \frac{\frac{1+1.258}{2} \times \frac{8.314}{2\times10^{-3}\times300}}{\gamma - 1} \times \left(700^{\frac{\gamma-1}{\gamma}} - 1\right) = 19.30\text{MJ/kg}.$$

(iii) Comparison with LHV
Since for H_2, $LHV = 120$ MJ/kg, the ratio of isothermal compression work to LHV is 7.266%, while that of the isentropic compression work is 16.09%.

Problem 6.3

Typical characteristics of various metal-hydride materials (1–4) for hydrogen storage are listed in the following table [350]. Evaluate the energy density for these storage systems.

- Solution

	(1)	(2)	(3)	(4)
Material density (g/cm^3)	6.2	1.25	1.26	0.66
Porosity (%)	50	50	50	50
Mass storage capacity (%)	1.8	5.55	6.5	11.5

a) Energy density for storage
For hydrogen storage, the energy density is given by:

$$\frac{E_{ht}}{V_{ht}} = \frac{H_h\,\xi}{\gamma_{ht}}.$$

where $\xi = P\% \cdot M\%$:

γ_{ht} denotes the reciprocal of material density
$P\%$ denotes the porosity of material
$M\%$ denotes the mass storage capacity

b) Material 1
$\frac{E_{ht}}{V_{ht}}|_{(1)} = 6.2 \times 50\% \times 1.8\% \times 33.33\text{kW h/kg} = 1.860\,\text{kW h/l}$
c) Material 2
$\frac{E_{ht}}{V_{ht}}|_{(2)} = 1.25 \times 50\% \times 5.55\% \times 33.33\text{kW h/kg} = 1.156\,\text{kW h/l}$
d) Material 3
$\frac{E_{ht}}{V_{ht}}|_{(3)} = 1.26 \times 50\% \times 6.5\% \times 33.33\text{kW h/kg} = 1.365\,\text{kW h/l}$
e) Material 4
$\frac{E_{ht}}{V_{ht}}|_{(4)} = 0.66 \times 50\% \times 11.5\% \times 33.33\text{kW h/kg} = 1.265\,\text{kW h/l}$
Discussion
Despite an increase in the gravimetric storage capacity, the energy density still decreases when the material changes from (1) to (4). This is because the size and porosity of the material also matters a lot. Even the specific energy may decrease if the necessary system becomes significantly larger and thus heavier (ancillaries, etc.).

Problem 6.4

Evaluate the increase of energy density obtained with the cryo-compressed tank (CcH2) concept operated at 77 K with respect to conventional, ambient-temperature pressurized tanks.

- Solution

Assume the energy density is evaluated through the storage capacity(hydrogen density) ρ_h, using the method of Problem 6.1.
a) Compressed tank: 350 bar, 300 K
Density has been calculated in Problem 6.1, where

$$\rho_h = 22.31\text{kg/m}^3.$$

b) Cyro-compressed tank: 350 bar, 77 K
Molar specific volume can be calculated by solving:

$$350 \times 10^5 = \frac{8.314 \times 77}{\tilde{V} - 1.837 \times 10^{-5}} - \frac{9.251 \times 10^{-5}}{\sqrt{77} \cdot \tilde{V}\,(\tilde{V} + 1.837 \times 10^{-5})}$$

Therefore,

$$\tilde{V} = 3.666 \times 10^{-5}\,\mathrm{m}^3/\mathrm{mol}$$

$$\rho_h = \frac{M_h}{\tilde{V}} = \frac{2 \times 10^{-3}}{3.666 \times 10^{-5}} = 54.56\mathrm{kg/m}^3$$

c) Cyro-compressed tank: 700 bar, 77 K
Molar specific volume can be calculated by solving:

$$700 \times 10^5 = \frac{8.314 \times 77}{\tilde{V} - 1.837 \times 10^{-5}} - \frac{9.251 \times 10^{-5}}{\sqrt{77} \cdot \tilde{V}\,(\tilde{V} + 1.837 \times 10^{-5})}$$

Therefore,

$$\tilde{V} = 2.751 \times 10^{-5}\,\mathrm{m}^3/\mathrm{mol}$$

$$\rho_h = \frac{M_h}{\tilde{V}} = \frac{2 \times 10^{-3}}{2.751 \times 10^{-5}} = 72.70\mathrm{kg/m}^3$$

d) Liquid cyrogenic tank
Check thermodynamic tables, we find:

$$\rho_h = 71\mathrm{kg/m}^3.$$

Discussion

- Cyrogenic pressurized tank gives twice the hydrogen storage density if compared with the normal pressurized tank with ambient temperature.
- Cyrogenic liquidified storage gives comparable result if compared with the cyro-genic pressurized storage.

Problem 6.5

Explain the different values of γ_{ht} in Table 6.2.1, for storage tanks pressurized at 350 bar. Note that γ_{ht} denotes the reciprocal of the material density.

- Solution

a) Calculate thickness of housing:
According to equation (5.33)

$$w = \frac{p \cdot d}{4 \cdot \sigma}$$

where d is the diameter of the tank shell, while σ is the maximum tensile stress.

b) Derive the γ_{ht} for spherical shape:

$$\begin{aligned} m_{ht} = \rho\,\pi\,d^2\omega = \rho\cdot\pi d^2\cdot\frac{p\,d}{4\,\sigma} \\ = \rho\cdot\frac{1}{6}\pi\,d^3\cdot\frac{6\,p}{4\,\sigma} \\ = \frac{3}{2}\,\rho V\frac{p}{\sigma} \end{aligned}$$

Thus, the reciprocal of density is

$$\gamma_{ht} = \frac{V}{m_{ht}} = \frac{2\,\sigma}{3\,\rho p}$$

c) Evaluate and explain different γ_{ht} According to Table 6.2.1, the reciprocal of density can be calculated for different materials:

Material	ρ [kg/l]	σ [MPa]
Steel	8.0	460
Aluminum	2.7	210
Magnesium-composite	1.9	1000

$$\begin{aligned} \gamma_{ht,steel} &= \frac{2\times 460\times 10^6}{3\times 8\times 350\times 10^5} \\ &= 1.095\,\mathrm{l/kg}. \\ \gamma_{ht,alu} &= \frac{2\times 210\times 10^6}{3\times 2.7\times 350\times 10^5} \\ &= 1.481\,\mathrm{l/kg}. \\ \gamma_{ht,composite} &= \frac{2\times 1000\times 10^6}{3\times 1.9\times 350\times 10^5} \\ &= 10.03\,\mathrm{l/kg}. \end{aligned}$$

Discussion

- The differece in γ_{ht} is caused by differences in tensile strength σ and material density ρ.
- In the energy density calculation,

$$\frac{E_{ht}}{V_{ht}} = \frac{H_h\,\xi}{\gamma_{ht}}.$$

the numerator focuses on the mass fraction of hydrogen that is stored in the system, while the denominator includes the effect of material density and geometric shape.

- As a further step in the future, increasing pressurized level (towards 700bar), or choosing material with higher tensile strength (till 6000MPa) can improve the storage system and will be gradually introduced.

Problem 6.6

Evaluate the storage pressure and the specific strength (ratio of tensile strength to density) of the tank material that would be necessary to meet the 2015 DOE targets of Table 6.2.1 with gaseous hydrogen.

- Solution

a) Calculate the 2015 DOE target

$$\begin{cases} \frac{E_{ht}}{m_{ht}} = 3.0\,\text{kWh /kg} \\ \frac{E_{ht}}{V_{ht}} = 2.7\,\text{kWh /l} \\ \xi_{ht} = 9.0\% \end{cases} \tag{10.30}$$

b) Calculate the pressurized level

$$\begin{cases} \rho_h = \frac{E_{ht}}{V_{ht} \cdot H_h} = \frac{2.7\times10^3\times3.6\times10^6}{120\times10^6} = 81\text{kg/m}^3 \\ \tilde{V} = \frac{M_h}{\rho_h} = \frac{2\times10^{-3}}{81} = 2.469\times10^{-5}\text{m}^3/\text{mol} \end{cases} \tag{10.31}$$

Solving the RK-equation from Problem 6.1,

$$\begin{aligned} p &= \frac{8.314\times300}{\tilde{V} - 1.837\times10^{-5}} - \frac{9.251\times10^{-5}}{\sqrt{300}\times\tilde{V}\,(\tilde{V} + 1.837\times10^{-5}} \\ &= 394.6\text{MPa}. \end{aligned}$$

c) Calculate tensile specific length of material
Density reciprocal:

$$\begin{aligned} \gamma_{ht} &= \frac{H_l\,\xi_{ht}}{\left(\frac{E_{ht}}{V_{ht}}\right)} \\ &= \frac{33.33\text{kWh/kg}\times0.09}{2.7\text{kWh/l}} \\ &= 1.111\text{l/kg}. \end{aligned}$$

Therefore, the tensile specific length is:

$$\begin{aligned} \left(\frac{\sigma}{\rho}\right) &= \frac{3}{2}\gamma_{ht}p\big|_{p\approx395\text{MPa}} \\ &= \frac{3}{2}\times1.111\times10^{-3}\times395\times10^6 \\ &= 657.1\text{kNm/kg}. \end{aligned}$$

which is larger than that of magnesium-based composite fiber:

$$\left(\frac{\sigma}{\rho}\right)_{compsite} = \frac{1000}{1.9} = 526.3\mathrm{kN\,m/kg}$$

<u>**Discussion**</u>

- As for current technology level, both the pressurized level and the housing material tensile strength have to be improved a lot to meet the 2015 DOE requirement.
- Carbon fiber can probably have a specific tensile strength of a few thousands kN m/kg, which might be a probably candidate.

Problem 6.7

Explain the explicitness of the number of cells N in (6.72).

- Solution

Assume the first approximation in **(6.72)** holds:

$$P_{aux(t)} = P_0 + N \cdot \kappa_{aux} \cdot I_{fc}(t)$$

According to the semi-empirical data, which sugests a linear dependency between P_{aux} and P_{st} for first order approximation:

$$\begin{aligned} P_{aux} &\propto P_{st} \\ P_{aux} &= C_0 + C_1 \cdot P_{st} \\ &= C_0 + C_1 \cdot U_{fc}(t) \cdot I_{fc}(t) \cdot N. \end{aligned}$$

As the main component of the auxiliary poewr, P_{st} contains the number of cell explicitly.

Problem 6.8

For the fuel cell stack of Fig. 6.11, find (i) the maximum output power $P_{fcs,max}$, (ii) the current $I_{fc,P}$ at which this power is yielded, and (iii) the current $I_{fc,\eta}$ that maximizes the overall efficiency. Compare the result with the curves shown in the figure.

- Solution

Assume: According to the fuel cell power & efficiency graph: $N = 250, u_{rev} = 1.23\,V, u_{OC} = 0.82\,V,\ A_{fc} = 200\,cm^2, R_{fc} = 0.0024\,\Omega, P_0 = 100\,W\ \kappa_{aux} = 0.05\,V$

a) Find the optimal current $I_{fc,P}$
As is known from (6.79), cell power can be described as:

$$P_{fcs} = (N \cdot U_{oc} - N \cdot \kappa_a ux)\, I_{fc} - P_0 - N \cdot R_{fc}\, I_{fc^2(t)}$$

At optimum, set derivative $\frac{\partial P_{fcs}}{\partial I_{fc}}|_{I_{fc,P}} = 0$, we have:

$$\begin{aligned} N \cdot U_{oc} - N \cdot \kappa_{aux} &= 2\, N\, R_{fc}\, I_{fc,P}(t) \\ I_{fc,P} = I^*_{fc} &= \frac{N\, U_{oc} - N\, \kappa_{aux}}{2\, N\, R_{fc}} \\ &= \frac{0.82 - 0.05\text{V}}{2 \times 0.024\Omega} \\ &= 160.4 \;\; \text{A} \end{aligned}$$

b) Find the maximum power of one single cell

$$\begin{aligned} P_{fcs} &= (N \cdot U_{oc} - N \cdot \kappa_a ux)\, I_{fc} - P_0 - N \cdot R_{fc}\, I_{fc^2(t)} \\ &= 250 \times (0.82 - 0.05) \times 160.4 - 100 - 250 \times 0.0024 \times 160.4^2 \\ &= 15.34\text{kW} \end{aligned}$$

c) Find the optimal current in terms of maximum efficiency
The cell efficiency can be described as:

$$\eta_{st}(I_{fc}) = \eta_{id} \frac{U_{oc}}{U_{rev}} \left(1 - \frac{R_{fc}}{I_{fc}} U_{oc} - \frac{P_0}{U_{oc}\, I_{fc}\, N} - \frac{\kappa_{aux}}{U_{oc}} \right)$$

As for the same reason, the maximum efficiency is achieved if the derivative is set to zero. $\frac{\partial \eta_{st}}{\partial I_{fc}}|_{I_{fc,\eta}} = 0 \text{gives} I^2_{fc,\eta} = \frac{P_0}{N\ R_{fc}}$ Thus, the optimal current in terms of maximum efficiency is:

$$I_{fc,\eta} = \sqrt{\frac{P_0}{N\ R_{fc}}} = \sqrt{\frac{100}{250 \times 0.0024}} = 12.91\text{A}$$

Problem 6.9

Calculate the same quantities as in Problem 6.8 for a small fuel cell stack powering a racing FCHEV (see Sect. 8.6). Use the quadratic expression (6.71) for P_{aux} and the following data: $N \cdot U_{oc} = 16.8$, $N \cdot R_{fc} = 0.137$, $P_0 = 19.89$, $\kappa_1 = 6.6$, $\kappa_2 = -0.024$.

- Solution

Assume: New data and new representation of P_{st}:

$$P_{aux} = P_0 + \kappa_1 \cdot I_{fc}(t) + \kappa_2 \cdot I_{fc}(t).$$

where $N \cdot U_{oc} = 16.8, N \cdot R_{fc} = 0.137,$
$P_0 = 19.89, \kappa_1 = 6.6, \kappa_2 = -0.024.$

a) Find the optimal current $I_{fc,P}$
   As is known from (6.79), cell power can be described as:

$$P_{fcs} = (N \cdot U_{oc} - \kappa_1)\, I_{fc} - P_0 - (N \cdot R_{fc} + \kappa_2) I_{fc}^2$$

   At optimum, set derivative $\frac{\partial P_{fcs}}{\partial I_{fc}}|_{I_{fc,P}} = 0$, we have:

$$\begin{aligned} I_{fc,P} = I_{fc}^* &\frac{N \cdot U_{oc} - \kappa_1}{2 \cdot (N \cdot R_{fc} + \kappa_2)} \\ &= \frac{16.8 - 6.6}{2\,(0.137 - 0.024)} \\ &= 45.13 \text{ A} \end{aligned}$$

b) Find the maximum power of one single cell

$$\begin{aligned} P_{fcs} &= (N \cdot U_{oc} - \kappa_1)\, I_{fc} - P_0 - (N \cdot R_{fc} + \kappa_2) I_{fc}^2 \\ &= (16.8 - 6.6) \times 45.13 - 19.89 - [0.137 + (-0.024)] \times 45.13^2 \\ &= 210.3 \text{W} \end{aligned}$$

c) Find the optimal current in terms of maximum efficiency
   The cell efficiency can be described as:

$$\begin{aligned} \eta_{st}(I_{fc}) &= \frac{P_{fcs}(I_{fc})}{N \cdot U_{id} \cdot I_{fc}} \\ &= \eta_{id} \frac{U_{oc}}{U_{rev}} \left(1 - \frac{P_0 + \kappa_1\, I_{fc} + \kappa_2\, I_{fc}^2}{U_{oc}\, I_{fc}\, N} - \frac{R_{fc} I_{fc}^2}{U_{oc}\, I_{fc}} \right) \end{aligned}$$

   As for the same reason, the maximum efficiency is achieved if the derivative is set to zero. $\frac{\partial \eta_{st}}{\partial I_{fc}}|_{I_{fc,\eta}} = 0 \text{gives} I_{fc,\eta}^2 = \frac{P_0}{\kappa_2 + N\ R_{fc}}$ Thus, the optimal current in terms of maximum efficiency is:

$$I_{fc,\eta} = \sqrt{\frac{P_0}{\kappa_2 + N\ R_{fc}}} = \sqrt{\frac{19.89}{0.1370.024}} = 13.25 \text{A}$$

Reformers

Problem 6.10

Derive (6.95).

