ODEs $\longrightarrow$ AEs
PDEs $\longrightarrow$ ODEs

} mathematical methods / graphical methods

careful — beautiful

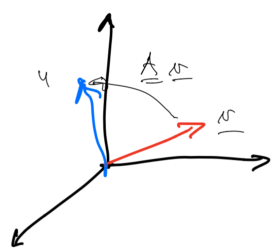
Varrus & Montidelli "Mathematical methods"

ch. 1, 2, 5, 6 — 1, 5, 6, 7, 11, 12, 13, 15, 16, 17

$\underline{\underline{A}} = [a_{ij}]$ $n \times n$ matrix — square matrix

$i, j = 1, \dots, n$

$\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\quad \underline{v} \in \mathbb{R}^n \quad \underline{v}^T = [v_1 \; \dots \; v_n]$



$n = 3$

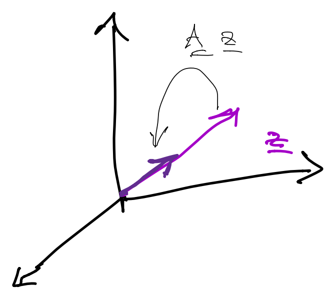
$\mathbb{R}^3$

$\underline{\underline{A}}\,\underline{v} = \underline{u}$

↑ operation

$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underline{\underline{A}}$ $\quad n \times n \quad a_{ij} \in \mathbb{R}$



is this possible?

$$\boxed{\underline{\underline{A}}\,\underline{z} = \lambda\,\underline{z}}$$

eigenvalue problem

non trivial solution

$\underline{z} \neq \underline{0}$

$$\boxed{(\underline{\underline{A}} - \lambda\underline{\underline{I}})\,\underline{z} = \underline{0}}$$

system of linear AEs in z_i

homogeneous

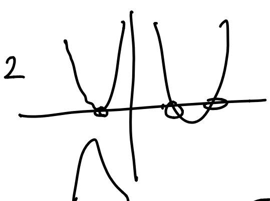
$$|\underline{\underline{A}} - \lambda\underline{\underline{I}}| = \boxed{0 = P_n(\lambda)} \Longrightarrow \lambda_i \begin{cases} \in \mathbb{R} \\ \in \mathbb{C} \end{cases}$$

up to n distinct solutions

eigenvalue λ_j $\longleftrightarrow$ $\underline{z}_j$ eigenvector corresponding to λ_j

for $\underline{\underline{A}}$ — $\underline{w}_j$ left eigenvector

$n = 2$



new eigenvalue problem

$$\underline{w}^T \underline{\underline{A}} = \eta\, \underline{w}^T \qquad (\ )^T = (\ )^T \Rightarrow \underline{\underline{A}}^T \underline{w} = \eta\, \underline{w}$$

$$\left|\underline{\underline{A}} - \eta\, \underline{\underline{I}}\right| = \left|\underline{\underline{A}}^T - \eta\, \underline{\underline{I}}\right| = 0 \longleftarrow \left(\underline{\underline{A}}^T - \eta\, \underline{\underline{I}}\right)\underline{w} = \underline{0}$$

$$\eta_j = \lambda_j$$

1.13 bi-orthogonality property

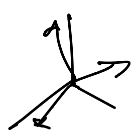
$$\underline{u} \perp \underline{v} \iff \underline{u}^T \underline{v} = \sum_{i=1}^{n} u_i v_i = 0$$

$$\boxed{\underline{w}_i^T \underline{z}_j = \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}}$$

$$\begin{cases} \underline{w}_i^T \underline{\underline{A}}\, \underline{z}_j = \lambda_j\, \underline{w}_i^T \underline{z}_j \\ \underline{w}_i^T \underline{\underline{A}}\, \underline{z}_j = \lambda_i\, \underline{w}_i^T \underline{z}_j \end{cases} \quad (-) \qquad 0 = (\lambda_j - \lambda_i)\, \underline{w}_i^T \underline{z}_j$$

1.12

$\underline{\underline{A}}$ $n \times n$ $\quad \lambda_j$ eigenvalues $\begin{pmatrix} \lambda_j \neq 0 \\ \lambda_j \text{ distinct} \end{pmatrix}$

