# Approximation of Continuous-Time Infinite-Horizon Optimal Control Problems Arising in Model Predictive Control

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Abstract— This article presents two different approximations to linear infinite-horizon optimal control problems arising in model predictive control. The dynamics are approximated using a Galerkin approach with parametrized state and input trajectories. It is shown that the first approximation represents an upper bound on the optimal cost of the underlying infinite dimensional optimal control problem, whereas the second approximation results in a lower bound. We analyze the convergence of the costs and the corresponding optimizers as the number of basis functions tends to infinity. The results can be used to quantify the approximation quality with respect to the underlying infinite dimensional optimal control problem.

#### I. INTRODUCTION

Model predictive control (MPC) is one of few control strategies that take input and state constraints fully into account and is, therefore, applicable to a wide range of challenging control problems. The underlying principle is the following: At each time step, an optimal control problem is solved, which includes a prediction of the system's state over a certain time horizon subject to the current state as an initial condition. The first portion of the obtained input trajectory is applied to the system before repeating the optimization in the next time step, resulting in an implicit feedback law. The standard MPC formulation is based on a discrete-time representation of the dynamics, includes a finite time horizon, and often requires a combination of terminal state constraint and terminal cost to guarantee closed-loop stability, [1]. In [2], an alternative approach has been proposed, which avoids truncating the time horizon by relying on a continuous-time approximation of the underlying optimal control problem. As a result, stability guarantees were shown to arise naturally from the problem formulation.

The approach from [2] was found to yield an upper bound on the optimal cost of the underlying optimal control problem. Herein, we show that a slight variation of the approximation from [2] leads to a lower bound. We will analyze both approximations with respect to convergence of the optimal costs and the corresponding optimizers as the basis function complexity is increased. By combining both approaches, the approximation quality with respect to the underlying infinite dimensional optimal control problem can be quantified.

*Related work:* The infinite-horizon optimal control problems arising in MPC are typically formulated in discrete time over a finite time horizon, [3]. Truncating the time horizon leads inevitably to issues with stability and recursive feasibility, which are, for example, addressed in [4]–[6]. These issues are alleviated with the parametrized infinite-horizon formulation presented in [2], which is further analyzed herein.

A different approach is proposed in [7] and [8], where strictly monotonic transformations from the time domain  $[0,\infty)$  to the bounded interval [-1,1) are used to reformulate the infinite-horizon optimal control problem as a finitehorizon problem. The resulting optimal control problem is discretized using pseudospectral methods via Legendre-Gauss (in [8]) and Legendre-Gauss-Radau (in [7] and [8]) collocation. Due to the transformation of the infinite time domain to [-1, 1), linear dynamics on  $[0, \infty)$  are transformed to nonlinear dynamics on [-1, 1), and as a result, non-convex finite dimensional optimization problems are obtained. The dynamics are not guaranteed to be fulfilled exactly, which renders reasoning about closed-loop stability in the context of predictive control more involved. This is due to the fact that arguments for showing closed-loop stability rely often on exact open-loop predictions of the state trajectory, see [1] or [2].

The underlying infinite dimensional infinite-horizon problem has proven to be difficult to analyze, as, for example, the transversality conditions of the maximum principle cannot be extended directly to the infinite-horizon problem, see [9, Ch. 3.7, Ch. 6.5], [10], and [11]. We circumvent these problems by assuming from the outset that unique solutions to the underlying infinite dimensional optimal control problem exist, and restrict the state trajectory, the time derivative of the state trajectory, and the input trajectory to be square integrable, similar to [11]. The finite dimensional approximations, which will be introduced in the following, are convex problems with a strictly convex cost and a closed domain, and therefore the corresponding minimizers are unique, provided that feasible trajectories exist.

The parametrization of input trajectories using Kauz or Laguerre basis functions, similar to the ones used herein has been proposed in [12] and [13] for solving MPC problems. It has been argued that such a parametrization results in an optimization problem with fewer variables offering computational benefits. Still, the finite-horizon formulation is retained in [12] and [13].

The authors of [14] use polynomials for approximating continuous linear programs. Similar to the approach presented herein, duality is exploited for constructing approximations yielding upper and lower bounds on the underlying continuous linear program. The resulting semi-infinite

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constraints are reformulated using sum-of-squares techniques yielding semidefinite programs. Although to some extent related, the optimal control problem that is considered in the following cannot be cast as a continuous linear program. Moreover, we treat equality constraints (in the form of linear ordinary differential equations) and inequality constraints differently. In addition, we present an axiomatic treatment for dealing with the inequality constraints, and as a consequence, we do not require the inequality constraints to be polyhedral.

Thus, in contrast to previous work, the approach presented herein tackles the infinite-horizon problem directly by means of a parametric description of input and state trajectories. A detailed analysis regarding convergence of the corresponding optimization problems is presented.

*Outline:* Sec. II introduces the underlying optimal control problem we seek to approximate. The two finite dimensional approximations are subsequently presented. The convergence results are discussed in Sec. III, and are underpinned with a numerical example in Sec. IV. The article concludes with a summary in Sec. V.

