## Autonomous systems

$\underline{\dot{y}} = \underline{f}(\underline{y})$ $\left(\underline{\dot{y}} = \underline{\underline{A}}\,\underline{y}\right)$

no explicit dependence on t

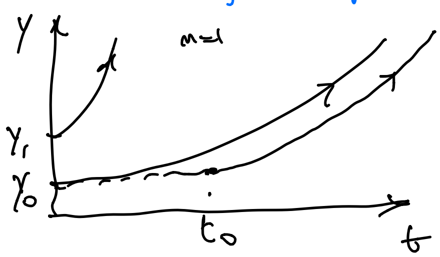
$$\boxed{\begin{array}{c} \underline{y} \in \mathbb{R}^n \\ \underline{y} = \underline{y}(t) \end{array}}$$

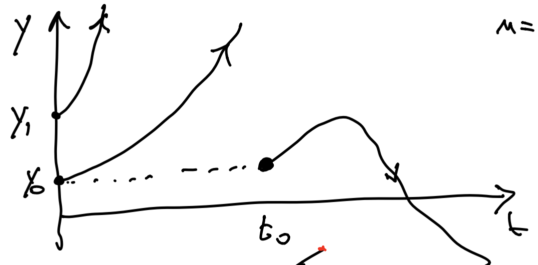
solutions

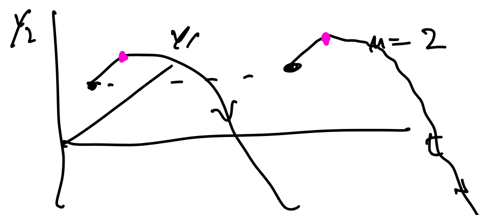
## Non-autonomous systems

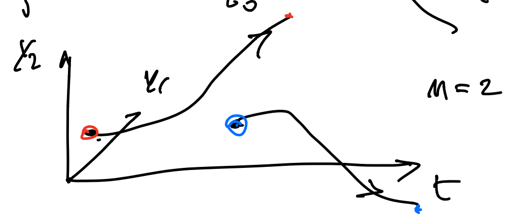
$\underline{\dot{y}} = \underline{f}(\underline{y}, t)$

explicit dependence on t



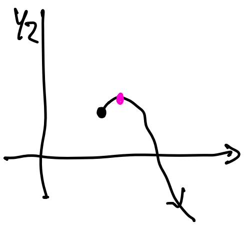



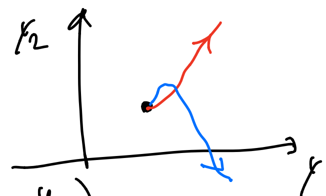




$\boxed{\text{trajectory}}$

phase plane $(y_1, y_2)$




trajectories cannot intersect!

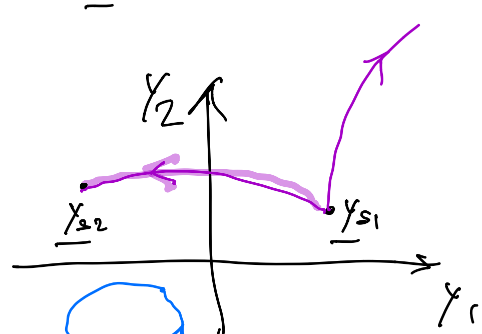
solutions cannot intersect in $\mathbb{R}^n \times \mathbb{R}$

## Types of trajectories

(1) $\underline{\dot{y}} = \underline{f}(\underline{y})$ $\quad \underline{f}(\underline{y}_s) = \underline{0}$ $\quad \underline{y}_s$ is steady state

$\underline{y}_0 = \underline{y}_s$ $\quad \underline{y}(t) = \underline{y}_s$

$\boxed{\text{POINT}}$



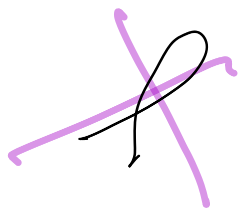
(2) $\boxed{\text{open lines}}$ $\quad \underline{y}_s$ can be limit for $t \to \pm\infty$

$$\underline{y}(t) \xrightarrow[t\to+\infty]{} \underline{y}_{s2} \qquad \underline{y}(t) \xrightarrow[t\to-\infty]{} \underline{y}_{s1}$$