- Solution

As for a Methanol Reformer, the chemical reactions can be concluded as follows:

$$CH_3OH + H_2O \rightarrow CO_2 + 3H_2, \Delta h_R = 58.4\text{kJ/mol}$$

Assume $n_m, n_s, n_{CO_2}, n_{H_2}$ denotes the number of moles of methanol, water, carbon-dioxide and hydrogen at time t, respectively.

a) Methanol
Using the extent of reaction:

$$x = \frac{n_m(0) - n_m(t)}{n_m(0)},$$

we have:

$$n_m = n_m(0)\,(1 - x).$$

b) Watersteam

$$\begin{aligned} n_s &= n_s(0) - (n_m(0) - n_m(t)) \\ &= n_s(0) - n_m(0) \cdot x \\ &= n_m(0)(\sigma - x) \end{aligned}$$

where σ denotes the ratio of the moles of water to the moles of methanol, which is the feed-in gas ratio and is assumed to be known at the beginning of reaction.

c) Hydrogen

$$n_h = 3 \cdot x \cdot n_m(0).$$

d) Carbon-dioxide

$$n + CO_2 = 1 \cdot x \cdot n_m(0).$$

e) Derivation of the methanol evolution
Substitutes all the molar number into the molar fraction, we have:

$$\begin{aligned} C_m(x) &= \frac{n_m}{n_m + n_s + n_h + n_{CO_2}} \\ &= \frac{n_m(0)(1-x)}{n_m(0)(1-x) + n_m(0)(\sigma - x) + 4 \cdot x \cdot n_m(0)} \\ &= \frac{1-x}{(1+\sigma) + 2\,x} \end{aligned}$$

Given that at the beginning of the reforming process, the feed-in gas ratio is predefined $\frac{n_s(0)}{n_m(0)} = \sigma$. Thus,

$$\begin{aligned} C_m(0) &= \frac{n_m(0)}{n_m(0) + n_s(0)} = \frac{1}{1+\sigma} \\ C_m(x) &= (1+\sigma)\frac{1-x}{1+\sigma+2 \cdot x} C_m(0). \end{aligned}$$

which exactly gives (6.95).

Supervisory Control Algorithms

Driver's Intepretation

Problem 7.1

An ICE-based powertrain has the following characteristics: $\gamma = \{15.0, 8.1, 5.3, 3.9, 3.1, 2.6\}$, wheel radius $r_w = 0.32\,\text{m}$, rated engine power $P_{e,max} = 92\,\text{kW}$ at a speed $\omega_{e,max} = 524\,\text{rad/s}$, engine braking torque $T_{e,min} = 20\,\text{Nm}$. Build a driver's interpretation map. Then, follow a torque control structure to generate an engine torque setpoint for a driver pedal request of 50% at a vehicle speed of 100 km/h and fourth gear.

- Solution

The curve describing the maximum force available at the wheels consists of a first part that reproduces the engine maximum torque curve (not known in this exercise) for the 1st gear, and a second part that is the envelop of the maximum-power engine points at different gears. The maximum-power engine point is at $\omega_e = 524\,\text{rad/s}$ and $T_e = 92 \cdot 10^3/524 = 176\,\text{Nm}$.

The vehicle speed at gear i is related to the engine speed by the equation $v_i = r_w/\gamma_i \cdot \omega_e$, for $i = 1, \ldots, 6$. The force at the wheels is $F_{t,i} = \gamma_i/r_w \cdot T_e$. Since $v = 100/3.6 = 27.8\,\text{m/s}$ is greater than $v_1 = 524 \cdot 0.32/15 = 11.2\,\text{m/s}$, the maximum-power range is active. Therefore, $F_t = 92 \cdot 10^3/V$ corresponds to 100% accelerator pedal. The maximum brake power is calculated from the engine data as $20 \cdot 524 = 10.5\,\text{kW}$. Thus $F_t = 10.5 \cdot 10^3/V$ corresponds to 0% accelerator pedal. At the current speed, the maximum force is 3312 N, the minimum force is 378 N. For a pedal depression of 50%, assuming linear interpolation, we have a force request $F_t = -378 + (3312 + 378) \cdot 0.5 = 1467\,\text{N}$. Assuming 4th gear, the engine torque is $T_e = 0.32/3.9 \cdot 1467 = 120\,\text{Nm}$ at a speed $\omega_e = 27.8 \cdot 3.9/0.32 = 339\,\text{rad/s} = 3235\,\text{rpm}$.

Problem 7.2

Add an electric machine to the powertrain of Problem 7.1, having the following characteristics: maximum torque $T_{m,max} = 140\,\text{Nm}$, base speed $\omega_b = 300\,\text{rad/s}$, maximum power $P_{m,max} = 42\,\text{kW}$. Calculate the total torque demand for the same driving situation as in Problem 7.1 if power assist is authorized at each vehicle speed. Assume coupled regenerative braking.

- Solution

The vehicle speed corresponding to the base speed of the motor is $300 \cdot 0.32/11 = 8.7\,\text{m/s} = 31.4\,\text{km/h}$, thus lower than v_1 (see Problem 7.1). Therefore, the actual vehicle speed corresponds to the maximum-power range of the motor. Summing the two powers yields $92+42 = 134\,\text{kW}$. The minimum force at the wheels does not change with respect to the ICE case. The maximum force is now $F_t = 134 \cdot 10^3/27.8 = 4820\,\text{N}$. For a 50% pedal position, the force demand is $F_t = -378 + (4820 + 378) \cdot 0.5 = 2221\,\text{N}$, which corresponds to a total powertrain torque of $T_t = 2221 \cdot 0.32 = 711\,\text{N}$. In the 4th gear, that would correspond to an engine torque $T_e = 711/3.9 = 182\,\text{Nm}$, which is very close to its maximum torque.

Problem 7.3

Propose a driver's interpretation function for a BEV whose motor and battery have the same data as in Problem 4.9. Assume a coupled braking circuit. Calculate the torque setpoint for (i) $\omega_m = 0\,\text{rad/s}$ and 0% pedal depression, (ii) $\omega_m = 100\,\text{rad/s}$ and 0% pedal depression, (iii) $\omega_m = 250\,\text{rad/s}$ and 50% pedal depression.

- Solution

The 0% pedal position corresponds to the minimum between $I_{a,min}$ as calculated in Problem 4.35 and a negative torque for which the driver has a similar feeling than with an ICE-based powertrain. Fix this torque to, say, 1/10 of the maximum torque. Thus

$$I_{a,min} = \begin{cases} -\kappa_a \cdot \dfrac{\omega_m}{2 \cdot R_a} & \text{for } \omega_m < \dfrac{2 \cdot R_a \cdot (0.1 \cdot I_{max})}{\kappa_a}, \\ -0.1 \cdot I_{max} & \text{for } \omega_m < \dfrac{P_{max}}{\kappa_a \cdot 0.1 \cdot I_{max}}, \\ -\dfrac{P_{max}}{\kappa_a \cdot \omega_m} & \text{else.} \end{cases}$$

while for 100% acceleration

$$I_{a,max} = \begin{cases} I_{max} & \text{for } \omega_m < \dfrac{P_{max}}{\kappa_a \cdot I_{max}}, \\ \dfrac{P_{max}}{\kappa_a \cdot \omega_m} & \text{else.} \end{cases}$$

The curve $I_a = f(\alpha, \omega_m)$ is obtained by interpolation between these two limits, $I_a = I_{a,min} + \alpha \cdot (I_{a,max} - I_{a,min})$.

For the case (i), $I_{a,min} = 0$ and $I_a = T_m = 0$.

For the case (ii), $\omega_m = 100\,\text{rad/s}$ and $\omega_m > \frac{2 \cdot R_a \cdot (0.1 \cdot I_{max})}{\kappa_a} = 2 \cdot 0.05 \cdot (0.1 \cdot 88)/0.25 = 3.52\,\text{rad/s}$. Thus $I_a = I_{a,min} = -0.1 \cdot 88 = -8.8\,\text{A}$, $T_m = -2.2\,\text{Nm}$.

For the case (iii), $\omega_m = 250\,\text{rad/s}$ and $\omega_m < 4 \cdot 10^3/(0.25 \cdot 0.1 \cdot 88) = 1818\,\text{rad/s}$. Thus $T_m = -2.2 + 0.5 \cdot (22 + 2.2) = 9.9\,\text{Nm}$.

Problem 7.4

Derive a PI model for a human driver of an electric vehicle that tries to follow a prescribed drive cycle acting on the acceleration pedal. Derive a gain-scheduling tuning of the PI parameters. Vehicle data: mass $m_v = 1360\,\text{kg}$, $\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d = 0.25\,\text{N}/(\text{m/s})^2$. Evaluate the PI coefficients for a vehicle speed of $20\,\text{m/s}$. Assume perfect recuperation (decoupled braking).

- Solution

Vehicle dynamics can be generally written as

$$\frac{dv}{dt} = \frac{F_t - c_0 - c_2 \cdot v^2}{m_v}$$

The driver is sensitive to the difference between $v(t)$ and $v_s(t)$. Its action is on the acceleration and brake pedals. Define the general driver's output as the force F_t (positive for traction, negative for braking). Thus

$$\frac{dv}{dt} = \frac{u - c_0 - c_2 \cdot v^2}{m_v},$$

$$u = K_p \cdot (v - v_s) + K_i \cdot \int (v - v_s)\, dt.$$

Linearize around an operating point v_s, then define $u_s = c_0 + c_2 \cdot v_s^2$, $z = v - v_s$, $w = u - u_s$. Then evaluate

$$\frac{dz}{dt} = \frac{w - c_2 \cdot (v^2 - v_s^2)}{m_v} \approx \frac{w - c_2 \cdot 2 \cdot v_s \cdot (v - v_s)}{m_v} = \frac{w - 2 \cdot c_2 \cdot v_s \cdot z}{m_v} = K \cdot w - a \cdot z,$$

where $K = 1/m_v$ and $a = 2 \cdot c_2 \cdot v_s / m_v$.

Now close the loop with the driver regulator

$$w = K_p \cdot z + K_i \cdot \int z\, dt$$

Obtain

$$s^2 \cdot z = K \cdot K_p \cdot s \cdot z + K \cdot K_i \cdot z - a \cdot s \cdot z \rightarrow z(s) \cdot (s^2 + (a - K \cdot K_p) \cdot s - K \cdot K_i) = 0$$

By pole-placement, target at

$$\omega_n = \sqrt{(-K \cdot K_i)} = 1 \ \text{rad/s},$$

and

$$\zeta = (a - K \cdot K_p)/(2 \cdot \omega_n) = 0.7$$

With the numerical values, $K = 1/1360 = 7.35 \cdot 10^{-4}$, $a = 2 \cdot 0.25 \cdot v_s/1360 = 3.7 \cdot 10^{-4} \cdot v_s$, $K_i = -1/K = -1360$, $K_p = (a - 0.7 \cdot 2)/K = (3.7 \cdot 10^{-4} \cdot v_s - 0.7 \cdot 2)/7.35 \cdot 10^{-4}$. For $v_s = 20\,\text{m/s}$ obtain $K_p = -1898\,\text{N}$.

Regenerative Braking Control

Problem 7.5

Derive an ideal law to split the braking effort between the two axles under the assumption that the adherence is the same at each wheel.

- Solution

Write force and torque balance equations for a two-wheel equivalent vehicle. We have four equations in the four unknowns N_1, N_2 (normal forces), F_1, and F_2 (longitudinal forces), where subscript 1 is for front wheels and subscript 2 for rear wheels, while the total required force F_t is known:

$$\begin{aligned} F_1 + F_2 &= F_t, \quad \text{(balance of logitudinal forces)} \\ N_1 + N_2 &= N, \quad \text{(balance of normal forces)} \\ N \cdot b &= F_t \cdot h + N_1 \cdot (a + b), \quad \text{(balance of momenta)} \\ \frac{F_1}{N_1} &= \frac{F_2}{N_2}, \quad \text{(equal adherence)} \end{aligned}$$

where $N = m_v \cdot g$ is the vehicle weight, a and b are the horizontal distances of the wheel axles from the center of gravity (CoG), and h the height of the CoG.

By combining these four equations, obtain

$$\begin{aligned} N_1 &= \frac{N \cdot b}{a+b} - \frac{F_t \cdot h}{a+b} \\ N_2 = N - N_1 &= \frac{N \cdot a}{a+b} + \frac{F_t \cdot h}{a+b} \end{aligned}$$

During braking, $F_t < 0$ and the front weight increases. Moreover, the equal adherence condition reads

$$F_1 \cdot N_2 = F_2 \cdot N_1. \tag{10.32}$$

For a given F_t, obtain the ideal split

$$F_1 \cdot (N \cdot a - F_t \cdot h) = (F - F_1) \cdot (N \cdot b - F_t \cdot h) \rightarrow F_1 \cdot N(a+b) = F_t \cdot (N \cdot b - F_t \cdot h),$$

or

$$F_1 = \frac{F_t \cdot b}{a+b} - \frac{F_t^2 \cdot h}{N \cdot (a+b)}.$$

In terms of torques

$$T_1 = \frac{T_t \cdot b}{a+b} - \frac{T_t^2 \cdot h}{N \cdot r_w \cdot (a+b)},$$

$$T_2 = \frac{T_t \cdot a}{a+b} + \frac{T_t^2 \cdot h}{N \cdot r_w \cdot (a+b)}.$$

The equal adherence curve $T_2 = T_2(T_1)$ is obtained by manipulating (**??**) as

$$F_1 \cdot (N \cdot a + F_t \cdot h) = F_2 \cdot (N \cdot b - F_t \cdot h)$$

or

$$T_1^2 \cdot h + 2 \cdot T_1 \cdot T_2 \cdot h + T_1 \cdot N \cdot a \cdot r_w - T_2 \cdot N \cdot b \cdot r_w + T_2^2 \cdot h = 0 \quad (10.33)$$

Problem 7.6

Consider a vehicle having an electric powertrain on the rear axle, with $T_{m,max} = 1540\,\text{Nm}$ (at the wheels), $P_{m,max} = 42\,\text{kW}$, and the following vehicle characteristics (see Problem 7.5): static weight distribution fraction $s = 0.40$, height of CG $h = 55\,\text{cm}$, wheelbase $l = 2.685\,\text{m}$, wheel radius $r_w = 0.32\,\text{m}$, vehicle mass $m_v = 1932\,\text{kg}$. Evaluate the regenerative braking torque and the frictional braking torque on the front and rear axles for a total braking torque $T_t = -1200\,\text{Nm}$, vehicle speed $v = 90\,\text{km/h}$, under (i) a maximum regeneration strategy, (ii) a constant braking distribution between the axles of 70%–30%, (iii) ideal braking as in the result of Problem 7.5, and (iv) a modified brake pedal that induces regenerative braking up to a deceleration of 0.05 g and then frictional braking with a constant braking distribution of 70%–30%.