#### **II. PROBLEM FORMULATION**

We present and analyze two approximations to the following optimal control problem,

$$J_{\infty} := \inf \frac{1}{2} ||x||_{2}^{2} + \frac{1}{2} ||u||_{2}^{2}$$
  
s.t.  $\dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_{0},$  (1)  
 $x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \quad \forall t \in [0, \infty),$   
 $x \in L_{n}^{2}, \quad u \in L_{m}^{2}, \quad \dot{x} \in L_{n}^{2},$ 

where  $\mathcal{X}$  and  $\mathcal{U}$  are closed and convex subsets of  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively, containing 0; the space of square integrable functions mapping from  $[0, \infty)$  to  $\mathbb{R}^q$  is denoted by  $L_q^2$ , where q is a positive integer; and the  $L_q^2$ -norm is defined as

$$L_q^2 \to \mathbb{R}, \quad x \to ||x||_2^2 := \int_0^\infty x^\mathsf{T} x \, \mathrm{d}t,$$
 (2)

where dt denotes the Lebesgue measure.<sup>1</sup>

Note that the change of variables  $U\hat{x} = x$  and  $V\hat{u} = u$ , with  $U \in \mathbb{R}^{n \times n}$  and  $V \in \mathbb{R}^{m \times m}$  invertible, transforms the running cost to the slightly more general quadratic objective

$$\frac{1}{2} \int_0^\infty \hat{x}^{\mathsf{T}} Q \hat{x} + \hat{u}^{\mathsf{T}} R \hat{u} \, \mathrm{d}t, \quad Q = U^{\mathsf{T}} U, R = V^{\mathsf{T}} V. \quad (3)$$

The approximations, which will be introduced in Sec. II-D are invariant with respect to bijective linear transformations and therefore all results remain valid if the objective function in (1) is replaced by (3). In addition, the results extend to the case where state and input trajectories are required to be elements of weighted  $L^2$ -spaces and (2) is replaced by a weighted  $L^2$ -norm, [11]. Moreover, parts of the results can be readily extended to more general running costs, for example, strongly convex cost functions. We will briefly comment on these extensions in due course.

We make the important assumption that  $J_{\infty}$  is finite. As a result, (1) reduces to an optimization over a closed convex and bounded set in the Banach space  $L_n^2 \times L_m^2$ , and therefore, due to the strong convexity of the objective function, the infimum is attained and the corresponding optimal input and state trajectories are unique, [15, p. 93, Thm. 26]. The approximations to (1) are obtained in three steps: 1) parametrization of input and state trajectories; 2) reformulation of the dynamics as an equality constraint for the parameters; 3) approximation of the state and input constraints. Each step is described in the following subsections.

# A. Parametrization of input and state trajectories

Input and state trajectories are approximated as a linear combination of basis functions  $\tau_i \in L_1^2$ , i = 1, 2, ..., that is

$$\widetilde{x}^{s}(t,\eta_{x}) := (I_{n} \otimes \tau^{s}(t))^{\mathsf{T}} \eta_{x}, 
\widetilde{u}^{s}(t,\eta_{u}) := (I_{m} \otimes \tau^{s}(t))^{\mathsf{T}} \eta_{u},$$
(4)

where  $\otimes$  denotes the Kronecker product,  $\eta_x \in \mathbb{R}^{ns}$  and  $\eta_u \in \mathbb{R}^{ms}$  are the parameter vectors, and  $\tau^s(t) := (\tau_1(t), \tau_2(t), \dots, \tau_s(t)) \in \mathbb{R}^s$ . In order to simplify notation we omit the superscript s in  $\tau^s$ ,  $\tilde{x}^s$ , and  $\tilde{u}^s$ , and simply write  $\tau$ ,  $\tilde{x}$ , and  $\tilde{u}$  whenever the number of basis functions is clear from context. Similarly, the dependence of  $\tilde{x}$  and  $\tilde{u}$  on  $\eta_x$  and  $\eta_u$  is frequently omitted. Throughout the article it is assumed that the basis functions are orthonormal (in the  $L^2$ -sense), which is without loss of generality.

As motivated in [2], the following assumptions on the basis functions are made:

- A1) They are linearly independent.
- A2) They fulfill  $\dot{\tau}(t) = M\tau(t)$  for all  $t \in [0, \infty)$ , for some matrix  $M \in \mathbb{R}^{s \times s}$ . The eigenvalues of M have strictly negative real parts.

An example of a set of basis functions fulfilling Assumptions A1 and A2 is provided in Sec. IV.

#### B. Finite dimensional representation of the dynamics

The dynamics can be reformulated using the fundamental lemma of the calculus of variations, [16, p. 18], leading to

$$\int_0^\infty \delta p^{\mathsf{T}} (Ax + Bu - \dot{x}) \mathrm{d}t - \delta p(0)^{\mathsf{T}} (x(0) - x_0) = 0, \quad (5)$$

for all variations  $\delta p \in L_n^2$  that are everywhere continuous except at t = 0. Note that (5) is equivalent to  $\dot{x}(t) = Ax(t) + Bu(t)$  for all  $t \in [0, \infty)$  (almost everywhere) and  $x(0) = x_0$ .

A finite dimensional representation of the dynamics is obtained via a Galerkin approach; that is, by restricting the variations  $\delta p$  to be linear combinations of the basis functions used for parametrizing input and state trajectories. We derive two variants, depending on whether  $\delta p$  is allowed to be discontinuous at t = 0 or not.