(3) $\boxed{\text{limit cycles}}$ – periodic solution

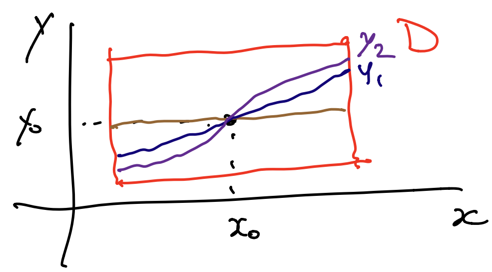
$$\exists\, T>0 \;:\; \underline{y}(t) = \underline{y}(t+T) \qquad \forall t \in \mathbb{R}$$

$$\text{but} \;:\; \underline{y}(t+s) \neq \underline{y}(t) \qquad 0<s<T$$



## Existence and uniqueness theorem for 1° order ODEs

$$\frac{dy}{dx} = f(x,y) \qquad y = y_0 \quad \text{for } x = x_0$$




Hypothesis:

$f(x,y)$ is continuous in D

$|x-x_0| \le a \qquad |y-y_0| \le b$

$|f(x,y)| \le M \qquad h = \min\left[a, \frac{b}{M}\right]$

Lipschitz condition $\quad |f(x,y_1) - f(x,y_2)| < K\,|y_1-y_2|$

Thesis $\exists$ unique solution

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y(x))\,dx = y(x) - y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x,y_{n-1}(x))\,dx$$

(I) for $n\to\infty$ $\quad \{y_n(x)\} \to y(x)$ continuous

(II) $y(x)$ must obey the ODE

(III) $y(x_0) = y_0$

(IV) unique

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = \frac{y_{n+1}-y_n}{x_{n+1}-x_n} = f(x_n, y_n)$$

$$y_{n+1} = y_n + f(x_n, y_n)\underbrace{(x_{n+1}-x_n)}_{\Delta x}$$

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## Population dynamics

single species: $y(t) \geq 0$ size of the population

$\alpha y > 0$ growth rate (net)

$\frac{dy}{dt} = \alpha y \qquad y(0) = y_0 \qquad y(t) = y_0 e^{\alpha t}$ $\boxed{\text{Malthus law}}$

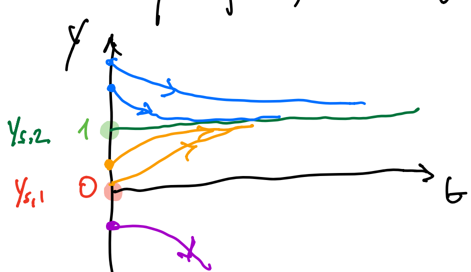
$\frac{dy}{dt} = g(y)\, y$ limited resources $\frac{dg}{dy} < 0 \qquad g(y) = \alpha - \beta y$

$\frac{dy}{dt} = (\alpha - \beta y)\, y \qquad \alpha = \beta = 1$

$\boxed{\dot{y} = (1-y)\, y}$ $\Longrightarrow$ $\boxed{\text{logistic equation}}$

$\dot{y} = f(y) \qquad f(y_s) = 0\ ? \qquad (1-y)\, y = 0$

$\boxed{y_{s1} = 0} \qquad \boxed{y_{s2} = 1}$




$y_0 > 1 \qquad \dot{y} < 0 \quad \ldots\ldots$

$y(t) \xrightarrow[t \to +\infty]{} 1^+$

$0 < y_0 < 1 \qquad \dot{y} > 0 \qquad y(t) \xrightarrow[t \to +\infty]{} 1^-$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{df}{dt} = \underbrace{\frac{df}{dy}}_{f'(y)}\frac{dy}{dt} = (1-2y)\,\dot{y}$$

$y_0 < 0 \qquad \dot{y} < 0$

non linear systems $\Rightarrow$ multiple steady-states w/ different stability character!

$y_{s1} \to$ asymptotically stable

$y_{s2} \to$ unstable

logistic equation has an analytical solution!!

$y_s$ : $f(y_s) = 0$ $\longrightarrow$ Taylor expansion in the vicinity of $y_s$

$$f(y) = \underbrace{f(y_s)}_{=0} + f'(y_s)(y-y_s) + \frac{f''(y_s)}{2}(y-y_s)^2 + \ldots \simeq$$

$y(0) = y_0$

$$\boxed{f(y) \simeq f'(y_s)(y-y_s)}$$

is a good approximation for $|y-y_s|$ very small

$$\boxed{\dot{y} = f(y)} \simeq f'(y_s)\underbrace{(y-y_s)}_{\text{deviation}}$$

$$\boxed{\dot{x} = a\, x}$$

s.s. $x = 0$ (or: $y = y_s$)

$x(0) = x_0 = y_0 - y_s$

$x(t) = x_0 e^{at}$

$$\begin{cases} a > 0 & x(t) \nearrow \Rightarrow y_s \text{ is unstable} \\ a < 0 & x(t) \xrightarrow[t \to +\infty]{} 0 \Rightarrow y_s \text{ is asymptotically stable} \end{cases}$$

logistic equation: $f(y) = (1-y)y$ $\boxed{y_{s1} = 0}$ $\boxed{y_{s2} = 1}$

$$f'(y) = \frac{df}{dy} = 1 - 2y = \begin{cases} 1 & \text{in } \boxed{y_{s1}(0)} \to \text{unstable} \\ -1 & \text{in } \boxed{y_{s2}(1)} \to \text{asympt. stable} \end{cases}$$

