Partial differential equations → more than one independent variable

typically, t, z

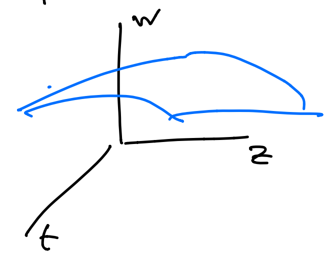
first order PDE → if w is the unknown

$$F\left(t,z,w,\frac{\partial w}{\partial t},\frac{\partial w}{\partial z}\right)=0$$

$$P(t,z,w)\,\frac{\partial w}{\partial t}+Q(t,z,w)\,\frac{\partial w}{\partial z}=R(t,z,w)$$

quasi-linear equation

↳ $w=w(t,z)$ → surface



① Example → PDE

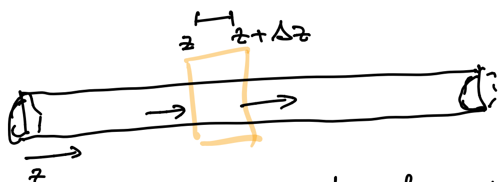
② Method of characteristics

③ Discussion about existence and uniqueness of the solution

↳ w/ examples

① Conservation law for an entity

$t \leftarrow t+\Delta t$



A: cross section

M: hold up $\left[\frac{mol}{m^3}\right]$

F: flux $\left[\frac{mol}{m^2 s}\right]$

1-dimensional domain (z)

evolution in time (t)

Assumption: $\begin{cases} M(w) \\ F(w) \end{cases}$

F is not a function of $\frac{\partial w}{\partial z}$

Conservation law: Accumulation = flow IN − flow OUT

$$\cancel{A}\,\Delta z\,\big(M(t+\Delta t)-M(t)\big)=\cancel{A}\,\Delta t\,\big(F(z)-F(z+\Delta z)\big)$$

finite form

$\frac{1}{\Delta z\,\Delta t}$ $\quad \Delta t,\Delta z \to 0$

$$\frac{\partial M(w)}{\partial t}+\frac{\partial F(w)}{\partial z}=0$$

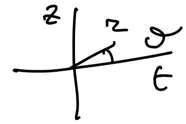
differential form

$\frac{dM}{dw} = M'$

$$\boxed{M'(w)\frac{\partial w}{\partial t} + F'(w)\frac{\partial w}{\partial z} = 0}$$

$P(t,z,w) = M'(w) \qquad Q(t,z,w) = F'(w) \qquad R(t,z,w) = 0$

$w = w(t,z)$: solution



directional derivative

$$\boxed{\frac{dw}{dr} = \cos\vartheta\,\frac{\partial w}{\partial t} + \sin\vartheta\,\frac{\partial w}{\partial z}}$$

$$\mathrm{tg}\,\vartheta = \frac{\sin\vartheta}{\cos\vartheta}$$

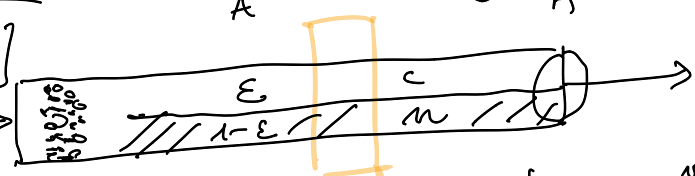
$$\frac{dw}{dr} = M'(w)\frac{\partial w}{\partial t} + F'(w)\frac{\partial w}{\partial z} = 0 \Rightarrow$$ w is constant along the direction

$$\lambda(w) = \mathrm{tg}\,\vartheta = \frac{F'(w)}{M'(w)}$$

$u = \frac{Q}{A} \qquad v = \frac{Q}{A\varepsilon} \qquad A$

u: superficial velocity

chromatographic column || $Q \left[\frac{m^3}{s}\right]$



$A\varepsilon$

v: interstitial velocity

$c, n \quad \left[\frac{mole}{m^3} ; \frac{g}{m^3}\right]$

chromatographic column

$$\varepsilon = \frac{\text{fluid phase volume}}{\text{total volume}}$$

two phases: fluid + solid (adsorbed phase)

equilibrium w.r.t. mass transfer

$w = c$ $\hookrightarrow n = f(c)$ at a given temperature

$$M(w) = \varepsilon c + (1-\varepsilon)n = \varepsilon c + (1-\varepsilon)f(c) = M(c)$$

$$F(w) = \varepsilon c\, v_{fluid} + (1-\varepsilon)n\,\underbrace{v_{solid}}_{=0} = \varepsilon c v = uc = F(c)$$