Variant 1: By choosing  $\delta \tilde{p} = (I_n \otimes \tau(t))^{\mathsf{T}} \delta \eta_p$ , for all  $t \in [0, \infty)$ , with  $\delta \eta_p \in \mathbb{R}^{ns}$ , (5) is approximated by

$$\int_0^\infty \delta \tilde{p}(t)^\mathsf{T} \left( A \tilde{x} + B \tilde{u} - \dot{\tilde{x}} \right) \mathrm{d}t - \delta \tilde{p}(0)^\mathsf{T} (\tilde{x}(0) - x_0) = 0,$$
(6)

<sup>&</sup>lt;sup>1</sup>The element  $x \in L_n^2$  is an equivalence class of functions that are equal in the almost everywhere sense. If  $x \in L_n^2$  and  $\dot{x} \in L_n^2$ , then x has a unique absolutely continuous representative. We refer to x(0) as the value this unique absolutely continuous representative takes at time t = 0.

which is required to hold for all variations  $\delta \eta_p \in \mathbb{R}^{ns}$ . This is equivalent to

$$\int_0^\infty (I_n \otimes \tau) \left( A\tilde{x} + B\tilde{u} - \dot{\tilde{x}} \right) dt - (I_n \otimes \tau(0)) \left( \tilde{x}(0) - x_0 \right) = 0; \quad (7)$$

Variant 2: Choosing

$$\delta \tilde{p}(t) := \begin{cases} \delta p_0 & t = 0, \\ (I_n \otimes \tau(t))^{\mathsf{T}} \delta \eta_p & t \in (0, \infty), \end{cases}$$
(8)

results in

$$\int_0^\infty \delta \tilde{p}^\mathsf{T} \left( A \tilde{x} + B \tilde{u} - \dot{\tilde{x}} \right) \mathrm{d}t - \delta p_0^\mathsf{T} (\tilde{x}(0) - x_0) = 0, \quad (9)$$

which has to hold for all variations  $\delta \eta_p \in \mathbb{R}^{ns}$  and  $\delta p_0 \in \mathbb{R}^n$ . This is equivalent to the two conditions

$$\int_0^\infty (I_n \otimes \tau) \left( A \tilde{x} + B \tilde{u} - \dot{\tilde{x}} \right) dt = 0, \quad \tilde{x}(0) - x_0 = 0.$$
(10)

Note that for variant 1, the variations  $\delta \tilde{p}$  are chosen to be continuous for all  $t \in [0, \infty)$ . The resulting state trajectory, as determined by (7), does not fulfill the initial condition exactly, which can be interpreted as discontinuity at t = 0. In contrast, the variations in variant 2 are chosen to be discontinuous at 0, which results in a finite dimensional representation of the dynamics where the initial condition is enforced exactly. It was shown in [2] that (10) imposes the dynamics strictly, i.e. if  $\tilde{x}$  and  $\tilde{u}$  fulfill (10), it holds that  $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$  for all  $t \in [0, \infty)$  and  $\tilde{x}(0) = x_0$ .

*Remark:* We restrict ourselves to two different choices for the variations  $\delta \tilde{p}$  that lead to the two approximation schemes (7) and (10). These choices are by no means unique. For the subsequent analysis it is crucial, however, that variant 2 leads to trajectories fulfilling the equations of motion exactly. Similarly, we exploit the fact that the projection (to be made precise below) of trajectories fulfilling the equations of motion exactly must be compatible with variant 1, that is, the projections fulfill conditions similar to (10). As a consequence, the subsequent results could potentially be transferred to trajectories defined piecewise on  $[0, \infty)$ , for example. A detailed and rigorous discussion of these extensions is, however, beyond the scope of this article.

# C. Finite dimensional representation of the constraint sets

We will introduce two finite dimensional approximations to (1) (they will be made precise in Sec. II-D). These are designed to yield upper and lower bounds of  $J_{\infty}$ , which become tighter as the number of basis functions increases. In particular, this is achieved by imposing requirements on the constraints of the two approximations as discussed below.

The constraints of the first approximation, bounding  $J_{\infty}$  from above are required to fulfill the following assumptions<sup>2</sup> B0)  $\mathcal{X}^s$  is closed and convex

B1)  $i_s(\mathcal{X}^s) \subset \mathcal{X}^{s+1}$ 

B2)  $\eta_x \in \mathcal{X}^s$  implies  $(I_n \otimes \tau(t))^{\mathsf{T}} \eta_x \in \mathcal{X}$  for all  $t \in [0, \infty)$ , where the inclusion  $i_s$ , mapping from  $\mathbb{R}^{ns}$  to  $\mathbb{R}^{n(s+1)}$  is defined by

$$\tilde{x}^s(t,\eta_x) = \tilde{x}^{s+1}(t, i_s(\eta_x)), \forall t \in [0,\infty), \ \forall \eta_x \in \mathbb{R}^{ns}.$$
(11)

Note that for n = 1 the function  $i_s$  maps a vector  $(x_1,\ldots,x_s) \in \mathbb{R}^s$  to  $(x_1,\ldots,x_s,0) \in \mathbb{R}^{s+1}$ . The motivation for introducing Assumptions B0-B2 is the following: Assumption B0 ensures that the resulting optimization problem is convex, and that corresponding minimizers exist, given the existence of feasible trajectories; Assumption B1 will be used to demonstrate that the optimal trajectories of the optimization over s basis functions are feasible for the optimization over s + 1 basis functions, which implies that corresponding optimal cost is monotonically decreasing in s; Assumption B2 ensures that the optimal trajectories of the optimization over s basis functions are feasible candidates for (1), which guarantees that the resulting optimal cost bounds  $J_\infty$  from above. These claims will be verified in detail in Sec. III. An example of a constraint set fulfilling Assumptions B0-B2 is given by

$$\{\eta_x \in \mathbb{R}^{ns} \mid (I_n \otimes \tau(t))^\mathsf{T} \eta_x \in \mathcal{X} \ \forall t \in [0, \infty)\}.$$
(12)

The given set is finite dimensional, closed, and convex, but in general not polyhedral. Arguably, one can construct polyhedral approximations without violating Assumptions B0-B2.