- Solution

At $v = 90\,\text{km/h}$ the maximum regenerative capability of the electric powertrain is

$$\frac{42 \cdot 10^3}{\frac{v}{3.6} \cdot \frac{1}{0.32}} = 537\,\text{Nm}.$$

The braking power is

$$\frac{90 \cdot 400}{3.6 \cdot 0.32} = 31.2\,\text{kW}.$$

In the case (i), $T_{rec} = -537\,\text{Nm}$; $T_1 = -1200 + 537 = -663\,\text{Nm}$, $T_2 = 0\,\text{Nm}$. The braking split ratio is 55%/45%. However, this value is above the equal-adherence curve, thus it is not admissible. The quantity T_{rec} should be limited as in the case (iii).

In the case (ii), $T_{rec} = -0.30 \cdot 1200 = -360\,\text{Nm}$, $T_1 = -1200 + 360 = -840\,\text{Nm}$, $T_2 = 0\,\text{Nm}$.

In the case (iii), using the formula of Problem 7.5,

$$T_{rec} = 0.40 \cdot (-1200) + \frac{1200^2 \cdot 0.55}{1932 \cdot 9.81 \cdot 0.32 \cdot 2.685} = -431\,\text{Nm},$$

and $T_1 = -1200 + 431 = -769\,\text{Nm}$, $T_2 = 0\,\text{Nm}$ (braking split ratio 64%/36%).

In the case (iv), the threshold torque is $m_v \cdot a \cdot r_w = 1932 \cdot (-0.05) \cdot 9.81 \cdot 0.32 = -300\,\text{Nm}$. Thus $T_{rec} = -300\,\text{Nm}$, $T_1 = 0.7 \cdot (-1200 + 300) = -630\,\text{Nm}$, $T_2 = -270\,\text{Nm}$.

Problem 7.7

Consider a conventional (coupled) braking system where $T_2 = k \cdot T_1$ ($T_1 < 0, T_2 < 0$). Calculate the maximum value of the adherence that can be obtained under the assumption of equal adherence between the axles (ideal distribution curve) and the corresponding total braking torque. Check what happens for higher braking torques. Then, calculate the limit value of k that can be achieved. Use the numerical values of Problem 7.6.

- Solution

The limit of conventional braking systems occurs when the practical braking split curve meets the ideal split curve. The latter is given by (**??**), while the former reads $T_2 = k \cdot T_1$. By combining these two equations, obtain

$$T_1 = \frac{-N \cdot r_w \cdot (a - k \cdot b)}{h \cdot (1+k)^2}$$

and

$$T_2 = \frac{-k \cdot N \cdot r_w \cdot (a - k \cdot b)}{h \cdot (1+k)^2}.$$

Since both T_1 and T_2 are negative quantities, a condition on k is that $a - k \cdot b) > 0$, or

$$k < \frac{a}{b}.$$

For $k = a/b$, the braking distribution is such that

$$\frac{T_1}{T_t} = \frac{1}{1+k}, \quad \frac{T_2}{T_t} = \frac{k}{1+k}.$$

The common value of the adherence factor ($\mu_1 = \mu_2 = \mu$) is obtained by calculating N_1 and N_2. First evaluate

$$T_t = T_1 + T_2 = (1+k) \cdot T_1 = \frac{-N \cdot r_w \cdot (a - k \cdot b)}{h \cdot (1+k)}.$$

Then, find

$$N_1 = \frac{N \cdot b - T_t \cdot h \cdot r_w}{a+b} = \frac{N \cdot b + N \cdot (a \cdot k - b)}{(1+k) \cdot (a+b)}, \quad N_2 = \frac{N \cdot k}{1+k}$$

and the adherence factor as

$$\mu = \frac{-T_1}{N_1 \cdot r_w} = \frac{-T_2}{N_2 \cdot r_w} = \frac{a - k \cdot b}{h \cdot (1+k)}.$$

With the data of Problem 7.6, the limit $k = 0.4/0.6 = 0.667$. For a value $k = 0.3/0.7 = 0.429$, the maximum adherence is 0.488. The limit values of T_1 and T_2 are -2055 Nm and -881 Nm. The maximum braking torque is $T_t = -2935$ Nm.

For higher braking torques, e.g., $T_t{-}3500$ Nm, the braking split ratio would be 2450/1050 Nm. The vertical forces would be $N_1 = \frac{1932 \cdot 9.81}{1+0.429} = 13632$ N, $N_2 = 1932 \cdot 9.81 - 13632 = 5321$ N. Thus $\mu_1 = 2450/13632/0.32 = 0.56$ while $\mu_2 = 1050/0.32/5321 = 0.62$. This circumstance $\mu_2 > \mu_1$ is potentially dangerous and should be avoided.

Problem 7.8

Derive the ideal braking distribution law as in Problem 7.5 when one axle in motoring while the other is braking (for instance, battery recharge mode in an HEV with an engine on the front axle and an electric machine on the rear axle). For simplicity, assume $a = b$.

- Solution

As in Problem 7.5,

$$F_1 + F_2 = F_t, \quad \text{(balance of logitudinal forces)}$$
$$N_1 + N_2 = N, \quad \text{(balance of normal forces)}$$
$$N \cdot b = F_t \cdot h + N_1 \cdot (a+b), \quad \text{(balance of momenta)}$$

but now

$$\frac{F1}{N1} = -\frac{F2}{N2},$$

the latter being the condition of equal adherence $\mu_1 = -\mu_2$. By combining these four equations, obtain

$$N_1 = \frac{N \cdot b}{a+b} - \frac{F_t \cdot h}{a+b}$$
$$N_2 = N - N_1 = \frac{N \cdot a}{a+b} + \frac{F_t \cdot h}{a+b}$$

$$F_1 \cdot N_2 = -F_2 \cdot N_1, \tag{10.34}$$

from whence

$$F_1 \cdot (N \cdot a + F_1 \cdot h + F_2 \cdot h) = -F_2 \cdot (N \cdot b - F_1 \cdot h - F_2 \cdot h) \rightarrow F_1 \cdot N \cdot a + h \cdot F_1^2 = -N \cdot b \cdot F_2 + h \cdot F_2^2.$$

There are two solutions to this equation for $a = b$. One is $F_2 = F_1 + \frac{N \cdot a}{h}$. The other is $F_2 = -F_1$, which does not imply the satisfaction of the total force F_t but constitutes a limit in the 2nd and 4th quadrant of the plane F_1—F_2.

From the first solution, find

$$T_1 = \frac{T_t}{2} - \frac{N \cdot a \cdot r_w}{2 \cdot h}, \quad T_2 = \frac{T_t}{2} + \frac{N \cdot a \cdot r_w}{2 \cdot h}.$$

Dynamic Coordination

Problem 7.9

In a series HEV the supervisory control yields engine torque and speed setpoints T_e and ω_e. Derive a simple generator controller in order to achieve the desired speed of the APU.

- Solution

The dynamics of the APU can be described by the simplified equation

$$\Theta_{apu} \cdot \frac{d\omega_e(t)}{dt} = T_e(t) - T_g(t).$$

The generator torque open-loop setpoint is $T_{g,sp} = T_{e,sp}$. However, in order to let the generator speed converge toward the value ω_e, at least a proportional correction should be added. Assuming $T_e(t) = T_{e,sp}(t)$,

$$T_g(t) = T_e(t) + k_p \cdot (\omega_g(t) - \omega_e(t)).$$

The closed-loop dynamics therefore reads

$$\Theta_{apu} \cdot s \cdot \tilde{\omega} = -K_p \cdot \tilde{\omega},$$

which converges to $\tilde{\omega} = 0$ with a time constant Θ_{apu}/k_p.

Problem 7.10

Derive the dynamic equations to control the generator torque in a simple PSD-based system like that of the Toyota Prius. Neglect the generator inertia.

- Solution

Manipulate the dynamic equations (4.163)–(4.164) with $\Theta_{sun} = 0$ to obtain

$$\Theta_{carrier} \cdot \frac{d\omega_e(t)}{dt} = T_e(t) - \frac{1+z}{z} \cdot T_g(t),$$

$$\Theta_{ring} \cdot \frac{d\omega_f(t)}{dt} = \frac{1}{z} \cdot T_g(t) + T_m(t) - T_f(t).$$

where $\Theta_{carrier}$ is represented by Θ_e and Θ_{ring} by the vehicle inertia.

The resulting dynamics for the engine speed is rather similar to that of Problem 7.9, except for the $\frac{1+z}{z}$ factor now multiplying the generator torque. It is laborious but straightforward to show that the same dynamic equation for ω_e (but not for ω_f!) applies also to the case where Θ_{sun} is not negligible.

Problem 7.11

In a post-transmission parallel HEV, in principle it is possible to compensate the torque gap at the wheels during a gear shift. Evaluate the time elapse after which the vehicle speed before the shift is recovered (t_r, recovery time) and the necessary electric energy for a downshift from 4th to 3rd gear occurring during a constant vehicle acceleration. Data: gear ratios including final gear $\gamma = 5, 4$, motor gear ratio $= \gamma_m = 11$, transmission efficiency $\eta_t = 0.97$, motor efficiency $\eta_m = 0.85$, wheel radius $r_w = 0.29\,\text{m}$, engine shift speed $\omega_e = 4500\,\text{rpm}$, shift duration $t_s = 1\,\text{s}$; acceleration $a = 0.5\,\text{m/s}^2$, vehicle mass $m_v = 1360\,\text{kg}$, $c_r = 0.009$, $c_d \cdot A_f = 0.5\,\text{m}^2$.

- Solution

Define the two time points t_1 and t_2 as the beginning and the end of the gear shift ($t_2 - t_1 = t_s$). Evaluate

$$v_1 \triangleq v(t_1) = \frac{\omega_{e,shift}}{\frac{\gamma_3}{r_w}} = \frac{\frac{4500\cdot\pi}{30}}{\frac{5}{0.29}} = 27\,\text{m/s} = 98\,\text{km/h}.$$

Without compensation, the vehicle speed decreases during the shift according to the law

$$\frac{dv(t)}{dt} = -g \cdot c_r - \frac{\rho_a \cdot A_f \cdot c_d \cdot v^2}{2 \cdot m_v},$$

or (2.17) with $\alpha = \sqrt{\frac{0.5\cdot 1.2}{1360\cdot 0.5}} = 0.015$, $\beta = \sqrt{9.8 \cdot 0.009} = 0.3$. The coasting velocity at t_2 is

$$\begin{aligned} v_2 &= \frac{\beta}{\alpha} \cdot \tan\left(\arctan\left(\frac{\alpha}{\beta} \cdot v_1\right) - \alpha \cdot \beta \cdot (t_2 - t_1)\right) \\ &= \frac{0.3}{0.015} \cdot \tan\left(\arctan\left(\frac{0.015}{0.3} \cdot 27\right) - 0.015 \cdot 0.3 \cdot 1\right) \\ &= 26.7\,\text{m/s} = 96\,\text{km/h}. \end{aligned}$$

After the engine is engaged again, the speed increases according to the linear law $v(t) = v_2 + a \cdot t$. The recovery time is

$$t_3 = \frac{v_1 - v_2}{a} = \frac{0.3}{0.5} = 0.6\,\text{s}.$$

Thus the total time lost is $t_r = t_s + t_3 = 1 + 0.6 = 1.6\,\text{s}$.

This time can be recuperated if the motor provides the missing torque during the shift. This torque is

$$T_m(t) = \frac{F_t(t)}{\frac{\gamma_m}{r_w}} = \frac{m_v \cdot a + m_v \cdot c_r + \rho_a \cdot c_d \cdot A_f \cdot v(t)^2}{2 \cdot \frac{\gamma_m}{r_w}}.$$

The energy provided by the motor is calculated from

$$\Delta E_m = \int \omega_m(t) \cdot T_m(t)dt = \frac{1}{\eta_t} \cdot \int F_t(t) \cdot v(t)dt =$$

$$= \frac{1}{\eta_t} \cdot \left((m_v \cdot a + m_v \cdot c_r) \cdot \int v(t)dt + \int 0.5 \cdot \rho_a \cdot c_d \cdot A_f \cdot v(t)^3 dt \right)$$

$$= \frac{1}{\eta_t} \cdot (m_v \cdot \frac{v_2^2 - v_1^2}{2} + \frac{m_v \cdot c_r}{2 \cdot a} \cdot (v_2^2 - v_1^2) + 1.2 \cdot c_d \cdot A_f \cdot (v_2^4 - v_1^4)/(8 \cdot a)),$$

where now $v_2 = v_1 + a \cdot t_s = 27 + 0.5 \cdot 1 = 27.5\,\text{m/s}$. Thus the energy is

$$\Delta E_m = \frac{1}{0.97} \cdot \left(1360 \cdot \frac{27.5^2 - 27^2}{2} + \frac{1360 \cdot 0.009}{2 \cdot 0.5} \cdot (27.5^2 - 27^2) + \right.$$

$$\left. +0.5 \cdot 1.2 \cdot 0.5 \cdot \frac{27.5^4 - 27^4}{4 \cdot 0.5} \right) = 25.7\,\text{kJ},$$

with an average power $\bar{P}_m = 25.7\,\text{kW}$.

Problem 7.12

Consider a parallel HEV with an electric machine mounted on the primary shaft of the gearbox with a reduction gear ratio γ_m. During a gear shift, the inertia of the motor sums up to the inertia of the primary shaft. To reduce the synchronization lag, the motor in principle could yield a torque to compensate its own inertia. Model this situation with simple equations. Then calculate the motor energy consumption for the following data: $\gamma_m = 3.3$, downshift from 4th to 3rd gear with $\gamma_3 = 5.5$, $\gamma_4 = 3.9$, vehicle speed $v = 60\,\text{km/h}$, motor inertia $\Theta_m = 0.07\,\text{kg/m}^2$.