$\underline{z}_j \in \mathbb{R}^n$  are a basis in $\mathbb{R}^n$

(1) linearly independent

$$\sum_{j=1}^{n} c_j \underline{z}_j = \underline{0} \iff c_j = 0 \ \forall j$$

(II) expansion of an arbitrary vector $\underline{x} \in \mathbb{R}^n$

$$\Rightarrow \quad \underline{x} = \sum_{j=1}^{n} c_j \underline{z}_j \qquad c_j = ?$$

$$\underline{w}_i^T \underline{x} = \sum_{j=1}^{n} c_j \underline{w}_i^T \underline{z}_j = c_i \underline{w}_i^T \underline{z}_i$$

$$c_i = \frac{\underline{w}_i^T \underline{x}}{\underline{w}_i^T \underline{z}_i} \quad \Rightarrow \quad \boxed{\underline{x} = \sum_{j=1}^{n} \left(\frac{\underline{w}_j^T \underline{x}}{\underline{w}_j^T \underline{z}_j} \right) \underline{z}_j}$$

(1.16) $\underline{\underline{A}}\,\underline{x} = \underline{b}$ $\qquad \underline{x} \in \mathbb{R}^n$ is unknown

$\underline{\underline{A}}, \underline{b}$ are known

distinct eigenvalues $\qquad |\underline{\underline{A}}| \neq 0$

$\underline{x}$ should be calculated as linear combination of the eigenvectors of $\underline{\underline{A}}$

$\underline{x} = \underline{\underline{A}}^{-1} \underline{b}$ $\qquad$ trivial, well known solution

$$\underline{b} = \sum_{j=1}^{n} \boxed{\beta_j}\, \underline{z}_j = \sum_{j=1}^{n} \boxed{\frac{\underline{w}_j^T \underline{b}}{\underline{w}_j^T \underline{z}_j}}\, \underline{z}_j$$

$$\underline{x} = \sum_{j=1}^{n} \alpha_j \underline{z}_j \qquad \alpha_j = ?$$

$$\underline{\underline{A}} \sum_{j=1}^{n} \alpha_j \underline{z}_j = \sum_{j=1}^{n} \beta_j \underline{z}_j$$

$$\sum_{j=1}^{n} \alpha_j \underline{\underline{A}} \underline{z}_j = \boxed{\sum_{j=1}^{n} \alpha_j \lambda_j \underline{z}_j = \sum_{j=1}^{n} \beta_j \underline{z}_j}$$

$$\alpha_j \lambda_j = \beta_j \qquad \alpha_j = \frac{\boxed{\beta_j}}{\lambda_j} \qquad \lambda_j \neq 0$$

$$\boxed{\underline{x} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \frac{\underline{w}_j^T \underline{b}}{\underline{w}_j^T \underline{z}_j} \underline{z}_j}$$

1.17 ODEs Linear ODEs $x(t) = x(0) \exp(at)$

$x \in \mathbb{R}$ $x(t)$ $t \in \mathbb{R}$

linear first order ODE $\dot{x} = \boxed{\frac{dx}{dt} = ax} + b$ $b=0$

$\underline{x} \in \mathbb{R}^n$ $\underline{x}(t)$ $\frac{dx_i}{dt} = b_i + a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n$

$\underline{\underline{A}} = [a_{ij}]$ $\underline{b} = [b_i]$ $i = 1, \dots, n$

initial condition

$$\frac{d\underline{x}}{dt} = \boxed{\dot{\underline{x}} = \underline{\underline{A}} \underline{x} + \underline{b}} \qquad \underline{x}(t=t_0) = \boxed{\underline{x}_0 = \underline{x}(0)}$$

$\underline{\underline{A}}$ $n \times n$ matrix, $|\underline{\underline{A}}| \neq 0$, distinct eigenvalues