The objective of the second approximation of (1) is to bound  $J_{\infty}$  from below. Its optimal cost is designed to be monotonically increasing. By combining both approximations, we can, therefore, quantify the suboptimality with respect to the underlying infinite dimensional problem (1). To that extent we require the constraints of the second approximation to fulfill

- C0)  $\tilde{\mathcal{X}}^s$  is closed and convex
- C1)  $\pi_s(\tilde{\mathcal{X}}^{s+1}) \subset \tilde{\mathcal{X}}^s$
- C2) For each  $x \in L^2_n$  with  $x(t) \in \mathcal{X}$  for all  $t \in [0, \infty)$  it holds that  $\pi^s(x) \in \tilde{\mathcal{X}}^s$ ,

where the projections  $\pi_s$  and  $\pi^s$  are defined as

$$\pi^{s}: L_{n}^{2} \to \mathbb{R}^{ns}, x \to \int_{0}^{\infty} (I_{n} \otimes \tau^{s}) x \, \mathrm{d}t, \tag{13}$$

$$\pi_s : \mathbb{R}^{n(s+1)} \to \mathbb{R}^{ns}, \ \eta_x \to \pi^s \left( \tilde{x}^{s+1}(\cdot, \eta_x) \right).$$
(14)

Note that the projection  $\pi^s$  maps an arbitrary square integrable function onto its first *s* Fourier coefficients, whereas, for n = 1, the function  $\pi_s$  maps  $(x_1, \ldots, x_s, x_{s+1}) \in \mathbb{R}^{s+1}$ to  $(x_1, \ldots, x_s) \in \mathbb{R}^s$ . The first assumption, C0, guarantees that the resulting optimization problem is convex, and that corresponding minimizers exist, given the existence of feasible solutions. Assumption C1 ensures that the optimal cost of the second approximation is monotonically increasing, whereas Assumption C2 guarantees that the solutions of the infinite dimensional problem (1) are feasible candidates for the optimization over *s* basis functions. This implies that the optimal cost of the second approximation is a lower bound on  $J_{\infty}$ . These observations will be explained further in Sec. III.

 $<sup>^2</sup> The assumptions are only listed for the state constraints <math display="inline">{\cal X}$  and are analogous for the input constraints  ${\cal U}.$ 

In case of polyhedral constraints, that is,  $x \in \mathcal{X}$  if and only if  $Dx \leq d$ ,  $D \in \mathbb{R}^{n_c \times n}$ ,  $d \in \mathbb{R}^{n_c}$ , where the inequality holds component wise, an example satisfying Assumptions C0-C2 is given by

$$\left\{ \eta_x \in \mathbb{R}^{ns} \mid \int_0^\infty \delta \tilde{p}^{\mathsf{T}} (D \tilde{x}^s(t, \eta_x)) - d) \, \mathrm{d}t \le 0, \\ \forall \delta \tilde{p} = (I_{n_c} \otimes \tau^s)^{\mathsf{T}} \delta \eta_p : \delta \tilde{p}(t) \ge 0, \forall t \in [0, \infty) \right\}.$$
(15)

We will use this set in the numerical example presented in Sec. IV and therefore show explicitly that Assumptions CO-C2 are fulfilled. It is straightforward to verify Assumption C0. By orthonormality of the basis functions, the inequality in (15) can be rewritten as

$$\delta \eta_p^{\mathsf{T}} \left( (D \otimes I_s) \eta_x - \int_0^\infty d \otimes \tau^s \mathrm{d}t \right) \le 0.$$
 (16)

Clearly, the inequality in (15) holds likewise for all positive test functions  $\delta \tilde{p}$ , which are spanned by the first s - 1 basis functions (instead of the first s basis functions), resulting in

$$\delta\eta_p^{\mathsf{T}}\left((D\otimes I_{s-1})\pi_s(\eta_x) - \int_0^\infty d\otimes \tau^{s-1}\mathrm{d}t\right) \le 0, \quad (17)$$

for all  $\delta \eta_p \in \mathbb{R}^{n(s-1)}$  such that  $(I_{n_c} \otimes \tau^{s-1}(t))^{\mathsf{T}} \eta_p \geq 0$  for all  $t \in [0, \infty)$ . Consequently, the set given by (15) satisfies Assumption C1. In addition, the constraint  $Dx(t) \leq d$  for all  $t \in [0, \infty)$  (almost everywhere), where x is square integrable, implies

$$\int_0^\infty \delta p^\mathsf{T} (Dx - d) \mathrm{d}t \le 0 \tag{18}$$

for all positive integrable test functions  $\delta p$ . Thus, the previous inequality holds certainly for test functions  $\delta \tilde{p}$  restricted to the span of the first *s* basis functions, which, by orthonormality of the basis functions and linearity of the constraint, leads to

$$\int_0^\infty \delta \tilde{p}^\mathsf{T} (D\pi^s(x) - d) \mathrm{d}t \le 0.$$
(19)

This asserts that Assumption C2 is fulfilled.

#### D. Resulting optimization problems

Combining the previous definitions leads to the following approximations of the original problem (1),

$$J_{s} := \inf \frac{1}{2} ||\tilde{x}||_{2}^{2} + \frac{1}{2} ||\tilde{u}||_{2}^{2}$$
  
s.t.  $\int_{0}^{\infty} (I_{n} \otimes \tau) \left( A\tilde{x} + B\tilde{u} - \dot{\tilde{x}} \right) dt = 0,$  (20)  
 $\tilde{x}(0) - x_{0} = 0, \quad \eta_{x} \in \mathcal{X}^{s}, \eta_{u} \in \mathcal{U}^{s},$ 

and

$$\tilde{J}_{s} := \inf \frac{1}{2} ||\tilde{x}||_{2}^{2} + \frac{1}{2} ||\tilde{u}||_{2}^{2}$$
s.t. 
$$\int_{0}^{\infty} (I_{n} \otimes \tau) \left(A\tilde{x} + B\tilde{u} - \dot{\tilde{x}}\right) dt$$

$$- (I_{n} \otimes \tau(0)) \left(\tilde{x}(0) - x_{0}\right) = 0,$$

$$\eta_{x} \in \tilde{\mathcal{X}}^{s}, \eta_{u} \in \tilde{\mathcal{U}}^{s}.$$
(21)