- Solution

During synchronization without motor assist, the dynamics of the primary shaft reads

$$\Theta_p \cdot \frac{d\omega_p(t)}{dt} = T_{m,i}(t) \cdot \gamma_m + k \cdot \left(\frac{v(t) \cdot \gamma_3}{r_w} - \omega_p(t) \right),$$

where the second right-hand term simulates the action of the synchronizer which is proportional to the difference between the secondary speed with the new gear ratio and the primary speed. The term $T_{m,i}$ is transmitted from the motor inertia

$$T_{m,i} = -\Theta_m \cdot \frac{d\omega_m}{dt}.$$

Since $\omega_m = \gamma_m \cdot \omega_p$,

$$(\Theta_p + \gamma_m^2 \cdot \Theta_m) \cdot \frac{d\omega_p(t)}{dt} = k \cdot \left(\frac{v(t) \cdot \gamma_3}{r_w} - \omega_p(t) \right)$$

and the primary shaft speed increases from the initial value $\omega_{p,4} = \frac{\gamma_4 \cdot v}{r_w}$ up to the new value $\omega_{p,3} = \frac{v \cdot \gamma_3}{r_w}$. The variation law is (ideally) asymptotic,

$$\omega_p(t) = \frac{\gamma_3 \cdot v(t)}{r_w} + \left(\frac{\gamma_4 \cdot v(t)}{r_w} - \frac{\gamma_3 \cdot v(t)}{r_w} \right) \cdot \exp\left(-\frac{k \cdot t}{\Theta} \right),$$

where $\Theta \triangleq \Theta_p + \Theta_m \cdot \gamma_m^2$. To decrease the motor inertia, one should apply a torque T_m such that

$$\Theta_m \cdot \frac{d\omega_m(t)}{dt} = T_m - T_{m,i}$$

such that $T_{m,i} = 0$, thus

$$\begin{aligned} T_m = \Theta_m \cdot \frac{d\omega_m(t)}{dt} &= \Theta_m \cdot \gamma_m \cdot \frac{d\omega_p(t)}{dt} \\ &= \frac{\Theta_m \cdot \gamma_m \cdot k \cdot \left(v(t) \cdot \frac{\gamma_4}{r_w} - \omega_p(t) \right)}{\Theta_p}, \end{aligned}$$

where $\omega_p(t)$ is still calculated with the equation above but with Θ_p instead of Θ.

The motor power is

$$\begin{aligned} P_m &= T_m \cdot \omega_m = \\ &= \Theta_m \cdot \gamma_m \cdot \frac{d\omega_p}{dt} \cdot \gamma_m \cdot \omega_p = \Theta_m \cdot \gamma_m^2 \cdot \omega_p \cdot \frac{d\omega_p}{dt}. \end{aligned}$$

The energy consumed results from the integral of P_m or

$$E_m = \Theta_m \cdot \gamma_m^2 \cdot \frac{\omega_{p,3}^2 - \omega_{p,4}^2}{2}.$$

Numerically,

$$\begin{aligned} \omega_{p,3} &= \frac{70 \cdot 5.5}{3.6 \cdot 0.32} = 334\,\mathrm{rad/s} \\ \omega_{p,4} &= \frac{70 \cdot 3.9}{3.6 \cdot 0.32} = 237\,\mathrm{rad/s} \\ E_m &= 0.07 \cdot 3.3^2 \cdot \frac{334^2 - 237^2}{2} = 21\,\mathrm{kJ} \end{aligned}$$

Heuristic Energy Management Strategies

Problem 7.13

Consider a pre-transmission, single-shaft parallel HEV with fixed gear reduction. System data: gear ratio including final gear ratio $\gamma = 4$, engine maximum torque curve $T_{e,max}(\omega_e) = 50 + 0.7 \cdot \omega_e - 1 \cdot 10^{-3} \cdot \omega_e^2$, motor maximum torque $T_{m,max} = 150\,\mathrm{Nm}$, motor maximum power $P_{m,max} = 25\,\mathrm{kW}$, vehicle data $c_D = 0.33$, $A_f = 2.5\,\mathrm{m}^2$, $c_r = 0.013$, $m_v = 1500\,\mathrm{kg}$, $\Theta_w = 0.25\,\mathrm{kg\,m}^2$, $r_w = 0.25\,\mathrm{m}$. Consider the simple, SOC-independent heuristic energy-management strategy:

- EV mode if $\omega_e < 1000\,\mathrm{rpm}$ or if $T_e < 40\,\mathrm{Nm}$,
- power assist mode if $T_e > T_{e,max}$,
- else, recharge mode if $T_e > 0$
- regenerative braking if $T_e < 0$.

Evaluate the scheduled mode, the engine torque, and the motor torque for the following driving situations: (i) $v = 17\,\mathrm{km/h}$, $a = 1.37\,\mathrm{m/s^2}$; (ii) $v = 38.76\,\mathrm{km/h}$, $a = 0.094\,\mathrm{m/s^2}$; (iii) $v = 28.8\,\mathrm{km/h}$, $a = 1.56\,\mathrm{m/s^2}$; (iv) $v = 95\,\mathrm{km/h}$, $a = 0.19\,\mathrm{m/s^2}$.

- Solution

For the case (i) the required propulsion force is

$$F_t = (m_v + \frac{\Theta_w}{r_w^2 \cdot \gamma^2}) \cdot a + m_v \cdot 9.81 \cdot c_r + \frac{1}{2} \cdot \rho_a \cdot c_D \cdot A_f \cdot v^2 =$$
$$= 1503 \cdot 1.369 + 191.3 + 0.47 \cdot 4.75^2 = 2257\,\mathrm{N}.$$

The engine torque would be $T_e = F_t \cdot \frac{r_w}{\gamma} = 2257 \cdot 0.25/4 = 141\,\mathrm{Nm}$. The engine speed would be $\omega_e = v \cdot \frac{\gamma}{r_w} = 75.9\,\mathrm{rad/s} = 725\,\mathrm{rpm}$. The mode selected would be the ZEV ($\omega_e < 1000\,\mathrm{rpm}$ and $T_e > 0$). The base speed is $25 \cdot 10^3/150 = 167\,\mathrm{rad/s}$. The motor torque $T_m = 141\,\mathrm{Nm}$ is lower than the motor maximum torque at 725 rpm, which is 150 Nm.

For the case (ii), the required force is $1503 \cdot 0.094 + 191.3 + 0.47 \cdot (38.76/3.6)^2 = 387\,\mathrm{Nm}$. The engine torque would be of $387 \cdot 0.25/4 = 24.2\,\mathrm{Nm}$. The engine speed would be $38.76/0.25 \cdot 4 = 172\,\mathrm{rad/s} = 1645\,\mathrm{rpm}$. The mode selected would be again the ZEV ($T_e < 40\,\mathrm{Nm}$ with $\omega_e > 1000\,\mathrm{rpm}$). At 1645 rpm (higher than the motor base speed) the motor maximum torque is $25 \cdot 10^3/172 = 145\,\mathrm{Nm}$. Thus the ZEV mode is feasible.

For the case (iii), the required force is $1503 \cdot 1.56 + 191.3 + 0.47 \cdot (28.8/3.6)^2 = 2535\,\mathrm{N}$. The engine torque would be of 160 Nm. The engine speed would be of 1230 rpm. The engine max torque at that speed would be $T_{e,max} = 50 + 0.7 \cdot 128 - 1 \cdot 10^{-3} \cdot 128^2 = 123\,\mathrm{Nm}$. Thus the mode selected would be the boost ($T_e > T_{e,max}$ and $\omega_e > 1000\,\mathrm{rpm}$). The motor torque would be $160 - 123 = 37\,\mathrm{Nm}$, which is lower than the motor max torque.

For the case (iv), the required force is 805 N. The engine torque would be of 50 Nm. The engine speed would be 4033 rpm = 422 rad/s. Thus the selected mode would be the battery recharge ($T_e > 40\,\mathrm{Nm}$ and $\omega_e > 1000\,\mathrm{rpm}$). The maximum generating torque at 422 rad/s is $-25 \cdot 10^3/422 = -59\,\mathrm{Nm}$. Thus the maximum engine torque could be $50 + 59 = 119\,\mathrm{Nm}$ (feasible because the maximum engine torque is 167 Nm).

Problem 7.14

Consider the following SOC-dependent energy-management heuristic strategy:

- engine on with $P_e = P_t - P_{t,b}(\xi)$ if $P_t > P_{e,start}(\xi)$,
- else, engine off,

with the definitions $P_{t,b} = -P_{m,max} + 2 \cdot P_{m,max}/(\xi_{hi} - \xi_{lo}) \cdot (\xi - \xi_{lo})$, $P_{e,start} = P_{m,max}/(\xi_{hi} - \xi_{lo}) \cdot (\xi - \xi_{lo})$ and the numerical values $\xi_{hi} = 80\%$, $\xi_{lo} = 40\%$. Assume a unit-efficiency motor operation. Perform again the calculations of Problem 7.13, for $\xi = \{55, 70\}\%$.

- Solution

For the case (ii), engine possible speed and torque $\omega_e = 172\,\text{rad/s}$, $T_e = 24.2\,\text{Nm}$ would lead to $P_e = 4.16\,\text{kW}$. Since $P_{m,max} = 25\,\text{kW}$, $P_{e,max} = 24.2\,\text{kW}$, evaluate

$$\begin{aligned} P_{e,start} = \frac{\xi - 40}{80 - 40} \cdot P_{m,max} &= 9.4\,\text{kW} \quad \text{for} \quad \xi = 55\%, \\ &= 18.75\,\text{kW} \quad \text{for} \quad \xi = 70\%. \end{aligned}$$

Now, $P_e < P_{e,start}$, thus the selected mode is ZEV.

For the case (iii), $\omega_e = 129\,\text{rad/s}$, $T_e = 160\,\text{Nm}$ would lead to $P_e = 20.6\,\text{kW}$. In this case $P_{m,max} = 150 \cdot 129 = 19.3\,\text{kW}$, $P_{e,max} = 15.95\,\text{kW}$, thus $P_{e,start} = \{7.2, 14.5\}\,\text{kW}$ for the two SOC values. In both cases, $P_e > P_{e,start}$. Evaluate $P_{t,b} = -19.3 + (\xi - 40)/40 \cdot (19.3 \cdot 2) = \{-4.8; 9.7\}\,\text{kW}$. Therefore, P_e would be $\{25.4; 10.9\}\,\text{kW}$. After saturation, $P_e = \{15.95; 10.9\}$ and obtain as a difference $P_m = \{4.65; 9.7\}$ (boost mode).

Fot the case (iv), $\omega_e = 422\,\text{rad/s}$, $T_e = 50\,\text{Nm}$ would lead to $P_e = 21\,\text{kW}$. Since $P_{m,max} = 25\,\text{kW}$ and $P_{e,max} = 70.6\,\text{kW}$, evaluate $P_{e,start} = \{9.4; 18.75\}\,\text{kW}$. In both cases, $P_e > P_{e,start}$. Evaluate $P_{t,b} = \{-0.55; 18.2\}\,\text{kW}$ and find $P_e = \{21.55; 2.8\}\,\text{kW}$. No saturation is needed and $P_m = \{-0.55, 18.2\}$ thus the selected mode is recharge, resp., boost.

Problem 7.15

Give an intepretation of the heuristic energy-management strategy of Problem 7.14 in terms of equivalent "cost" of the battery power with respect to the fuel power. Assume a Willans-type engine model with constant parameters e and P_0 and a unit-efficiency electric machine.

- Solution

The heuristic rule reads (P_t is the demand power)

$$\begin{aligned} P_e &= 0 \quad \text{if} \quad P_t > P_{e,start} + P_{t,b} \\ P_e &= P_t - P_{t,b} \quad \text{if} \quad P_t > P_{e,start} + P_{t,b}. \end{aligned}$$

Define an "equivalent" power consumption as $H = P_e + s \cdot P_m$, where s is the equivalence factor. Using a Willans engine model,

$$H = \frac{P0}{e} \cdot (P_{mt} > 0) + \frac{P_e}{e} + s \cdot (P_t - P_e).$$

Note that:

- For $Pe = 0$, $H = s \cdot P_t$.
- For $P_e = P_t - P_{t,b}$, $H = \frac{P_0}{e} + \frac{P_t - P_{t,b}}{e} + s \cdot P_{t,b}$.

The switching condition for which $P_e = 0$ is preferable is that $s \cdot P_t < \frac{P_0}{e} + \frac{P_t - P_{t,b}}{e} + s \cdot P_{t,b}$, that is, $P_t \cdot \left(s - \frac{1}{e}\right) < \frac{P_0}{e} + P_{t,b} \cdot \left(s - \frac{1}{e}\right)$, or $P_t < \frac{P_0}{e \cdot s - 1} + P_{t,b}$.

By comparing this switching condition with the heuristic rule, derive $P_{e,start}$ as

$$P_{e,start} = \frac{P_0}{e \cdot s - 1} + P_{t,b},$$

from whence derive

$$s \cdot e = \frac{1 + P_0}{P_{e,start} - P_{t,b}}$$

as the equivalence rule between the two strategies (i.e., between s and $P_{e,start}$).

Optimal Energy Management Strategies

Problem 7.16

Derive the exact formulation of the Euler-Lagrange equation (7.14) if the equivalent-circuit parameters of the battery are affine functions of SoC as described by (4.64) and (4.66). Consider the following system and operating point: battery capacity $Q_b = 6.5\,\text{Ah}$, nominal open-circuit voltage $U_{oc} = 250\,\text{V}$, nominal internal resistance $R_i = 0.3\,\Omega$, electric power $P_b = 15\,\text{kW}$, variation of the open-circuit voltage with respect to SOC $\kappa_2 = 20\,\text{V}$, and variation of the internal resistance $\kappa_4 = -0.1\,\Omega$. Evaluate the characteristic time constant associated with the variation of the Lagrange multiplier and assess the constant-μ approximation.