# Nonlinear systems, autonomous

$\underline{y} \in \mathbb{R}^n \qquad \underline{f}(\underline{y}) \in \mathbb{R}^n$

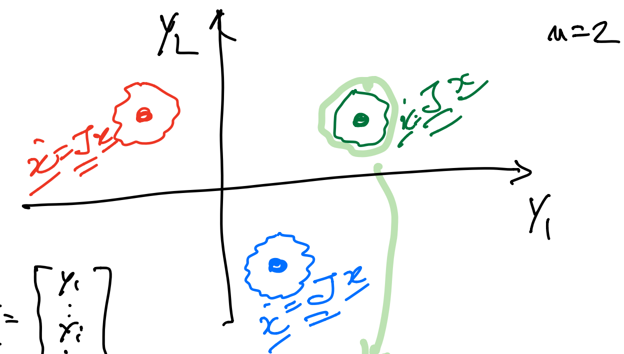
$\dot{\underline{y}} = \underline{f}(\underline{y}) \qquad \underline{y}(0) = \underline{y}_0$

steady states $\quad \underline{y}_s : \underline{f}(\underline{y}_s) = \underline{0} \quad$ definition

if $\underline{y}_0 = \underline{y}_s \iff \underline{y}(t) = \underline{y}_s$

perturbation of $\underline{y}_s$

$\underline{x} = \underline{y} - \underline{y}_s$




Taylor expansion for $y_i$, $\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_n \end{bmatrix}$

$$\dot{y}_i = f_i(\underline{y}) = \underbrace{f_i(\underline{y}_s)}_{=0} + \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}\bigg|_{\underline{y}_s} (y_j - y_{j,s}) + \underbrace{O\left(\|\underline{y} - \underline{y}_s\|^2\right)}_{\simeq 0} \qquad (i = 1, \dots, n)$$

$$\underbrace{\dot{\underline{y}}}_{\dot{x}} = \underline{\underline{J}}\bigg|_{\underline{y}_s} \underbrace{(\underline{y} - \underline{y}_s)}_{\underline{x}} + O\left(\|\underline{y} - \underline{y}_s\|^2\right) \qquad \underline{\underline{J}} = \left[\frac{\partial f_i}{\partial y_j}\right]$$

$$\boxed{\dot{\underline{x}} = \underline{\underline{J}}_s \, \underline{x}}$$

linearized system

$\underline{x}_s = \underline{0}$

$$\underline{x}(t) = \sum_{i=1}^{n} \frac{\underline{w}_i^T \underline{x}_0}{\underline{w}_i^T \underline{z}_i} \underline{z}_i e^{\lambda_i t}$$

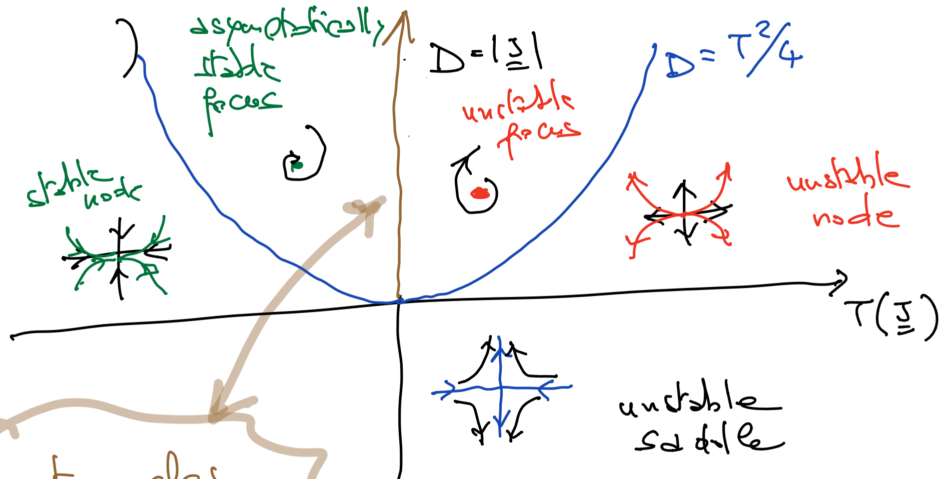
## Stability theorem for nonlinear systems $\boxed{2.13}$

1. when s.s. of linearized system is asymptotically stable, so is the s.s. of the nonlinear system
2. when s.s. of linearized system is unstable, so is the s.s. of the nonlinear system
3. when s.s. of linearized system is stable (periodic solutions) then nothing can be said about the s.s. of the nonlinear system (purely imaginary eigenvalues