Both problems are convex and have linear equality constraints. The objective functions are strictly convex, which asserts the existence of unique minimizers provided that the corresponding problem is feasible. We will analyze both problems and show that they yield upper and lower bounds on the optimum cost of the original problem (1). Under favorable circumstances,  $J_s$  is a monotonically decreasing sequence approaching  $J_\infty$  from above and  $J_s$  a monotonically increasing sequence approaching  $J_{\infty}$  from below. If certain additional conditions are met (to be made precise in the following), both  $J_s$  and  $\tilde{J}_s$  converge to  $J_{\infty}$ . We will use convergence of the optimal cost  $J_s$  to argue that the corresponding optimal trajectories converge as the number of basis functions increases. Provided that Assumptions B0 and B2 are fulfilled, the optimal state and input trajectories,  $\tilde{x}$  and  $\tilde{u}$ , corresponding to (20) respect the state constraints  $\tilde{x}(t) \in \mathcal{X}$  and the input constraints  $\tilde{u}(t) \in \mathcal{U}$  for all times and achieve the cost  $J_s$  on the nominal system. It was shown in [2] that solving (20) repeatedly, and applying each time the first portion of the obtained input trajectory, offers inherent closed-loop stability and recursive feasibility in the context of MPC.

#### III. MAIN RESULTS

We will discuss the following main result:

Theorem 3.1: Let  $N_0$  be such that  $J_{N_0}$  and  $\tilde{J}_{N_0}$  are finite. 1) If Assumptions B0, B1, and B2 hold, then the sequence  $J_s$  is monotonically decreasing for  $s \ge N_0$ , converges as  $s \to \infty$ , and is bounded below by  $J_{\infty}$ . The corresponding optimizers  $\tilde{x}^s$  and  $\tilde{u}^s$  converge (strongly) in  $L_n^2$ , respectively  $L_m^2$  as  $s \to \infty$ .

2) If Assumptions C0, C1, and C2 are fulfilled, then  $\tilde{J}_s$  is monotonically increasing for  $s \ge 1$ , converges as  $s \to \infty$ , and is bounded above by  $J_{\infty}$ .

Instead of presenting a formal proof of this result, which can be found in [17], we discuss and highlight the underlying ideas. The section is divided into three parts. The first is devoted to the problem (20), summarizes the results from [2], and establishes the convergence of the corresponding optimal trajectories. The problem (21) is analyzed in the second part, where it is shown that the dual of (21) fulfills the adjoint equations exactly, which is exploited for demonstrating that  $\tilde{J}_s$  is monotonically increasing and bounded above by  $J_{\infty}$ . The third part establishes conditions under which  $\lim_{s\to\infty} J_s = \lim_{s\to\infty} \tilde{J}_s$  is fulfilled.

# A. Analysis of (20)

We start by summarizing the results from [2].

Proposition 3.2: Let  $N_0$  be such that  $J_{N_0}$  is finite and let Assumptions B0, B1, and B2 be fulfilled. Then the sequence  $J_s$  is monotonically decreasing for  $s \ge N_0$ , converges as  $s \to \infty$ , and is bounded below by  $J_{\infty}$ .

*Proof:* The proof is based on the fact that the equality constraint of (20) imposes the dynamics exactly. Thus, in combination with Assumption B2, the optimal trajectories of (20) fulfill the equations of motion and the constraints  $\tilde{x}(t) \in \mathcal{X}, \ \tilde{u}(t) \in \mathcal{U}$  for all times  $t \in [0, \infty)$ , and are

therefore feasible candidates for the infinite dimensional problem (1). Hence, it follows that  $J_s \ge J_{\infty}$  for all  $s \ge N_0$ . Assumption B1 implies that the feasible trajectories of (20) satisfy the constraints  $i_s(\eta_x) \in \mathcal{X}^{s+1}$  and  $i_s(\eta_u) \in \mathcal{U}^{s+1}$  (the corresponding parameter vectors are simply augmented with zeros). As a result, the optimal trajectories of (20) are feasible candidates for the optimization over s + 1 basis functions, which implies  $J_s \ge J_{s+1}$ . Thus, the sequence  $J_s$  is monotonically decreasing, bounded above by  $J_{N_0}$ , bounded below by  $J_{\infty}$ , and therefore converges.

In the context of MPC we are mainly interested in the optimal policy  $\tilde{u}$ , which is obtained by solving (20) with a finite number of basis functions. Thus, demonstrating convergence of the trajectories  $\tilde{x}^s$  and  $\tilde{u}^s$  guarantees that the optimal input trajectory from (1) can be approximated arbitrarily accurately using a sufficiently large number of basis functions (provided that  $\lim_{s\to\infty} J_s = J_{\infty}$ ).

Proposition 3.3: Let  $N_0$  be such that  $J_{N_0}$  is finite and let Assumptions B0 and B1 be fulfilled. Then, the optimal trajectories  $\tilde{x}^s$  and  $\tilde{u}^s$  of (20) converge (strongly) in  $L_n^2$ , respectively  $L_m^2$ .