- Solution

The Hamiltonian function is

$$H = \overset{*}{m}_f + \mu \cdot \dot{x},$$

where $\dot{x} = -I_b/Q_0$. The Euler–Lagrange equation is written as

$$\dot{\mu} = -\frac{\partial H}{\partial x} = \frac{\mu}{Q_0} \cdot \frac{\partial I_b}{\partial x}$$

For an equivalent circuit model,

$$
\begin{aligned}
I_b &= \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\frac{U_{oc}^2}{4 \cdot R_i^2} - \frac{P_b}{R_i}} \triangleq \frac{U_{oc}}{2 \cdot R_i} - A \\
\frac{\partial I_b}{\partial U_{oc}} &= \frac{1}{2 \cdot R_i} - \frac{1}{2 \cdot A} \cdot \frac{U_{oc}}{2 \cdot R_i^2} \\
\frac{\partial I_b}{\partial R_i} &= -\frac{U_{oc}}{2 \cdot R_i^2} - \frac{1}{2 \cdot A} \cdot \left(-\frac{U_{oc}^2}{2 \cdot R_i^3} + \frac{P_b}{R_i^2} \right)
\end{aligned}
$$

Numerically,

$$
A = \sqrt{\frac{250^2}{(2 \cdot 0.3)^2} - \frac{15 \cdot 10^3}{0.3}} = 351.58 \text{ A}
$$

$$
\begin{aligned}
\frac{\partial I_b}{\partial U_{oc}} &= \frac{1}{2 \cdot 0.3} - \frac{1}{2 \cdot 351.58} \cdot \frac{250}{2 \cdot 0.3^2} = -0.31 \text{ A/V} \\
\frac{\partial I_b}{\partial R_i} &= -\frac{250}{2 \cdot 0.3^2} - \frac{1}{2 \cdot 351.58} \cdot \left(-\frac{250^2}{2 \cdot 0.3^3} + \frac{15 \cdot 10^3}{0.3^2} \right) = 20 \text{ A}/\Omega \\
\frac{\partial I_b}{\partial x} &= \frac{\partial I_b}{\partial U_{oc}} \cdot \frac{\partial U_{oc}}{\partial x} + \frac{\partial I_b}{\partial R_i} \cdot \frac{\partial R_i}{\partial x} = -0.31 \cdot 20 + 20 \cdot (-0.1) = -8.2 \text{ A/-}
\end{aligned}
$$

and finally obtain

$$
\frac{\dot{\mu}}{\mu} = \frac{\partial I_b}{\partial x} \cdot \frac{1}{Q_0} = -\frac{8.1}{6.5 \cdot 3600} = -\frac{1}{2854 \text{ s}}.
$$

Problem 4.36

Starting from the results of Problems 7.16, 4.26, find an approximated expression for the variation of the Lagrange multiplier. Evaluate the error with respect to the exact solution.

- Solution

Using the result of Problem 4.26,

$$
I_b \approx \hat{I} = \frac{P_b}{U_{oc}} + 2 \cdot \frac{R_i}{U_{oc}^3} \cdot P_b^2,
$$

thus

$$
\begin{aligned}
\frac{\partial \hat{I}}{\partial U_{oc}} &= -\frac{P_b}{U_{oc}^2} - 6 \cdot \frac{R_i}{U_{oc}^4} \cdot P_b^2 \\
\frac{\partial \hat{I}}{\partial R_i} &= \frac{2}{U_{oc}^3} \cdot P_b^2
\end{aligned}
$$

With the numerical values of Problem 7.16 compared to the exact solution,

$$\frac{\partial \hat{I}}{\partial U_{oc}} = -\frac{15 \cdot 10^3}{250^2} - 6 \cdot \frac{0.3}{250^4} \cdot \left(15 \cdot 10^3\right)^2 = -0.34 \text{ A/V} \qquad (10\% \text{ error})$$

$$\frac{\partial \hat{I}}{\partial R_i} = \frac{2}{250^3} \cdot \left(15 \cdot 10^3\right)^2 = 28.8 \text{ A}/\Omega \qquad (50\% \text{ error})$$

$$\frac{\partial \hat{I}}{\partial x} = -0.34 \cdot 20 + 29 \cdot (-0.1) = -9.7 \text{ A} \qquad (20\% \text{ error})$$

Problem 7.18

At low temperature operation, the variation of the internal parameters of a battery can be significant. Develop a version of the ECMS where variations of internal resistance, via the parameter κ_3 of (4.66), with temperature are accounted for. Cell data: $\kappa_1 = 3.4\,\text{V}$, $\kappa_2 = 0.5\,\text{V}$, $\kappa_4 = 0$, and

$$\kappa_3 = 0.015 - \vartheta_b \cdot \frac{0.01}{40},$$

with ϑ_b in °C. The nominal SOC is $\xi = 0.5$, temperature $\vartheta = 25\,°\text{C}$, power $P_b = 0.1 \cdot P_{b,max}$, thermal capacitance $C_{t,b} = 300$ J/K, thermal conductance $1/R_{th} = 0.5\,\text{W/K}$, and capacity $Q_b = 6\,\text{Ah}$. Evaluate the time constant of the adjoint states.

- Solution

The quantity to be minimized is still the fuel consumption rate. The SOC variation is still proportional to the current. However the latter varies as a function of SOC and temperature. Thus the state equation for the temperature must be taken into account. The Hamiltonian is

$$H = \overset{*}{m}_f + \mu \cdot \frac{\partial \xi}{\partial t} + \nu \cdot \frac{\partial \vartheta}{\partial t},$$

where $\partial\xi/\partial t = -I_b/Q_0$ and

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial \tilde{\vartheta}}{\partial t} = \left(R_i \cdot I_b^2 - \alpha \cdot \tilde{\vartheta}\right) \cdot \frac{1}{C_{t,b}},$$

where $\tilde{\vartheta} \triangleq \vartheta - \vartheta_{amb}$. The Euler–Lagrange equations read

$$\dot{\mu} = -\frac{\partial H}{\partial \xi} = \mu \cdot \frac{1}{Q_0} \cdot \frac{\partial I_b}{\partial \xi}$$

$$\dot{\nu} = -\frac{\partial H}{\partial \vartheta} = -\nu \cdot \left(2 \cdot R_i \cdot I_b \cdot \frac{\partial I_b}{\partial \vartheta} - \alpha\right) \cdot \frac{1}{C_{t,b}}$$

The quantity $\partial I_b/\partial\xi$ is calculated as

$$\frac{\partial I_b}{\partial \xi} = \frac{\partial I_b}{\partial U_{oc}} \cdot \frac{\partial U_{oc}}{\partial \xi} + \frac{\partial I_b}{\partial R_i} \cdot \frac{\partial R_i}{\partial \xi} = \frac{\partial I_b}{\partial U_{oc}} \cdot \kappa_2 + \frac{\partial I_b}{\partial R_i} \cdot \kappa_4$$

while

$$\frac{\partial I_b}{\partial \vartheta} = \frac{\partial}{\partial R_i} \cdot \frac{\partial R_i}{\partial \vartheta} = \frac{\partial I_b}{\partial R_i} \cdot \frac{\partial \kappa_3}{\partial \vartheta}$$

With the numerical values for the open-circuit voltage and the internal resistance are calculated as

$$\begin{aligned} U_{oc} &= \kappa_1 + \kappa_2 \cdot q = N \cdot (3.4 + 0.5 \cdot 0.5) = 3.65 \cdot N \text{ V} \\ R_i &= \kappa_3 + \kappa_4 \cdot q = N \cdot \left(0.015 - 0.01 \cdot \frac{25}{40}\right) = 0.009 \cdot N \ \Omega, \end{aligned}$$

while the maximum power can be calculated with equation (4.74)

$$\begin{aligned} P_{b,max} &= \frac{U_{oc}^2}{4 \cdot R_i} = \frac{3.65^2 \cdot N^2}{4 \cdot 0.09 \cdot N} = 370 \cdot N \text{ W} \\ P_e &= 0.1 \cdot P_{b,max} = 37 \cdot N \text{ W}. \end{aligned}$$

This results in the numerical value for the current

$$\begin{aligned} A &= \sqrt{\frac{U_{oc}^2}{4 \cdot R_i^2} - \frac{P_b}{R_i}} = \sqrt{\frac{3.65^2}{4 \cdot 0.009^2} - \frac{37}{0.009}} = 192 \text{ A} \\ I_b &= \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\left(\frac{U_{oc}}{2 \cdot R_i}\right)^2 - \frac{P_b}{R_i}} = \\ &= \frac{3.65}{2 \cdot 0.009} - \sqrt{\left(\frac{3.65}{2 \cdot 0.009}\right)^2 - \frac{37}{0.009}} = 10.5 \text{ A} \end{aligned}$$

The variations in the current relative to vriations in the states $\{\xi, \vartheta\}$ are now

$$\begin{aligned} \frac{\partial I_b}{\partial U_{oc}} &= \frac{1}{2 \cdot R_i} \cdot \left(1 - \frac{U_{oc}}{2 \cdot R_i \cdot A}\right) = \\ &= \frac{1}{2 \cdot 0.009 \cdot N} \cdot \left(1 - \frac{3.65}{2 \cdot 0.009 \cdot 192}\right) = -\frac{3.12}{N} \text{ A/V} \\ \frac{\partial I_b}{\partial R_i} &= -\frac{U_{oc}}{2 \cdot R_i^2} - \frac{1}{2 \cdot A} \cdot \left(\frac{P_b}{R_i^2} - \frac{U_{oc}^2}{2 \cdot R_i^3}\right) = \\ &= -\frac{3.65}{2 \cdot 0.009^2} - \frac{1}{2 \cdot 192} \cdot \left(\frac{37}{0.009^2} - \frac{3.65^2}{2 \cdot 0.009^3}\right) = \frac{75.2}{N} \text{ A}/\Omega \\ \frac{\partial I_b}{\partial \xi} &= -3.12 \cdot 0.5 + 75.2 \cdot 0 = -1.56 \text{ A} \\ \frac{\partial I_b}{\partial \vartheta} &= 75.2 \cdot \frac{-0.01}{40} = -0.019 \text{ A/K} \end{aligned}$$

Following time constants of the lagrange mutlipliers are obtained

$$\frac{\dot{\mu}}{\mu} = \frac{1}{Q_0} \cdot \frac{\partial I_b}{\partial \xi} = -\frac{1.56}{6 \cdot 3600} = -7.2 \cdot 10^{-5}\ {}^1\!/\!{}_\mathrm{s}$$

$$\frac{\dot{\nu}}{\nu} = -\frac{2 \cdot R_i \cdot I_b \cdot \frac{\partial I_b}{\partial \vartheta} - \alpha}{C_{t,b}} = -\frac{-2 \cdot 0.009 \cdot 10.5 \cdot 0.019 - 5 \cdot 10^{-1}}{300} = 0.0017\ {}^1\!/\!{}_\mathrm{s}$$

Problem 7.19

Find the optimal-control formulation (Hamiltonian function and Euler-Lagrange equation) of the energy management of a hybrid powertrain with an ICE and a supercapacitor. Find under which approximation the costate is time-invariant.

- Solution

The state equation (4.117) of the supercapacitor reads

$$\frac{d}{dt} U_{sc}^2 \cdot \left(1 - \frac{R_{sc} \cdot P_{sc}}{U_{sc}^2}\right) = -\frac{2 \cdot P_{sc}}{C_{sc}}.$$

Define $x \triangleq C_{sc} \cdot U_{sc}^2 = 2 \cdot E_{sc}$ as the state variable, then

$$\frac{dx}{dt} = -\frac{2 \cdot P_{sc}}{1 - R_{sc} \cdot P_{sc} \cdot \frac{C_{sc}}{x}}.$$

In this way the Hamiltonian can be built in power terms as

$$H = P_f + s \cdot P_{ech},$$

where $P_{ech} = dx/dt$. The Euler–Lagrange equation reads

$$\frac{ds}{dt} = -\frac{\partial H}{\partial x} = -s \cdot \frac{\partial P_{ech}}{\partial x},$$

where

$$\frac{\partial P_{ech}}{\partial x} = -\frac{2 \cdot P_{sc}^2 \cdot R_{sc} \cdot C_{sc}}{(x - P_{sc} \cdot R_{sc} \cdot C_{sc})^2},$$

which depends on x, thus is not constant. Only if one neglects the resistance R_{sc}, then $\partial P_{ech}/\partial x$ is zero.

Problem 7.20

Formulate the energy-optimal energy management in the case of a double-source electric powertrain, with a battery and a supercapacitor.

- Solution

In this case the optimization criterion is the minimization of the battery consumption, i.e.,

$$L = P_{ech} = U_{oc} \cdot I_b,$$

while the global constraint is over the supercapacitor SOC or voltage,

$$x(t_f) = \frac{1}{2} \cdot C_{sc} \cdot U_{sc}^2(t_f) = x(0).$$

Therefore the Hamiltonian reads

$$H = U_{oc} \cdot I_b(P_b) + s \cdot P_{sc},$$

where $P_{sc} = \frac{dx}{dt}$, and the Euler–Lagrange equation is

$$\frac{ds}{dt} = -\frac{\partial H}{\partial x} = -s \frac{\partial P_{sc}}{\partial x},$$

for whose development see Problem 7.19. The global constraint over the state x can be used to find the unknown initial value of s. However, the constraints locally applied to the state x are even more critical in this case.

Problem 7.21

Formulate the optimal energy management for a parallel HEV that includes engine temperature variations. Assume that the cold-engine fuel consumption is given by an equation of the type

$$\overset{*}{m}_f(T_e, \omega_e, \vartheta_e) = \overset{*}{m}_{f,w}(T_e, \omega_e) \cdot f(T_e, \omega_e, \vartheta_e),$$

where ϑ_e is one engine relevant temperature and $\overset{*}{m}_{f,w}$ is the warm-engine fuel consumption. Moreover, assume an engine temperature dynamic of the type

$$C_{t,e} \cdot \dot{\vartheta}_e = P_{heat}(T_e, \omega_e, \vartheta_e) - \alpha \cdot (\vartheta_e - \vartheta_{amb}) .$$

- Solution

The cost function is $L = \overset{*}{m}_f$. There exist two state variables, namely, the SOC of the battery ξ and the engine temperature ϑ_e. Thus the Hamiltonian is

$$H = \overset{*}{m}_f\ (T_e, \omega_e, \vartheta_e) + \mu \cdot \frac{d\xi}{dt} + \nu \cdot \frac{d\vartheta_e}{dt},$$

where $\dot{\xi} = g(T_e, t, \xi)$ and $\dot{\vartheta}$ is given in the problem text.

The Euler–Lagrange equations read

$$\dot{\mu} = -\frac{\partial H}{\partial \xi} = -\mu \cdot \frac{\partial \dot{\xi}}{\partial \xi} \qquad \text{as usual (see Problem 7.16)}$$

$$\dot{\nu} = -\frac{\partial H}{\partial \vartheta_e} = -\frac{\partial \overset{*}{m}_f}{\partial \vartheta_e} - \nu \cdot \frac{\partial \dot{\vartheta}_e}{\partial \vartheta_e} = -\overset{*}{m}_{f,w} \cdot \frac{\partial f}{\partial \vartheta_e} - \frac{\nu}{C_{t,e}} \cdot \left(\frac{\partial P_{heat}}{\partial \vartheta_e} - \alpha \right)$$

Define $\nu = -\bar{\nu} \cdot C_{t,e}$ to have a third term in the Hamiltonian that has the units of a power. In that case, the state variable would be the thermal energy accumulated E_{th}. Since there is no constraint over the state ϑ_e (or E_{th}), the terminal value of the second Lagrange multiplier ν must be $\nu(T) = 0$.

Problem 7.22

Evaluate the optimal gear ratio profile during an ICE-based vehicle acceleration from rest to v_f on a flat road. Use (i) the acceleration time t_f and (ii) the fuel consumption m_f as the performance index. Make the following simplifying assumptions: constant engine parameters $T_e = T_{e,max}$, e, P_0, continuously variable gear ratio, linearized vehicle dynamics

$$\dot{v} = \frac{F_t}{m_v} - b \cdot v,$$

where $F_t = u \cdot T_e$ and $u = \gamma/r_w$. Verify the solution given by optimal control theory by analyzing the dependency of the criterion on the gear ratio. Numerical data: $b = 10^{-2}$, $u = \gamma/r_w \in [u_{min}, u_{max}] = [2, 12]$, $m_v = 1000\,\text{kg}$, $v_f = 100\,\text{km/h}$, $T_e = 150\,\text{Nm}$, $e = 0.4$, $P_0 = 2\,\text{kW}$.