$|\underline{\underline{A}}| = \prod_{j=1}^{n} \lambda_j$

$$\underline{x}(t) = \underline{y}(t) + \underline{k} \qquad \dot{\underline{y}} = \underline{\underline{A}}(\underline{y}+\underline{k}) + \underline{b} =$$

$$= \underline{\underline{A}}\,\underline{y} + \underbrace{\underline{\underline{A}}\,\underline{k} + \underline{b}}_{=\underline{0}}$$

$$\underline{k} = -\underline{\underline{A}}^{-1}\underline{b}$$

$$\underline{y}(t) = \underline{x}(t) + \underline{\underline{A}}^{-1}\underline{b}$$

$$\underline{x}(0) = \underline{x}_0 = \underline{y}(0) + \underline{k}$$

$$\underline{y}(0) = \underline{x}_0 - \underline{k} = \underline{x}_0 + \underline{\underline{A}}^{-1}\underline{b} = \underline{y}_0$$

$$\boxed{\begin{aligned} \dot{\underline{y}} &= \underline{\underline{A}}\,\underline{y} \\ \underline{y}(0) &= \underline{y}_0 \end{aligned}}$$

homogeneous system of linear equations

$\underline{y} \in \mathbb{R}^n$ $\quad \underline{\underline{A}}$ with $|\underline{\underline{A}}| \neq 0$ distinct eigenvalues

solution is $\underline{y}(t)$ that fulfills the ODEs and the I.C.

$$\underline{y}(t) = \underline{v}\exp(a t) = \quad \underline{\underline{\text{hypothesis}}} \quad = \underline{z}_j \exp(\lambda_j t)$$

$$a\,\underline{v}\,\cancel{\exp(at)} = (\underline{\underline{A}}\,\underline{v})\,\cancel{\exp(at)}$$

eigenvalue problem

$$\underline{y}(t) = \sum_{j=1}^{n} c_j\,\underline{z}_j \exp(\lambda_j t)$$

c_j : scalar to be determined using I.C.

$$\underline{y}(0) = \underline{y}_0 = \sum_{j=1}^{n} c_j\,\underline{z}_j \qquad c_j = \frac{\underline{w}_j^T \underline{y}_0}{\underline{w}_j^T \underline{z}_j}$$

$$\boxed{\underline{y}(t) = \sum_{j=1}^{n} \frac{\underline{w}_j^T \underline{y}_0}{\underline{w}_j^T \underline{z}_j}\, \underline{z}_j \exp(\lambda_j t)}$$!!!

system's dynamic

decomposition in modes

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

cofactor $A_{ij} = (-1)^{i+j} \left| \begin{matrix}\text{complement} \\ \text{of } a_{ij}\end{matrix} \right|$

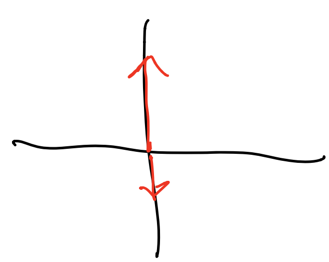
$(n-1)\times(n-1)$ matrix

$$A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

adjoint matrix of $\underline{\underline{A}}$

$$\text{adj } \underline{\underline{A}} = \left[A_{ji} \right] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example



$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \lambda_j, \ \underline{z}_j, \ \underline{w}_j$$

$$0 = \left| \underline{\underline{A}} - \lambda \underline{\underline{I}} \right| = \begin{vmatrix} 1-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = -(1-\lambda)(1+\lambda) = \begin{cases} \lambda_2 = 1 \\ \lambda_1 = -1 \end{cases}$$

$$\lambda_1 = -1 \rightarrow \underline{z}_1, \ \underline{w}_1$$

$$\underline{\underline{A}} - \lambda_1 \underline{\underline{I}} = \begin{bmatrix} 1+1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \operatorname{adj}(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}) = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \begin{matrix} \\ -\underline{w}_1^T \end{matrix}$$

$$\underline{z}_1$$

$$\lambda_2 = 1 \rightarrow \underline{z}_2, \ \underline{w}_2$$

$$\underline{\underline{A}} - \lambda_2 \underline{\underline{I}} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \qquad \operatorname{adj}(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}}) = \begin{bmatrix} -2 & 0 \\ -1 & 0 \end{bmatrix} \begin{matrix} \\ -\underline{w}_2^T \end{matrix}$$

$$\underline{z}_2$$

$$\underline{w}_1^T \underline{z}_1 = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -2$$

$$\underline{w}_1^T \underline{z}_2 = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = 2 - 2 = 0$$

$$\underline{w}_2^T \underline{z}_2 = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = 2$$

$$\underline{w}_2^T \underline{z}_1 = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 + 0 = 0$$

$$\underline{\underline{A}}\, \underline{z}_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\underline{\underline{A}}\, \underline{z}_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} (1)$$