**Proof:** From Assumption B1 it can be inferred that  $J_s$  is monotonically decreasing, bounded above by  $J_{N_0}$ , bounded below by 0, and therefore converges, see Prop. 3.2. The argument in Prop. 3.2 asserts that the trajectories corresponding to (20), denoted by  $\tilde{x}^s$  and  $\tilde{u}^s$ , are feasible candidates for the optimization over s + 1 basis functions. Moreover, the candidates  $1/2(\tilde{x}^s + \tilde{x}^{s+1})$  and  $1/2(\tilde{u}^s + \tilde{u}^{s+1})$ , where  $\tilde{x}^{s+1}$  and  $\tilde{u}^{s+1}$  correspond to the optimization over s + 1 basis functions, which follows by convexity of the constraint sets and linearity of the dynamics. Hence, they achieve a cost larger than  $J_{s+1}$ . As a consequence, the identity

$$\begin{split} ||\frac{1}{2}(\tilde{x}^{s} + \tilde{x}^{s+1})||_{2}^{2} = \\ \frac{1}{2}||\tilde{x}^{s}||_{2}^{2} + \frac{1}{2}||\tilde{x}^{s+1}||_{2}^{2} - \frac{1}{4}||\tilde{x}^{s} - \tilde{x}^{s+1}||_{2}^{2}, \end{split}$$

and the fact that the objective function is quadratic, bounds  $||\tilde{x}^s - \tilde{x}^{s+1}||_2^2$  and  $||\tilde{u}^s - \tilde{u}^{s+1}||_2^2$  by  $4(J_s - J_{s+1})$ . Combined with the observation that  $J_s$  is converging, it implies that the optimal trajectories of (20) form Cauchy sequences in  $L_n^2$ , respectively  $L_m^2$ . Both spaces,  $L_n^2$  and  $L_m^2$  are complete, [18, p. 67], and therefore the sequence of optimal trajectories converges. A formal proof can be found in [17].

## B. Analysis of (21)

We will show that under favorable circumstances (to be made precise below) the optimal cost of (21) bounds  $J_{\infty}$  from below and is monotonically increasing in s.

Proposition 3.4: Let Assumptions C0 and C2 be fulfilled. Then  $\tilde{J}_s \leq J_{\infty}$  holds for all  $s \geq 1$ .

**Proof:** The proof relies on the fact that the optimal trajectories of (1), denoted by x and u, can be used to generate feasible trajectories for (21). It can be verified, see [17], that the candidates  $\eta_x := \pi^s(x)$  and  $\eta_u := \pi^s(u)$  fulfill the equality constraint in (21). Together with Assumption C2 this guarantees that  $\eta_x$  and  $\eta_u$  are feasible candidates for (21). Moreover, it follows from Bessel's inequality, [19, p. 51], that

$$|\pi^s(x)|_2^2 \le ||x||_2^2, \tag{22}$$

is fulfilled for all  $x \in L_n^2$ , where  $|\cdot|_2$  denotes the Euclidean norm. Thus, the feasible candidates  $\eta_x$  and  $\eta_u$  achieve a smaller cost than x and u, which implies  $\tilde{J}_s \leq J_\infty$  for all  $s \geq 1$ .

In order to establish that the sequence  $\tilde{J}_s$  is monotonically increasing, we work with the dual problem. It turns out that the finite dimensional representations of the adjoint equations are fulfilled exactly by (21). We use this fact to construct feasible candidates for the optimization over s + 1 basis functions.

Proposition 3.5: Let Assumptions C0, C1, and C2 be fulfilled. Then  $\tilde{J}_s$  is monotonically increasing and bounded above by  $J_{\infty}$  for all  $s \ge 1$ .

*Proof:* Lagrange duality is used to derive the dual of (21), which is given by

$$\tilde{J}_{s} = \sup_{\eta_{p}} -I_{\varphi_{s}}^{*}(\tilde{v}) - I_{\psi_{s}}^{*}(-B^{\mathsf{T}}\tilde{p}) + \tilde{p}(0)^{\mathsf{T}}x_{0},$$
s.t. 
$$\int_{0}^{\infty} (I_{n} \otimes \tau) \left(\dot{\tilde{p}} + A^{\mathsf{T}}\tilde{p} + \tilde{v}\right) \mathrm{d}t = 0,$$
(23)

where  $\tilde{v} := (I_n \otimes \tau^s)^{\mathsf{T}} \eta_v$  and  $\tilde{p} := (I_n \otimes \tau^s)^{\mathsf{T}} \eta_p$ , see [17]. The functions  $I_{\varphi_s}^*$  and  $I_{\psi_s}^*$  are the convex-conjugates, [20, p. 473], of

$$I_{\varphi_s}(x) := \begin{cases} \frac{1}{2} ||\tilde{x}^s(t, \pi^s(x))||_2^2 & \pi^s(x) \in \tilde{\mathcal{X}}^s, \\ \infty & \text{otherwise,} \end{cases}$$
(24)

$$I_{\psi_s}(u) := \begin{cases} \frac{1}{2} ||\tilde{u}^s(t, \pi^s(u))||_2^2 & \pi^s(u) \in \tilde{\mathcal{U}}^s, \\ \infty & \text{otherwise,} \end{cases}$$
(25)

(where notation is slightly abused to denote both projections  $L_n^2 \to \mathbb{R}^{ns}$  and  $L_m^2 \to \mathbb{R}^{ms}$ , as defined in Sec. II-C, by  $\pi^s$ ). The equality constraint in (23) implies that trajectories  $\tilde{v}$  and  $\tilde{p}$  with cost  $\tilde{J}_s$  satisfy the adjoint equations exactly, see [2]. As a result, they are feasible candidates for the optimization over s + 1 basis functions. From Assumption C1 and the quadratic running cost it follows that  $I_{\varphi_s}(x) \leq I_{\varphi_{s+1}}(x)$  for all  $x \in L_n^2$  and similarly  $I_{\psi_s}(u) \leq I_{\psi_{s+1}}(u)$  for all  $u \in L_m^2$ . The convex-conjugation reverses ordering, [20, p. 475], which concludes that the candidates  $\tilde{v}$  and  $\tilde{p}$  are feasible candidates to the optimization problem over s + 1 basis functions with higher corresponding cost, and therefore  $\tilde{J}_{s+1} \geq \tilde{J}_s$ . A formal proof can be found in [17].

non-expansive with respect to the projection  $\pi^s$ , respectively  $\pi_s$ .