$\underline{y}, \underline{f} \in \mathbb{R}^2$

$\dot{\underline{x}} = \underline{\underline{J}}\,\underline{x}$

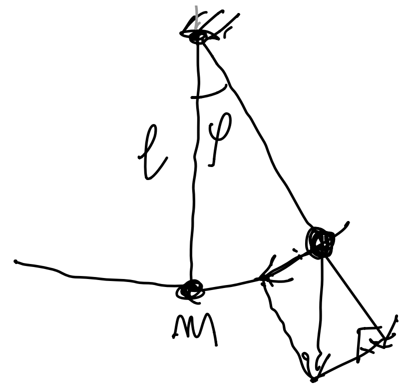
$\underline{x} = \underline{y} - \underline{y}_s$




↳ limit cycles

↳ Hopf bifurcation theorem

# Pendulum with attrition




$$m\ell\ddot{\varphi} = -mg\sin\varphi - h\ell\dot{\varphi}$$

$$\ddot{\varphi} = -\frac{g}{\ell}\sin\varphi - \frac{h}{m}\dot{\varphi}$$

$$\tau = t/t_2 \qquad \frac{d}{dt} = \frac{1}{t_2}\frac{d}{d\tau}$$

$$\frac{d^2\varphi}{d\tau^2} + t_2^2\frac{g}{\ell}\sin\varphi + \frac{t_2 h}{m}\frac{d\varphi}{d\tau} = 0$$

$$t_2^2 = \frac{\ell}{g} \qquad K = \frac{t_2 h}{m} = \frac{h}{m}\sqrt{\frac{\ell}{g}} = K$$

$$\boxed{\ddot{\varphi} + \sin\varphi + K\dot{\varphi} = 0}$$

$$x_1 = \varphi$$
$$x_2 = \dot{\varphi} = \dot{x}_1$$

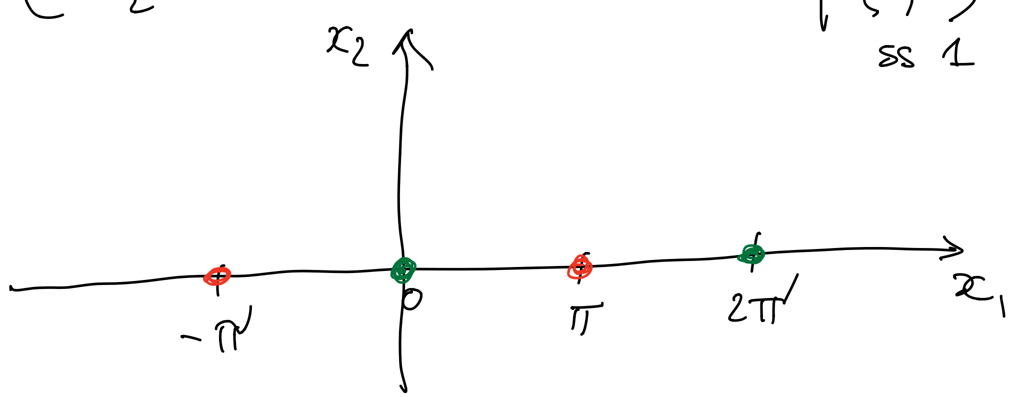
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 - Kx_2 \end{cases}$$

S.S. $x_2 = 0$

$\sin x_1 = 0$

$x_1 = k\pi \quad k \in \mathbb{Z}$

$(0,0)$ ss 1 $\qquad (\pi, 0)$ ss 2

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$$\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -k \end{bmatrix} \qquad \underline{\underline{J}}_1 = \begin{bmatrix} 0 & 1 \\ -1 & -k \end{bmatrix} \qquad \underline{\underline{J}}_2 = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$$

SS 2 $\quad T_2 = -k \quad D_2 = -1 \quad$ saddle

$\varphi = \pi$

SS 1 $\quad T_2 = -k \quad D_2 = 1 \quad$ stable ?

$\varphi = 0$

$$\lambda_i = \frac{1}{2}\left(-k \pm \sqrt{k^2 - 4}\right)$$

$k$

$k > 2$: $\lambda_2 < \lambda_1 < 0$ real distinct eigenvalues $\quad$ asymptotically stable node

$2$: $k = 2 \longrightarrow \lambda_1 = \lambda_2 \rightsquigarrow$ ? $\quad$ not generic

$0 < k < 2$: complex eigenvalues with $\mathrm{Re}(\lambda) < 0 \quad$ " " focus

$0$: $k = 0 \longrightarrow \lambda_j = \pm i \longrightarrow$ periodic solution

pendulum without attrition