- Solution

Fuel power consumption:

$$P_f = \frac{T_e \cdot \omega_e + P_0}{e} = \frac{T_e \cdot u \cdot v + P_0}{e}.$$

Case (i)

$$J = \int_0^{t_f} dt = t_f$$
$$L = 1$$
$$H = 1 + \mu \cdot \left(\frac{u \cdot T_e}{m_v} - b \cdot v \right)$$
$$s = \frac{\partial H}{\partial u} = \mu \cdot \frac{T_e}{m_v}$$
$$u^o = \begin{cases} u_{min} & \text{if } s < 0 \\ u_{max} & \text{if } s > 0 \end{cases}$$

Euler–Lagrange equation:

$$\dot{\mu} = -\frac{\partial H}{\partial v} = \mu \cdot b,$$

thus

$$\mu(t) = \mu(0) \cdot e^{b \cdot t}$$

Use the condition that $H \equiv 0$ for "free final time" problems, to find that $\mu(0)$ must be negative,

$$\mu(0) = -\frac{1}{\frac{u \cdot T_e}{m_v}},$$

since the quantity $u \cdot T_e / m_v$ is positive. Thus $\mu(t)$ is always negative and $s(t)$ is negative as well. Consequently,

$$u^o(t) = u_{max}.$$

Intuitively, the longest gear is to accelerate in the least time possible. Of course, speed limits of the engine force the gear shift as soon as the upper limit is reached. The criterion $J = t_f$ is obtained after having calculated

$$\mu(t_f) = -\frac{1}{\frac{u_{max} \cdot T_e}{m_v} - b \cdot v_f}.$$

Then

$$t_f = \frac{1}{b} \cdot \ln\left(\frac{\mu(t_f)}{\mu(0)}\right) = \frac{1}{b} \cdot \ln\left(\frac{1}{1-x}\right),$$

where $x = b \cdot v_f \cdot m_v / (u_{max} \cdot T_e)$. It is easy to verify that t_f is a decreasing function of u. Also verify that

$$v(t) = \frac{u \cdot T_e}{m_v \cdot b} \cdot \left(1 - e^{-b \cdot t}\right)$$

and

$$v(t_f) = v_f \Rightarrow e^{-b \cdot t_f} = 1 - \frac{b \cdot v_f \cdot m_v}{u_{max} \cdot T_e} = 1 - x \Rightarrow e^{b \cdot t_f} = \frac{1}{1-x}$$

as in the previous equation.

Case (ii)

$$\begin{aligned}
J &= \int_0^{t_f} \frac{T_e \cdot u \cdot v + P_0}{e} dt \\
L &= \frac{T_e \cdot u \cdot v}{e} \\
H &= \frac{T_e \cdot u \cdot v}{e} + \mu \cdot \left(\frac{u \cdot T_e}{m_v} - b \cdot v\right) \\
s &= \frac{\partial H}{\partial u} = \frac{T_e \cdot v}{e} + \mu \cdot \frac{T_e}{m_v} = \left(\frac{v}{e} + \frac{\mu}{m_v}\right) \cdot T_e \Rightarrow s = \frac{v}{e} + \frac{\mu}{m_v}
\end{aligned}$$

Euler–Lagrange equation:

$$\begin{aligned}
\dot{\mu} &= -\frac{T_e \cdot u}{e} + \mu \cdot b \\
\mu(t) &= \frac{u \cdot T_e}{b \cdot e} + \left(\mu(0) - \frac{u \cdot T_e}{b \cdot e}\right) \cdot e^{b \cdot t}
\end{aligned}$$

Again, the Hamiltonian must be constantly zero for the optimal solution:

$$H(0) = \mu(0) \cdot \frac{u \cdot T_e}{m_v} = 0 \Rightarrow \mu(0) = 0$$

Thus

$$\mu(t) = \frac{u \cdot T_e}{e \cdot b} \cdot \left(1 - e^{b \cdot t}\right).$$

Moreover, from the state equation for v and the constancy of the control u (either u_{min} or u_{max}), obtain

$$\begin{aligned} v(t) &= \frac{u \cdot T_e}{b \cdot m_v} \cdot \left(1 - e^{-b \cdot t}\right) \\ s(t) &= \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot \left(1 - e^{-b \cdot t}\right) + \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot \left(1 - e^{b \cdot t}\right) \end{aligned}$$

Analyze $s(t)$:

$$\begin{aligned} s(0) &= 0 \\ s(\infty) &= -\infty \\ \dot{s}(0) &= \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot b + \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot (-b) = 0 \end{aligned}$$

Thus $s(t)$ is always negative (except for $t = 0$). Consequently,

$$u^o(t) = u_{max}.$$

Verify the criterion

$$J = \int L\, dt = \frac{u \cdot T_e}{e} \cdot \int \frac{u \cdot T_e}{b \cdot m_v} \cdot \left(1 - e^{-b \cdot t}\right) dt = \frac{u^2 \cdot T_e^2}{e \cdot b \cdot m_v} \cdot \left(t_f + \frac{1}{b} \cdot e^{-b \cdot t_f} - \frac{1}{b}\right).$$

However,

$$e^{-b \cdot t_f} = 1 - \frac{v_f \cdot b \cdot m_v}{u \cdot T_e}$$

and thus

$$J = \frac{u^2 \cdot T_e^2}{e \cdot b \cdot m_v} \cdot \left(t_f + \frac{1}{b} - \frac{v_f \cdot m_v}{u \cdot T_e} - \frac{1}{b}\right) = \frac{u^2 \cdot T_e^2}{e \cdot b \cdot m_v} \cdot t_f - \frac{u \cdot T_e \cdot v_f}{e \cdot b}$$

After inserting the expression for t_f, it is easy to see that J is a decreasing function of u.

ECMS

Problem 7.23

Consider a parallel HEV. The engine is a Willans machine with $e = 0.3$ and $P_{e,0} = 2\,\text{kW}$. The electric drivetrain has a constant efficiency $\eta_{el} = 0.8$ and a maximum/minimum power $P_{m,max/min} = \pm 20\,\text{kW}$. Calculate for which values of the equivalence factor s a purely electric drive and a full recharge, respectively, are optimal for a power demand $P_t = 20$ kW.

- Solution

If $u = P_e/P_d$, i.e., the ratio between the power delivered by the IC engine and the power demand, then

$$H(P_t, s, u) = \frac{u \cdot P_t + P_{e,0}}{e} \cdot h(u) + s \cdot (1-u) \cdot P_t \cdot \eta_{el}^{\text{sign}(u-1)},$$

where $h(.)$ is the unit step fuction. Since

$$\frac{\partial H}{\partial u} = \frac{P_t}{e} - s \cdot P_t \cdot \eta_{el}^{\text{sign}(u-1)}$$

is piecewise constant, the optimal u^o is either at $u = 0$, $u = 1$, or at $u = u_{max}$ (discontinuities). The three values of the Hamiltonian are

$$\begin{aligned} H(0) &= s \cdot \frac{P_t}{\eta_{el}} \\ H(1) &= \frac{P_t}{e} + \frac{P_{e,0}}{e} \\ H(u_{max}) &= \frac{P_t}{e} \cdot u_{max} + s \cdot \eta_{el} \cdot (1 - u_{max}) \cdot P_t + \frac{P_{e,0}}{e}. \end{aligned}$$

The value of u_{max} is such that $(1 - u_{max}) \cdot P_t = P_{m,min}$, which leads to $u_{max} = 2$. After inspection, the purely electric drive ($u = 0$) is selected when

$$H(0) < H(1) \quad \Rightarrow \quad s < \frac{\eta_{el}}{e} \cdot \frac{P_t + P_{e,0}}{P_t} = \frac{0.8}{0.3} \cdot 1.1 = 2.9$$

and

$$H(0) < H(2) \quad \Rightarrow \quad s < \frac{2 \cdot P_t + P_{e,0}}{P_t} \cdot \frac{1}{e \cdot \left(\eta_{el} + \frac{1}{\eta_{el}}\right)} = \frac{2.1}{0.3 \cdot \left(0.8 + \frac{1}{0.8}\right)} = 3.4.$$

Thus, s must be lower than 2.9 for the purely electric drive to be optimal.

The full recharge is optimal when

$$H(2) < H(1) \quad \Rightarrow \quad \frac{2 \cdot P_t}{e} - s \cdot \eta_{el} \cdot P_t < \frac{P_t}{e} \quad \Rightarrow \quad s > \frac{1}{e \cdot \eta_{el}} = 4.2.$$

and

$$H(2) < H(0) \quad \Rightarrow \quad s > 3.4$$

Thus full recharge is optimal when $s > 4.2$). For $2.9 < s < 4.2$, the purely ICE operation is optimal.

Problem 7.24

Derive a look-up table yielding the optimal engine torque T_e of a post-transmission parallel hybrid as a function of ω_w, T_t and s. Use the following engine model,

$$P_f = \begin{cases} \frac{P_{e,0}+P_e}{e}, & \text{for } T_e > 0 \\ 0, & \text{for } T_e > 0 \end{cases}$$

with the following parameters:

$$\frac{1}{e} = \begin{cases} 1.21 \cdot 10^{-5} \cdot \omega_e^2 - 0.0053 \cdot \omega_e + 2.94, & T_e < \min(T_{e,max}, T_{e,turbo}) \\ -1.63 \cdot 1^{-4} \cdot \omega_e^2 - 0.0876 \cdot \omega_e - 6.80, & T_e > T_{e,turbo} \end{cases}$$

$$\frac{P_{e,0}}{e} = \begin{cases} 0.166 \cdot \omega_e^2 + 1.174 \cdot \omega_e + 4.59 \cdot 10^3, & T_e < \min(T_{e,max}, T_{e,turbo}) \\ 5.19 \cdot \omega_e^2 - 2.83 \cdot 10^3 \cdot \omega_e + 2.27 \cdot 10^5, & T_e > T_{e,turbo} \end{cases}$$

with $T_{e,turbo} = 200\,\text{Nm}$ and $T_{e,max} = -0.0038 \cdot \omega_e^2 + 2.32 \cdot \omega_e - 79\,\text{Nm}$. Use the motor model

$$\begin{aligned} P_m &= \omega_m \cdot T_m + (0.0012 \cdot \omega_m + 0.0179) \cdot T_m^2 + \left(-0.0002 \cdot \omega_m^2 + 0.789 \cdot \omega_m + 384\right) = \\ &= \omega_m \cdot T_m + a(\omega_m) \cdot T_m^2 + c(\omega_m) \end{aligned}$$

with $P_{m,max} = 42\,\text{kW}$ and $T_{m,max} = 140\,\text{Nm}$, and the battery model of Problem 4.26 with $R_i = 0$. Find the optimal T_e for $T_t = 1000\,\text{Nm}$, $\omega_w = 39\,\text{rad/s}$, $\gamma_m = 11$, $\gamma = 8.1$ (including the final gear), and $s = 2.8$.

- Solution

The unconstrained optimum of Problem 7.25 must fulfil the constraints

$$T_m > T_{m,min} = -T_{m,max} \text{ and } T_e < T_{e,max}$$

Moreover, the coefficient $1/e$ changes across $T_e = 200$ Nm.

The motor limits at $\omega_m = \omega_r \cdot \gamma_m = 429$ rad/s are $\pm 98\,\text{Nm}$ (both in motoring and generating). These two values correspond to $T_e = -10\,\text{Nm}$ and $T_e = 256\,\text{Nm}$, respectively. At $\omega_e = \omega_w \cdot \gamma = 315$ rad/s, the engine maximum torque is

$$T_{e,max} = -0.0038 \cdot 315^2 + 2.32 \cdot 315 - 79 = 274 \text{ Nm}.$$

Summarizing, the admissible T_e range is between -10 Nm and 256 Nm, with a discontinuity at 200 Nm.

For the assigned operating point, assume first that the optimal solution is below the turbocharging limit. Thus $1/e = 2.47$ ($e = 0.40$), $a = 0.53$. Consequently, from (**??**) of Problem 7.25 find

$$T_m = \frac{315 \cdot 2.47 - \frac{8.1}{11} \cdot 2.8 \cdot 429}{2 \cdot 0.53} = -48.7\,\text{Nm}$$

and

$$T_e = \frac{T_t - \gamma_m \cdot T_m}{\gamma} = 189.6\,\text{Nm}$$

which is a point below the turbocharging limit. To confirm this result, test the other set of Willans parameters. In particular, $1/e = 4.65$ ($e = 0.21$). Consequently,

$$T_m = \frac{315 \cdot 4.65 - \frac{8.1}{11} \cdot 2.8 \cdot 429}{2 \cdot 0.53} = 547\,\text{Nm}$$

which is clearly beyond the motor limit. Thus the optimal point is $T_e = 190\,\text{Nm}$, $T_m = -49\,\text{Nm}$.

For a further verification, calculate the Hamiltonian for $T_e = -10\,\text{Nm}$, $T_e = 0\,\text{Nm}$, $T_e = 190\,\text{Nm}$, $T_e = 200\,\text{Nm}$ and $T_e = 256\,\text{Nm}$ ($a = 0.53$, $c = 686$):

$$\begin{aligned}
&\text{For } T_e = -10,\ \frac{1}{e} = \frac{P_{e,0}}{e} = 0,\ T_m = 98 && \Rightarrow H = 1.34 \cdot 10^5 \\
&\text{For } T_e = 0,\ \frac{1}{e} = 2.47,\ \frac{P_{e,0}}{e} = 21.43 \cdot 10^3,\ T_m = 91 && \Rightarrow H = 1.23 \cdot 10^5 \\
&\text{For } T_e = 190,\ \frac{1}{e} = 2.47,\ \frac{P_{e,0}}{e} = 21.43 \cdot 10^3,\ T_m = -49 && \Rightarrow H = 1.159 \cdot 10^5 \\
&\text{For } T_e = 200,\ \frac{1}{e} = 2.47,\ \frac{P_{e,0}}{e} = 21.43 \cdot 10^3,\ T_m = -56 && \Rightarrow H = 1.163 \cdot 10^5 \\
&\text{For } T_e = 256,\ \frac{1}{e} = 4.65,\ \frac{P_{e,0}}{e} = -1.49 \cdot 10^5,\ T_m = -98 && \Rightarrow H = 1.25 \cdot 10^5
\end{aligned}$$

which confirms the bounded optimum at 190 Nm.

Problem 7.25

Find the unconstrained optimal engine torque for a post-transmission parallel hybrid with an engine model of the type

$$P_f = \frac{P_{e,0} + P_e}{e},$$

an electric machine model of the type

$$P_m = \omega_m \cdot T_m + a \cdot T_m^2 + c,$$

and the battery model of Problem 4.26,

$$P_{ech} = P_b + P_b^2 \cdot \frac{2 \cdot R_i}{U_{oc}^2},$$

where $P_b = P_m$. Neglect the SOC influence.