$$\frac{\partial}{\partial t}\Big(\varepsilon c + (1-\varepsilon) f(c)\Big) + \frac{\partial}{\partial z}\left(\varepsilon v c\right) = 0$$

$$\varepsilon \frac{\partial c}{\partial t} + (1-\varepsilon)\frac{\partial f}{\partial t} + \varepsilon v \frac{\partial c}{\partial z} = 0$$

$$\boxed{\Big(\varepsilon + (1-\varepsilon) f'(c)\Big)\frac{\partial c}{\partial t} + \varepsilon v \frac{\partial c}{\partial z} = 0} \quad \leftarrow \quad \text{PDE}$$

$$M'(c)\frac{\partial c}{\partial t} + F'(c)\frac{\partial c}{\partial z} = 0$$

② Method of characteristics

$$\boxed{P(t,z,w)\frac{\partial w}{\partial t} + Q(t,z,w)\frac{\partial w}{\partial z} = R(t,z,w)}$$

initial line ≡ specification (1D object)

solution $w = w(t,z)$ surface (2D object)

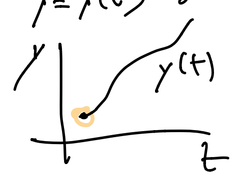
verification → substitution of $w(t,z)$ in PDE

↳ identity!!

$\frac{dy}{dt} = f(y)$ ODE

$y(t_0) = y_0$ initial point

$y = y(t)$ solution

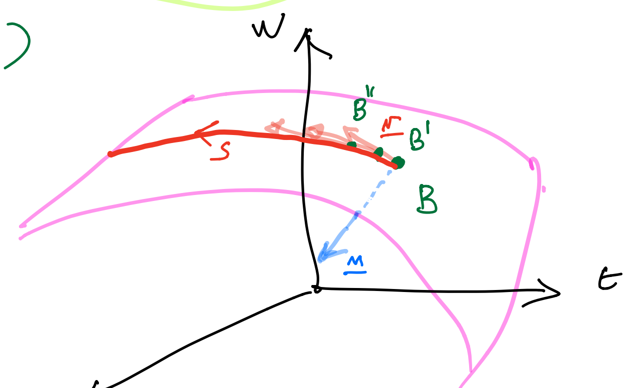


$B \to w \quad w(t,z) \; : \; B(t,z,w)$

$$\underline{v} = \begin{bmatrix} Q(t,z,w) \\ P(t,z,w) \\ R(t,z,w) \end{bmatrix}$$

$$\underline{n} = \begin{bmatrix} \frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial t} \\ -1 \end{bmatrix}$$

Mathematic



Thesis: $\underline{n} \perp \underline{v}$

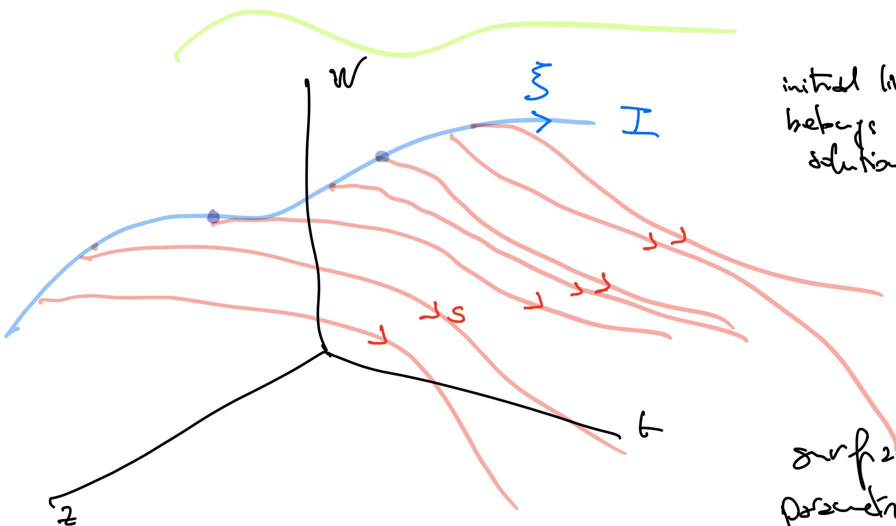
$$\underline{n}^T \underline{v} = \underline{n} \cdot \underline{v} = \boxed{Q \frac{\partial w}{\partial z} + P \frac{\partial w}{\partial t} - R = 0} \equiv \text{PDE}$$

vector $\underline{v}$ is on tangent plane

$$\begin{bmatrix} dz \\ dt \\ dw \end{bmatrix} = \underline{v}\, ds \longrightarrow \begin{cases} \dfrac{dz}{ds} = Q(t,z,w) \\ \dfrac{dt}{ds} = P(t,z,w) \\ \dfrac{dw}{ds} = R(t,z,w) \end{cases} \qquad \begin{cases} z = z(s) \\ t = t(s) \\ w = w(s) \end{cases}$$

characteristic curve

characteristic differential equation



initial line belongs to solution surface

char. diff. eqs.