C. Convergence to  $J_{\infty}$ 

Next, we would like to establish that  $\lim_{s\to\infty} \tilde{J}_s = \lim_{s\to\infty} J_s$ . In order to do so, we need the following assumptions:

- D0)  $\limsup_{s\to\infty} \hat{\mathcal{X}}_s \subset \liminf_{s\to\infty} \mathcal{X}^s$ ,
- D1) The basis functions  $\tau_i$ , i = 1, 2, ..., are dense in  $C_0^{\infty}$  (in the topology of uniform convergence).<sup>3</sup>

Proposition 3.6: Let  $N_0$  be such that  $J_{N_0}$  is finite and let Assumptions B0-D1 be fulfilled. Then,  $\lim_{s\to\infty} \tilde{J}_s = \lim_{s\to\infty} J_s$  holds.

**Proof:** Assumption D1 can be used to construct a sequence in the span of the basis functions, which converges uniformly to an arbitrary test function  $\delta p \in C_0^{\infty}$ . This implies, by the fundamental lemma of the calculus of variations [16, p. 18], that in the limit as  $s \to \infty$ , the solutions  $\tilde{x}$  and  $\tilde{u}$  to (21) fulfill the equations of motion and the initial condition exactly. Combined with Assumption D0, it follows that  $\tilde{x}$  and  $\tilde{u}$  are feasible candidates for (20) (in the limit as  $s \to \infty$ ) and therefore  $\lim_{s\to\infty} J_s \leq \lim_{s\to\infty} \tilde{J}_s$ . Combined with Thm. 3.1 this leads to the desired result. A formal proof can be found in [17].

## IV. NUMERICAL EXAMPLE

In the following we illustrate the results on a numerical example. We consider the system governed by

$$2\ddot{r} + \ddot{\varphi} = u,$$
  
$$\ddot{r} + \ddot{\varphi} = 9.81\varphi,$$
 (26)

whose open-loop poles are located at  $0, 0, \pm 4.429$  rad/s. These dynamics can be obtained by linearizing the invertedpendulum-on-a-cart-system around the upright equilibrium, where r corresponds to the cart position,  $\varphi$  to the pendulum angle, and u to the normalized force applied to the cart. We define the state vector as  $x := (r, \dot{r}, \varphi, \dot{\varphi})^{\mathsf{T}}$  and consider the task of driving the system from  $x(0) = (1, 0, 0, 0)^{\mathsf{T}}$  back to the origin. We penalize input and state deviations with the following cost

$$\int_0^\infty \frac{1}{2} x^\mathsf{T} x + \frac{0.05}{2} u^2 \, \mathrm{d}t. \tag{27}$$

The basis functions  $\tau$  are designed to be orthonormal and spanned by

$$\tau \in \exp(-\nu t) \operatorname{span}(1, t, t^2, \dots, t^{s-1}),$$
(28)

where  $\nu$  is set to 3 rad/s (this corresponds approximately to the closed-loop poles of an LQR design). Note that the Theorem of Stone-Weierstrass, [21, p. 122], states that the basis functions given by (28) are dense in the set of continuous functions vanishing at infinity. The set of smooth compactly supported functions is contained in the set of continuous functions vanishing at infinity, [18, p. 70] and as a result, Assumption D1 holds. In the simplest case, we assume input and state constraints to be absent, i.e.  $\mathcal{X} = \mathbb{R}^n$  and  $\mathcal{U} = \mathbb{R}^m$ . It follows immediately that  $J_s$  is monotonically decreasing,  $\tilde{J}_s$  monotonically increasing, provided that solutions to (20) exist, and that (20) and (21) converge to  $J_{\infty}$ . The quadratic running cost asserts that the optimal trajectories corresponding to  $J_s$  will converge strongly. The optimum costs  $J_s$  and  $\tilde{J}_s$  are depicted in Fig. 1 (top).

Next, the input u is restricted to lie within [-0.5, 0.5], and the set  $\mathcal{U}^s$  is defined as

$$\mathcal{U}^{s} := \{ \eta_{u} \in \mathbb{R}^{s} | \tau(t)^{\mathsf{T}} \eta_{u} \in [-0.5, 0.5], \forall t \in [0, \infty) \}.$$
(29)