- Solution

The optimal operating point is the pair (T_e, γ). The optimal T_e is calculated for each transmission ratio γ. For each γ the engine speed $\omega_e = \gamma \cdot \omega_w$ and the motor speed $\omega_m = \gamma_m \cdot \omega_w$ are fixed (γ_m is usually a constant). Thus the coefficients $1/e$, $P_{e,0}/e$, a, and c are also fixed.

The Hamiltonian is

$$\begin{aligned} H &= P_f + s \cdot P_{ech} = \\ &= \frac{1}{e} \cdot \omega_e \cdot T_e + \frac{P_{e,0}}{e} + \\ &\quad + s \cdot \left(\left(\omega_m \cdot T_m + a \cdot T_m^2 + c\right) + \frac{2 \cdot R_i}{U_{oc}^2} \cdot \left(\omega_m \cdot T_m + a \cdot T_m^2 + c\right)^2 \right). \end{aligned}$$

To find the optimal T_e, differentiate the Hamiltonian

$$\begin{aligned} \frac{\partial H}{\partial T_e} &= \frac{\omega_e}{e} - s \cdot \left(1 + \frac{2 \cdot R_i}{U_{oc}^2} \cdot 2 \cdot \left(\omega_m \cdot T_m + a \cdot T_m^2 + c\right) \right) \cdot (\omega_m + 2 \cdot a \cdot T_m) \cdot \gamma = \\ &= 0 \end{aligned}$$

since $\gamma_m \cdot T_m = T_t - \gamma \cdot T_e$ and $\partial T_m / \partial T_e = -\gamma / \gamma_m$. If one neglects the loss term in the battery model, the resulting equation is

$$\frac{\omega_e}{e} = \frac{\gamma}{\gamma_m} \cdot s \cdot (\omega_m + 2 \cdot a \cdot T_m)$$

and the optimal solution would be

$$T_m = \frac{\frac{\omega_e}{e} - \frac{\gamma}{\gamma_m} \cdot s \cdot \omega_m}{2 \cdot a} \cdot \frac{1}{s} \cdot \frac{\gamma_m}{\gamma},$$

from whence

$$T_e = \frac{T_t - \gamma_m \cdot T_m}{\gamma} = \frac{T_t}{\gamma} - \left(\frac{\gamma_m^2}{2 \cdot a \cdot e \cdot s \cdot \gamma} - \frac{\gamma_m^2}{2 \cdot a \cdot \gamma} \right) \cdot \omega_w.$$

Problem 7.26

Use the result of Problem 4.21 and a simplified battery model $P_{ech} = P_b$ to derive an analytical solution of the optimal energy management of a series hybrid. Following Problem 4.21, consider the engine Willans parameter varying as

$$\frac{1}{e} = \begin{cases} 0 & \text{for} \quad P_g = 0 \\ 4.01 & \text{for} \quad 0 < P_g \leq 14 \cdot 0.92 \cdot 10^3 \\ 3.36 & \text{for} \quad 14 \cdot 0.92 \cdot 10^3 < P_g \leq 62 \cdot 0.92 \cdot 10^3 \\ 3.89 & \text{for} \quad 62 \cdot 0.92 \cdot 10^3 < P_g \leq 68 \cdot 0.92 \cdot 10^3, \end{cases}$$

and $\eta_g = 0.92$.

- Solution

The Hamiltonian is

$$H = P_f + s \cdot P_{ech} = \frac{P_0}{e} + \frac{P_g}{\eta_g \cdot e} + s \cdot (P_m - P_g).$$

This function is affine in P_g, and

$$\frac{\partial H}{\partial P_g} = \frac{1}{\eta_g \cdot e} - s$$

However, the coefficient $1/e$ changes with P_g: Now, the possible solutions are at the discontinuity points. For low s, the optimum is at $P_g = 0^-$ and $H(0^-) = s \cdot P_m$. For increasing s, the optimum switches toward $P_g = P_1 \cdot \eta_g = 57$ kW, where P_1 is the engine power at $\omega_e = 1000$ rpm. The switching value of s is calculated by equating

$$s \cdot P_m = 214.8 \cdot 10^3 + s \cdot \left(P_m - 62 \cdot 10^3 \cdot \eta_g\right) \qquad \Rightarrow \qquad s = 3.76$$

For $s > 3.95$, the optimum shifts to $P_g = P_{max} \cdot \eta_g = 62.8$ kW.

Problem 7.27

Derive equations (7.24) – (7.25) from PMP.

- Solution

For a parallel hybrid with constant efficiencies η_f and η_e, the Hamiltonian is

$$H(s, P_e) = \begin{cases} \frac{P_e}{\eta_f} + s \cdot \frac{P_t - P_e}{\eta_e}, & \text{for } 0 < P_e < P_t \\ \frac{P_e}{\eta_f} + s \cdot \eta_e \cdot (P_t - P_e), & \text{for } P_t < P_e < P_{max} \end{cases}$$

that is, the dependency $H(P_e)$ is piecewise affine and consists of two segments. For small values of s, $\partial H/\partial P_e$ is always positive, thus the optimal value is $P_e = 0$.

The value of s for which $\partial H/\partial P_e = 0$ in the first segment is $s_1 = \eta_e/\eta_f$. Beyond this value, the first segment is increasing and the second is decreasing, thus the optimal control is $P_e = P_t$.

The value of s for which $\partial H/\partial P_e = 0$ in the second segment is $s_2 = 1/(\eta_e \cdot \eta_f)$. For higher values of s, the second segment is decreasing, thus the optimal control is $P_e = P_{max}$.

Summarizing, one recovers that $s = s_1$ or lower values lead to a battery discharge; $s = s_2$ or higher values lead to a battery recharge, while any value between s_1 and s_2 leads to pure ICE operation, thus charge-sustained operation (but no hybrid operation).

Problem 7.28

For the simple parallel HEV model of Problem 7.23, with $e = 0.4$, $P_{e,0} = 3\,\mathrm{kW}$, $\eta_{el} = 0.9$, find the conditions on P_t for which the ZEV mode, the ICE mode or the battery recharge with $P_b = -2 \cdot P_t$ are optimal, respectively.

- Solution

As opposed to the results of Problem 7.23 the solution is now to be found as a function of s. After inspection, three possibilities arise:

- If $s < s_1 = \frac{\eta_{el}}{e} = 3$, then only the purely electric mode ($P_e = 0$) could be optimal.
- If $s_1 < s < s_2 = \frac{1}{\eta_{el} \cdot e} = 3.7$, then the optimum is either the purely electric mode or the purely ICE operation. The switch is when

  $$P_t = \frac{\eta_{el} \cdot P_{e,0}}{s \cdot e - \eta_{el}} = \frac{27\ \mathrm{kW}}{4 \cdot s - 9},$$

  thus below which ZEV mode, above which ICE mode are optimal.
- If $s > s_2$, then again two possibilities arise, namely, the ZEV or the recharge mode. The switch is for

  $$P_t = \frac{\eta_{el} \cdot P_{e,0}}{s \cdot e - 2 \cdot \eta_{el} + s \cdot \eta_{el}^2 \cdot e} = \frac{2700\ \mathrm{kW}}{724 \cdot s - 1800}.$$

Problem 7.29

Use the result of Problem 7.28 to evaluate s over a drive cycle with the following characteristics: $\bar{E}_{trac} - \bar{E}_{rec} = \Delta\bar{E} = 0.183$ MJ, $\bar{E}_{trac} = 0.670$ MJ, $P_{max} = 18.9\,\mathrm{kW}$. Assume a linear relationship between cumulative energy and power demand. Then perform again the calculations for the data of Problem 7.23.

- Solution

The condition for s to be optimal is that the electrical energy is balanced over the cycle, thus

$$\eta_{el} \cdot \left(\Delta\bar{E} + \bar{E}_{chg}\right) = \frac{\bar{E}_{zev}}{\eta_{el}}$$

where $\bar{E}_{chg}$ is the mechanical energy demand during the recharge phase (the same quantity is sent from the engine to the generator because $u = 2$) and $\bar{E}_{zev}$ is the mechanical energy demand during the ZEV phase.

Assume first that $s > s_2$. Then two phases exist, ZEV or recharge. The switch power is

$$P_{lim} = \frac{\eta_{el} \cdot P_{e,0}}{s \cdot e - 2 \cdot \eta_{el} + s \cdot \eta_{el}^2 \cdot e}$$

and this relationship will be used to calculate s once P_{lim} has been found.

To calculate P_{lim}, observe that $\bar{E}_{zev}$ is the energy for power demand lower than P_{lim}. Since a linear relationship between energy and power demand is assumed,

$$E(P_t) = \frac{\bar{E}_{trac}}{P_{max}} \cdot P_t,$$

then

$$\bar{E}_{zev} = \frac{\bar{E}_{trac}}{P_{max}} \cdot P_{lim}.$$

Consequently, $\bar{E}_{chg} = \bar{E}_{trac} - \bar{E}_{zev}$. Using these equations, obtain

$$\begin{aligned}\eta_{el} \cdot \Delta\bar{E} + \eta_{el} \cdot \bar{E}_{trac} - \eta_{el} \cdot \frac{\bar{E}_{trac}}{P_{max}} \cdot P_{lim} &= \frac{\bar{E}_{trac}}{P_{max} \cdot \eta_{el}} \cdot P_{lim} \\ \Rightarrow \quad P_{lim} &= \frac{\eta_{el}^2}{1+\eta_{el}^2} \cdot \frac{\bar{E}_{trac} + \Delta\bar{E}}{\bar{E}_{trac}} \cdot P_{max} = 10.8 \text{ kW},\end{aligned}$$

from whence, calculate the charge-sustaining and optimal value of s as

$$s = \frac{2 \cdot P_{lim} + P_{e,0}}{e \cdot \left(\frac{P_{lim}}{\eta_{el}} + \eta_{el} \cdot P_{lim} \right)} = 2.83.$$

To verify the initial assumption,

$$s_2 = \frac{1}{\eta_{el} \cdot e} = 2.78 < s.$$

So the assumption was correct and the result is valid.

For the data of Problem 7.23, $e = 0.3$, $P_{e,0} = 2$ kW, $\eta_{el} = 0.8$, one would obtain $s_1 = 2.67$, $s_2 = 4.17$. Assuming $s > s_2$ would lead to $P_{lim,a} = 9.41$ kW and $s_a = 3.60$ which is not greater than s_2.

Assuming instead that $s_1 < s < s_2$, there is a switch between ZEV and ICE modes. Thus the energy balance is

$$\eta_{el} \cdot \Delta\bar{E} = \frac{\bar{E}_{zev}}{\eta_{el}}, \text{ or } P_{lim,b} = \frac{\eta_{el}^2 \cdot \Delta\bar{E} \cdot P_{max}}{\bar{E}_{trac}} = 3.3 \text{ kW}.$$

This switch power corresponds to

$$s_b = \frac{P_{lim,b} + P_{e,0}}{P_{lim,b}} \cdot \frac{\eta_{el}}{e} = 4.28,$$

which is not lower than s_2 as assumed.

Finally the only possible result could be $s = s_2$, so that the recharge and the ICE mode are equally optimal. A switch will be added in order to balance the battery energy. For $s = s_2$, the limit power for the ZEV mode is

$$\frac{\eta_{el}^2}{1-\eta_{el}^2} \cdot P_{e,0} = 3.56 \text{ kW}.$$

The corresponding ZEV energy is 0.126 MJ. To be balanced, a recharge energy

$$\bar{E}_{chg} = \frac{\bar{E}_{zev}}{\eta_{el}^2} - \Delta\bar{E} = 13.3 \text{ kJ}$$

is needed. Thus a further power limit

$$P_{max} \cdot \left(1 - \frac{\bar{E}_{chg}}{\bar{E}_{trac}}\right) = 15.4 \text{ kW}$$

can be taken, above which the recharge mode is selected and below which the ZEV mode is selected.

Implementation Issues

Problem 7.30

Consider a post-transmission parallel HEV with the following simplified data: motor transmission ratio $\gamma_m = 11$, wheel radius $r_w = 0.317\,\text{m}$, engine transmission ratio $\gamma = \{15.02, 8.09, 5.33, 3.93, 3.13, 2.59\}$, transmission efficiency $\eta_t = 0.95$. Consider the following driving situation: torque demand at the wheels $T_t = 378\,\text{Nm}$, vehicle speed $v = 69.25\,\text{km/h}$, engine on, electric consumers off, 4^{th} gear. In this situation, $T_{m,max} = 140\,\text{Nm}$, $P_{m,max} = 42\,\text{kW}$, $U_{b,min} = 300\,\text{V}$, $U_{b,max} = 420\,\text{V}$, $P_{e,start} = 3\,\text{kW}$, $P_m = 0.9345 \cdot T_m^2 + 673.97 \cdot T_m + 127.44$, $U_{oc} = 381.12\,\text{V}$, $R_i = 0.3648\,\Omega$ (discharge), $R_i = 0.3264\,\Omega$ (charge), Coulombic efficiency $\eta_c = 0.95$, $T_{e,max} = 269.7\,\text{Nm}$, $T_{e,min} = -20\,\text{Nm}$, fuel consumption

$$\begin{aligned} \overset{*}{m}_f &= (T_e - T_{e,min}) \cdot \left(2.9 \cdot 10^{-8} \cdot T_e + 1.112 \cdot 10^{-5}\right) = \\ &= 2.9 \cdot 10^{-8} \cdot T_e^2 + 1.17 \cdot 10^{-5} \cdot T_e + 2.225 \cdot 10^{-4} \end{aligned}$$

Find the engine and motor torque calculated by the ECMS. The current estimation of the equivalence factor is $s = 3$.