$$\boxed{\begin{cases} \dfrac{dz}{ds} = Q(t,z,w) \\ \dfrac{dt}{ds} = P(t,z,w) \\ \dfrac{dw}{ds} = R(t,z,w) \end{cases}} \longleftarrow \overset{\text{I}}{\begin{cases} z(0) = \psi(\xi) \\ t(0) = \varphi(\xi) \\ w(0) = \omega(\xi) \end{cases}} \Rightarrow$$

surface in parametric form

$$\boxed{\begin{cases} z = z(s, \xi) \\ t = t(s, \xi) \\ w = w(s, \xi) \end{cases}}$$

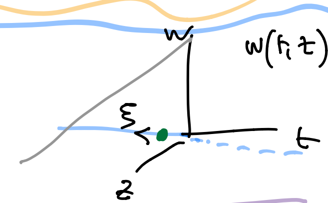
$$\boxed{s = s(t,z) \qquad \xi = \xi(t,z)}$$

$$\boxed{w = w\big(s(t,z), \xi(t,z)\big) = w(t,z)}$$

③ $\boxed{\frac{\partial w}{\partial t} + w\,\frac{\partial w}{\partial z} = 1}$

$P=1 \qquad \varphi = w \qquad R = 1$

$$\mathrm{I}:\begin{cases} t=0 \\ z=\xi \\ w=\alpha\xi \end{cases}$$



char. eqs.

$$\begin{cases} \frac{dt}{ds} = 1 \\ \frac{dz}{ds} = w \\ \frac{dw}{ds} = 1 \end{cases}$$

i.d. $s=0$

$$\begin{cases} t(0) = 0 \\ z(0) = \xi \\ w(0) = \alpha\xi \end{cases}$$

$$\begin{cases} \boxed{\begin{array}{l} t = s \\ z = \frac{s^2}{2} + \alpha\xi s + \xi \end{array}} \Leftrightarrow \boxed{\begin{array}{l} s = t \\ \xi = \frac{z - t^2/2}{1+\alpha t} \end{array}} \\ w = s + \alpha\xi \end{cases}$$

$$\boxed{w = t + \alpha\,\frac{z - t^2/2}{1+\alpha t}}$$

?

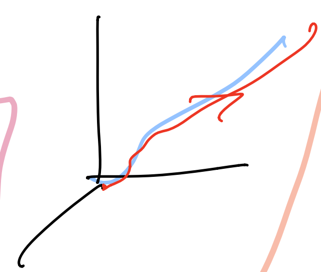
$$w_t = 1 + \alpha\,\frac{-t(1+\alpha t) - \left(z - \frac{t^2}{2}\right)\alpha}{(1+\alpha t)^2}$$

$$w_z = \alpha\,\frac{1}{1+\alpha t}$$

$$1 - \cancel{\frac{\alpha t}{1+\alpha t}} - \cancel{\frac{\alpha^2\left(z - t^2/2\right)}{(1+\alpha t)^2}} + \cancel{\frac{\alpha t}{1+\alpha t}} + \cancel{\frac{\alpha^2\left(z - t^2/2\right)}{(1+\alpha t)^2}} = 1$$

$$\begin{cases} \frac{dt}{ds} = 1 \\ \frac{dz}{ds} = w \\ \frac{dw}{ds} = 1 \end{cases}
\qquad
\begin{cases} t(0) = \xi \\ z(0) = \frac{\xi^2}{2} \\ w(0) = \xi \end{cases}$$

$$\begin{cases} \boxed{\begin{array}{l} t = s + \xi \\ z = \frac{s^2}{2} + \xi s + \frac{\xi^2}{2} = \frac{1}{2}(s+\xi)^2 \end{array}} \\ w = s + \xi \end{cases}$$



$$\mathrm{I}:\begin{cases} t = \xi \\ z = \frac{\xi^2}{2} \\ w = \xi \end{cases}$$

$$\boxed{\begin{array}{c} z = \frac{w^2}{2} + G(w-t) \\ \text{with } G(0) = 0 \end{array}}$$

$\frac{\partial}{\partial t}$: $\quad 0 = w\,w_t + G'(w_t - 1) \qquad w_t = \frac{G'}{w+G'}$

$\frac{\partial}{\partial z}$: $\quad 1 = w\,w_z + G' w_z \qquad w_z = \frac{1}{w+G'}$

$$\frac{G'}{w+G'} + w\,\frac{1}{w+G'} = 1$$