Note that the constraint  $u \in [-0.5, 0.5]$  can be rewritten as  $u \leq 0.5$  and  $-u \leq 0.5$ . Thus, in accordance to (15), we define  $\tilde{\mathcal{U}}^s$  as

$$\tilde{\mathcal{U}}^{s} := \Big\{ \eta_{u} \in \mathbb{R}^{s} \Big| \int_{0}^{\infty} \delta \tilde{p}^{\mathsf{T}} (D\tilde{u} - d) \, \mathrm{d}t \leq 0, \qquad (30)$$
$$\forall \delta \tilde{p} = (I_{2} \otimes \tau^{s})^{\mathsf{T}} \delta \eta_{p} : \delta \tilde{p}(t) \geq 0, \forall t \in [0, \infty) \Big\},$$

where  $D = (1, -1)^{\mathsf{T}}, d = (0.5, 0.5)^{\mathsf{T}}$ . The constraint set  $\mathcal{U}^s$  fulfills Assumptions B0, B1, and B2. This implies by Prop. 3.2 that  $J_s$  is monotonically decreasing for all  $s \ge N_0$ and is bounded below by  $J_{\infty}$ . In addition, the constraint set  $\mathcal{U}^s$  fulfills Assumptions C0, C1, and C2 and therefore, according to Prop. 3.4 and Prop. 3.5, the optimal cost  $J_s$ is monotonically increasing and bounded above by  $J_\infty$  for all  $s \geq 1$ . The optimal input  $\tilde{u}$  obtained by solving (20) is guaranteed to be feasible and achieves the cost  $J_s$  on the nominal system. In the context of MPC, applying the input  $\tilde{u}$  (in a receding manner) guarantees recursive feasibility and closed-loop stability. Moreover, from the fact that the optimal cost  $J_s$  corresponding to (21) represents a lower bound on  $J_{\infty}$ , it follows that the suboptimality with respect to the underlying infinite dimensional problem is bounded by  $J_s - J_s$ . This is particularly useful for determining the number of basis functions needed to achieve a certain performance. The optimum costs  $J_s$  and  $\tilde{J}_s$  are depicted in Fig. 1 (bottom). In addition, the optimal input trajectory corresponding to  $J_{16}$  is shown in Fig. 2.

In the numerical example the constraints  $\eta_u \in \mathcal{U}^s$  and  $\eta_u \in \tilde{\mathcal{U}}^s$  are imposed using an iterative procedure. We start by explaining how  $\eta_u \in \mathcal{U}^s$  is implemented: First (20) is solved, where the trajectory  $\tilde{u}$  is required to fulfill the constraints at the time instants  $t_i$ ,  $i = 1, 2, \ldots, s$ . These are defined by imposing  $\tau(t_i)^T \tau(t_j) = 0$ , for all  $i, j = 1, 2, \ldots, s, i \neq j, t_1 = 0$ . Constraint violations are checked a posteriori. If they occur, the set of constraint sampling instances is augmented accordingly and (20) is solved again. Repeating this procedure until no constraint violations occur guarantees  $\eta_u \in \mathcal{U}^s$ . A similar procedure is used for imposing  $\eta_u \in \tilde{\mathcal{U}}^s$ . The constraint (30) is simplified by allowing test functions  $\delta \tilde{p}$ , which are only required to be positive at certain time instants  $t_i$ ,  $i = 1, 2, \ldots, N$ , that is,

$$\sup_{\delta\eta_p \in \mathbb{R}^{2s}} \delta\eta_p^{\mathsf{T}} \left( (D \otimes I_s) \eta_u - \int_0^\infty d \otimes \tau \, \mathrm{d}t \right) \le 0 \quad (31)$$
  
s.t.  $(I_2 \otimes \tau(t_i))^{\mathsf{T}} \delta\eta_p \ge 0, \quad i = 1, 2, \dots, N.$ 

<sup>&</sup>lt;sup>3</sup>The set of smooth functions with compact support mapping from  $[0, \infty)$  to  $\mathbb{R}$  is denoted by  $C_0^{\infty}$ .



Fig. 1. Depicted are the optimal costs of (20) and (21). In the first plot (top) input and state constraints are absent, in the second plot (bottom) input constraints,  $u \in [-0.5, 0.5]$ , are included. In both cases  $J_s$  is monotonically increasing, whereas  $J_s$  is monotonically decreasing.



Fig. 2. Input trajectory obtained by solving (20) for s = 16. The input constraint is found to be active.

The constraint (31) can be reformulated using duality. In fact, (31) is equivalent to the existence of a vector  $\lambda \in \mathbb{R}^{2N}$  satisfying

$$\lambda \ge 0, \quad -T\lambda = (D \otimes I_s)\eta_u - \int_0^\infty d \otimes \tau \, \mathrm{d}t, \qquad (32)$$

where the matrix T is given by  $(I_2 \otimes \tau(t_1), \ldots, I_2 \otimes \tau(t_N)) \in \mathbb{R}^{2s \times 2N}$ , [20, p. 507]. Thus, the constraint  $\eta_u \in \tilde{\mathcal{U}}^s$  in (21) is initially replaced by (32), where the sampling instants  $t_i$ ,  $i = 1, 2, \ldots, s$  are again chosen such that  $\tau(t_i)^T \tau(t_j) = 0$ ,  $i, j = 1, 2, \ldots, s$ ,  $i \neq j$ ,  $t_1 = 0$ . The resulting problem is solved and constraint violations are checked a posteriori. This is done by solving (31) (with  $\eta_u$  fixed) and checking whether the resulting trajectory  $(I_2 \otimes \tau(t))^T \delta \eta_p$  is positive for all times  $t \in [0, \infty)$ . If this is the case, we can guarantee that  $\eta_u \in \tilde{\mathcal{U}}^s$ , if not, the corresponding time instants where the violations occur are added to the constraint sampling points  $t_i$  and (21) (with the constraint included by (32)) is solved again. Repeating this procedure until no constraint violations occur guarantees that  $\eta_u \in \tilde{\mathcal{U}}^s$ .

## V. CONCLUSION

We presented and analyzed two finite dimensional approximations to a class of infinite-horizon optimal control problems encountered in MPC. By exploiting suitable assumptions, the optimal costs of these approximations were found to bound the cost of the underlying infinite dimensional optimal control problem from above and from below. The results can be used to quantify the suboptimality of both approximations. The optimal trajectories of the first approach were found to respect input and state constraints, and achieve the corresponding cost on the nominal system.

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