- Solution

Calculate first the electric-mode Hamiltonian H_{ev}:

$$\begin{aligned} \omega_m &= \frac{v_v}{r_w} \cdot \gamma_m = 667 \text{ rad/s}, & \omega_b &= \frac{42 \cdot 10^3}{140} = 300 \text{ rad/s} \\ T_m(0) &= \frac{T_t}{\gamma_m} = 34.4 \text{ Nm}, & T_{m,max} &= \frac{P_{m,max}}{\omega_m} = 63 \text{ Nm} \end{aligned}$$

The motor speed ω_m is above the base speed ω_b, further the motor torque is smaller than the maximum possible motor torque. The eletrical power of the motor is calculated with the given relationship

$$P_m(0) = 0.9345 \cdot 34.4^2 + 673.97 \cdot 34.4 + 127.44 = 24.4 \text{ kW} = P_b(0).$$

The battery power is limited by

$$P_{b,max} = \min\left(\frac{U_{oc}^2}{4 \cdot R_i}, \left|\frac{U_{oc} \cdot U_{b,min} - U_{b,min^2}}{R_i}\right|\right) = 66.7 \text{ kW},$$
$$P_{b,min} = -\frac{U_{b,max}^2 - U_{oc} \cdot U_{b,max}}{R_i} = -50.0 \text{ kW}.$$

Thus the condition $P_{b,min} < P_b < P_{b,max}$ is fulfilled. Evaluate

$$I_b(0) = \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\frac{U_{oc}^2 - R_i \cdot 4 \cdot P_b}{4 \cdot Ri^2}} = 68.5 \text{ A},$$
$$P_{ech}(0) = U_{oc} \cdot I_b(0) = 26.1 \text{ kW}.$$

Thus the hamiltionian for the ZEV case is

$$H_{ev} = s \cdot P_{ech} = 3 \cdot 26.1 = 79.2\,\text{kW}.$$

Now calculate *hybrid* Hamiltonians. For simplicity take only three candidate values:

$$T_e(1) = T_{e,max}, \quad T_e(2) = \frac{T_t}{\gamma_e \cdot \eta_t} = 101.2 \text{ Nm}, \quad T_e(3) = T_{e,min}$$

The corrseponding fuel consumptions are

$$\overset{*}{m}_f\ (1) = (269.7 + 20) \cdot (2.9 \cdot 10^{-8} \cdot 269.7 + 1.112 \cdot 10^{-5}) = 5.49 \text{ g/s}$$
$$\overset{*}{m}_f\ (2) = (101.2 + 20) \cdot (2.9 \cdot 10^{-8} \cdot 101.2 + 1.112 \cdot 10^{-5}) = 1.70 \text{ g/s}$$
$$\overset{*}{m}_f\ (3) = 0 \text{ g/s}$$

The electric motor has to provide the folowing torque (note the role of the transmission efficiency):

$$T_m(1) = \frac{T_t - \gamma_e \cdot T_{e,max} \cdot \eta_t}{\gamma_m} = -57.2 \text{ Nm},$$
$$T_m(2) = 0 \text{ Nm},$$
$$T_m(3) = \frac{T_t - \frac{\gamma_e \cdot T_{e,min}}{\eta_t}}{\gamma_m} = 41.9 \text{ Nm}.$$

All three absolute values are lower than 63 Nm, so the constraints are not violated. Calculate the electric power

$$P_m(1) = 0.9345 \cdot 57.2^2 - 673.97 \cdot 57.2 + 127.44 = -35.37 \text{ kW} = P_b(1),$$
$$P_m(2) = 0.13 \text{ kW} = P_b(2),$$
$$P_m(3) = 0.9345 \cdot 41.9^2 + 673.97 \cdot 41.9 + 127.44 = 30 \text{ kW} = P_b(3).$$

All three values are between -50 kW and 66.7 kW, so the battery limit is not overstepped. Again, the battery current is derived with the same formula as above as

$$
\begin{aligned}
I_b(1) &= \frac{381.12}{2 \cdot 0.3264} - \sqrt{\frac{381.12^2 + 0.3264 \cdot 4 \cdot 35.37 \cdot 10^3}{4 \cdot 0.3264^2}} = -86.4 \text{ A}, \\
I_b(2) &= \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 127}{4 \cdot 0.3648^2}} = 0.33 \text{ A}, \\
I_b(3) &= \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 30 \cdot 10^3}{4 \cdot 0.3648^2}} = 85.8 \text{ A},
\end{aligned}
$$

which results in electrochemical power consumptions

$$
\begin{aligned}
P_{ech}(1) &= -381.12 \cdot 86.4 \cdot 0.95 = -31.28 \text{ kW}, \\
P_{ech}(2) &= \quad 381.12 \cdot 0.33 = 0.13 \text{ kW}, \\
P_{ech}(3) &= \quad 381.12 \cdot 85.8 = 32.7 \text{ kW}.
\end{aligned}
$$

Combining these results leads to

$$
\begin{aligned}
H_{hyb} &= 42.6 \cdot 10^6 \cdot \{5.49, 1.705, 0\} \cdot 10^{-3} + 3 \cdot \{-31.28, 0.13, 32.7\} \cdot 10^3 = \\
&= \{140, 73, 98\} \text{ kW}.
\end{aligned}
$$

Finally, the chosen operating point will be the pure ICE operation. Thus the engine will continue to stay on.

Problem 7.31

Solve again Problem 7.30 for the situation in which the engine is turned off.

- Solution

Nothing changes up to the calculation of $P_b(1, \ldots, 3)$. In order to account for the engine turning on phases, add $P_{e,start} = 3$ kW to the battery power:

$$
\begin{aligned}
P_b(1) &= -35.37 + 3 = -32.37 \text{ kW}, \\
P_b(2) &= 0.13 + 3 = 3.13 \text{ kW}, \\
P_b(3) &= 30 + 3 = 33 \text{ kW}.
\end{aligned}
$$

All three values are still admissible. Again the battery currents are calculated as

$$
\begin{aligned}
I_b(1) &= \frac{381.12}{2 \cdot 0.3264} - \sqrt{\frac{381.12^2 + 0.3264 \cdot 4 \cdot 32.37 \cdot 10^3}{4 \cdot 0.3264^2}} = -79.5 \text{ A}, \\
I_b(2) &= \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 3.13}{4 \cdot 0.3648^2}} = 8.27 \text{ A}, \\
I_b(3) &= \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 33 \cdot 10^3}{4 \cdot 0.3648^2}} = 95.3 \text{ A},
\end{aligned}
$$

which results in electrochemical power consumptions

$$\begin{aligned} P_{ech}(1) &= -381.12 \cdot 79.5 \cdot 0.95 = -28.79 \text{ kW}, \\ P_{ech}(2) &= \quad 381.12 \cdot 8.27 = 3.15 \text{ kW}, \\ P_{ech}(3) &= \quad 381.12 \cdot 94.2 = 36.3 \text{ kW}. \end{aligned}$$

Combining these results leads to

$$\begin{aligned} H_{hyb} &= 42.6 \cdot 10^6 \cdot \{5.49, 1.705, 0\} \cdot 10^{-3} + 3 \cdot \{-28.79, 3.15, 36.3\} \cdot 10^3 = \\ &= \{147.5, 82, 109\} \text{ kW} \end{aligned}$$

In this case $H_{ev} \leq \min(H_{hyb})$ and it is more convenient to keep the engine off and use the ZEV mode.

Problem 7.32

Consider an ECMS with a stop-start strategy implementation based on hysteresis thresholds. In order to start the engine, H_{hev} (see Problem 7.30) must fulfill the condition

$$H_{hev} < x_{on} \cdot H_{ev}.$$

At the previous calculation step, the lower Hamiltonian value was H_{ev}, thus the engine is off. At the current time step, the power demand is $P_t = 13.28\,\text{kW}$. The equivalence factor estimation is $s = 3.2813$. Calculate the mode selected for a hysteresis threshold x_{on} of (i) 95% and (ii) 90%, respectively. Use the following data and models: post-transmission parallel HEV architecture, transmission efficiency, $\eta_t = 0.95$, fuel consumption $P_f = 2.5446 \cdot P_e + 9.6525 \cdot 10^3$ if $P_e > 0$, electrochemical power

$$P_{ech} = \begin{cases} 1.2707 \cdot P_m + 2.7703 \cdot 10^3 - 2.014 \cdot 10^3, & \text{if } P_m > -595 \text{ W} \\ 0.7397 \cdot P_m + 2.4544 \cdot 10^3 - 2.014 \cdot 10^3, & \text{if } P_m < -595 \text{ W}. \end{cases}$$

The cost of engine start is $P_{e,start} = 2.014$ kW (in electrochemical power units).

- Solution

The Hamiltonian function is bilinear. Thus the optimum combination can be either at $P_e = 0\,\text{W}$ (ZEV mode), at $P_m = -595\,\text{W}$ (discontinuity), or at $P_e = P_{e,max}$. The engine power at the discontinuity is

$$P_{e,dis} = \frac{13.282 \cdot 10^3 + 595}{0.95} = 14.607 \text{ kW}.$$

However, a simple inspection of equations above shows that

$$\frac{\partial H}{\partial P_e} = \begin{cases} -1.1465, & \text{for } 0 < P_e < 14.607 \cdot 10^3 \text{ W} \\ 0.2388, & \text{for } P_e > 14.607 \cdot 10^3 \end{cases}$$

Thus the minimum of H is either at the discontinuity point or at the ZEV mode:

- for $P_e = 0$, $P_m = 13.282\,\text{kW}$ and $P_{ech} = 19.648 - 2.014\,\text{kW}$, thus

$$H_{ev} = 57.86 \text{ kW};$$

- for $P_e = 14.607\,\text{kW}$, $P_f = 46.821\,\text{kW}$, $P_m = -595\,\text{W}$, $P_{ech} = 2.014\,\text{kW}$, the engine must be started and

$$H_{hyb} = 53.4295 \text{ kW}.$$

In order to choose between H_{ev} and H_{hyb} the hysteresis threshold x_{on} must be considered:

- for (i) $x_{on} = 95\%$, the engine will be turned on if $53.43 \leq 0.95 \cdot 57.86 = 54.97$, which is true.
- for (ii) $x_{on} = 90\%$, the engine should be turned on if $53.43 \leq 0.90 \cdot 57.86 = 52.07$, which is not true.

  Thus in this case the engine should be kept off.

Problem 7.33

Consider an HEV under several repetitions of an elementary driving cycle. An ECMS has a PI adaptation of s as a function of SoC that yields a new estimation every cycle repetition. Assume that the overall behavior of the system on a *cycle-by-cycle* basis depends on s as follows:

$$\Delta\xi(n) = \xi(n) - \xi(n-1) = K_s \cdot (s\,(n) - s_0)\,,$$

where $\xi(n)$ is the SoC at the end of the n-th repetition, s_0 is the optimal value of s, $s(n)$ is the value adopted during the n-th cycle, and $K_s > 0$ is a constant depending on the particular system. Evaluate the stability and the dynamic characteristics of the controlled system on a cycle-by-cycle basis. Evaluate the influence of the integral term in the PI controller.

- Solution

Let $\Delta\xi(n)$ be the variation of SOC on the n-th *cycle*, i.e., in a first approximation, a quantity proportional to the electrochemical energy consumption of the cycle. The cycle-by-cycle dependency between SOC and s can be linearized as

$$\Delta\xi(n) = \xi(n) - \xi(n-1) = K_s \cdot (s\,(n) - s_0)\,,$$

where s_0 is the optimal theoretical value of s, $s(n)$ is the value adopted during the n-th cycle, and $K_s > 0$ is a constant depending on the particular system. Correspondingly, the adaptation rule for $s(n)$ reads

$$s(n) = s_p - k_p \cdot \xi(n-1) - k_i \cdot \sum_{i=0}^{n-1} \xi(i),$$

where $\Delta\xi(0) = 0$ by definition, $k_p > 0$, $k_i \geq 0$ and s_p initial value of s. By shifting from the discrete time to the continuous time, and by combining the two equations, one obtains

$$\frac{d^2 s}{dt^2} = -k_p \cdot K_s \cdot \frac{ds}{dt} - k_i \cdot K_s \cdot (s - s_0).$$

This dynamics is stable. The factor s converges to s_0 (unknown) as prescribed. Thus the quantity Δx converges to 0. The true SOC error, the integral of $\Delta\xi$, converges to zero as well (the prescribed value). The integral of the SOC error converges to the value $(s_p - s_0)/k_i$.

If $k_i = 0$, the SOC error does not vanish but it tends to $(s_p - s_0)/k_p$.

Problem 7.34

Compare (7.22) with (7.27) – (7.31). Under which assumptions are they equivalent?

- Solution

Combining (7.27) – (7.31), obtain

$$s(t) = s_{chg} + (s_{dis} - s_{chg}) \cdot p(t) \triangleq s_{chg} + (s_{dis} - s_{chg}) \cdot (p_0 + p_1(t) \cdot E_{ech}(t)) \, .$$

In the assumption that the difference $E_h - E_h(t) \triangleq \Delta E_h$ is kept constant (sliding horizon), the coefficient p_1 is constant as well and

$$s(t) = s_{chg} + (s_{dis} - s_{chg}) \cdot p_0 + (s_{dis} - s_{chg}) \cdot p_1 \cdot E_e(t).$$

The term $E_{ech}(t)$ is the electrochemical energy consumed. In terms of SOC,

$$E_{ech}(t) = Q_0 \cdot (\xi(0) - \xi(t)) \cdot U_{oc}(t)$$

and, assuming an averagely constant open-circuit voltage,

$$E_{ech}(t) = K \cdot (\xi(0) - \xi(t)) \, .$$

If $\xi(0) = \xi_t$, obtain (7.22) with

$$\begin{aligned} s_t &= s_{chg} + (s_{dis} - s_{chg}) \cdot p_0, \\ k_p &= (s_{dis} - s_{chg}) \cdot p_1 \cdot K. \end{aligned}$$

Now give a closer look to p_0 and p_1. To simplify the analysis, assume $u_r = u_l = 1$ and neglect λ. Thus,

$$p_0 = \frac{\frac{1}{\eta_e}}{\frac{1}{\eta_e} + \eta_e} \qquad \text{and} \qquad p_1 = \frac{1}{\left(\frac{1}{\eta_e} + \eta_e\right) \cdot \Delta E_h}.$$

Further impose (7.26)

$$\frac{1}{\eta_e} = s_{dis} \cdot \eta_f, \qquad \eta_e = s_{chg} \cdot \eta_f$$

to find

$$\begin{aligned} p_0 &= \frac{s_{dis}}{s_{dis} + s_{chg}} \\ p_1 &= \frac{1}{(s_{dis} + s_{chg}) \cdot \eta_f \cdot \Delta E_m} \\ s_t &= s_{chg} + (s_{dis} - s_{chg}) \cdot \frac{s_{dis}}{s_{dis} + s_{chg}} = \frac{s_{chg}^2 + s_{dis}^2}{s_{dis} + s_{chg}}. \end{aligned}$$

It is easy to show that under the aforementioned assumptions $s_{chg} < s_t < s_{dis}$, thus s_t plays the role of the constant optimal equivalence factor s_0.

Problem 7.35

Express s_0 as a function of u_r, u_l, s_{dis}, and s_{chg}. Evaluate s_0 for $s_{max} = 5$, $s_{min} = 2$, knowing from a cycle analysis that $u_r/\eta_e = 1.2$ and $u_l \cdot \eta_e = 1.8$.

- Solution

Using the method of Problem 7.34, but with $u_r \neq u_l$, obtain

$$s_0 = \frac{u_l \cdot s_c^2 + u_r \cdot s_d^2}{u_r \cdot s_d + u_l \cdot s_c}.$$

For the numerical case ($s_{max} = s_d$ and $s_{min} = s_c$.

$$\begin{aligned} \eta_e &= \sqrt{\frac{s_c}{s_d}} = 0.63 \\ u_r &= 0.76 \\ u_l &= 2.85 \\ s_0 &= \frac{2.85 \cdot 2^2 + 0.76 \cdot 5^2}{0.76 \cdot 5 + 2.85 \cdot 2} = 3.2. \end{aligned}$$

Remark: the value of u_r found shows that pure ZEV would not be allowed in this case.