Systems of linear ODEs, autonomous

$$\boxed{\dot{y} = A y} \qquad \boxed{y(0) = y_0} \qquad \boxed{y(t) = \sum_{j=1}^{n} \frac{w_j^T y_0}{w_j^T z_j} z_j \exp(\lambda_j t)}$$

$y \in \mathbb{R}^n$ $\qquad \lambda_j, \ z_j, \ w_j$

$A = [a_{ij}]$ $\qquad a_{ij} \in \mathbb{R}$

$\lambda_j \in \mathbb{R}$

$\lambda_j \in \mathbb{C}$ $\qquad \lambda_j = \alpha_j + i\beta_j$ $\qquad z_j = u + i v$

$\bar{\lambda}_j \in \mathbb{C}$ $\qquad \bar{\lambda}_j = \alpha_j - i\beta_j$ $\qquad \bar{z}_j = u - i v$

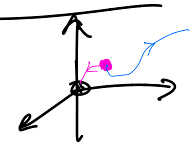
$w_j = r_j + i s_j$

$\bar{w}_j = r_j - i s_j$

$$A(u \pm i v) = (\alpha \pm i\beta)(u \pm i v)$$

$$\boxed{A u} \pm i \boxed{A v} = \boxed{(\alpha u - \beta v)} \pm i \boxed{(\beta u + \alpha v)}$$

Special case $\boxed{1}$ $\qquad \boxed{y_0 = 0}$ $\qquad y(t) = 0$



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s.s.: steady-state solution

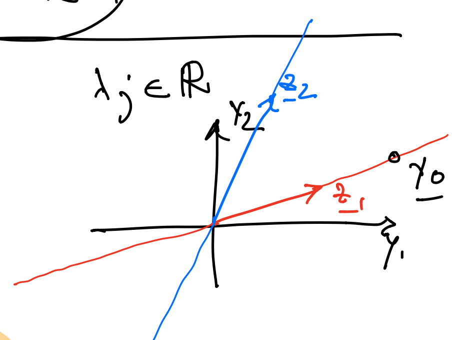
equilibrium solution

$|A| \neq 0$ $\qquad \exists$ one s.s. solution

stability of a s.s.?

Special case $\boxed{2}$ $\qquad y_0 = k z_i$ $\qquad \lambda_j \in \mathbb{R}$

$$\frac{w_j^T y_0}{w_j^T z_j} = \begin{cases} 0 & j \neq i \\ \neq 0 & j = i \end{cases}$$



$$y(t) = \frac{k \cancel{w_i^T z_i}}{\cancel{w_i^T z_i}} z_i \exp(\lambda_i t) = k z_i \exp(\lambda_i t)$$

invariant subspaces

special case [3] $\mathbb{R}^2$

$\lambda=\alpha+i\beta \qquad \underline{z}=\underline{u}+i\underline{v}$

$\bar\lambda=\alpha-i\beta \qquad \bar{\underline{z}}=\underline{u}-i\underline{v}$

$$\underline{y}(t)=c\,\underline{z}\exp(\lambda t)+\bar{c}\,\bar{\underline{z}}\exp(\bar\lambda t)\in\mathbb{R}^2$$

$c=\gamma+i\delta$

$\bar c=\gamma-i\delta$

$$\underline{y}(t)=(\gamma+i\delta)(\underline{u}+i\underline{v})\exp((\alpha+i\beta)t)+(\gamma-i\delta)(\underline{u}-i\underline{v})\exp((\alpha-i\beta)t)=$$

$$\exp((\alpha+i\beta)t)=\exp^{\alpha t}(\cos(\beta t)+i\sin(\beta t))$$

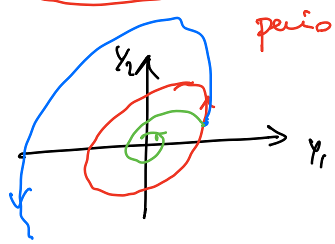
$$=2e^{\alpha t}\underbrace{\left[(\gamma\underline{u}-\delta\underline{v})\cos\beta t-(\delta\underline{u}+\gamma\underline{v})\sin\beta t\right]}_{\text{periodic function}}$$

Q.E.D.

$\alpha>0$

$\alpha=0$ periodic solution

$\alpha<0$



$T=\frac{2\pi}{\beta}$

$\mathbb{R}^3 \qquad \lambda\in\mathbb{R} \qquad \underline{z}, \underline{w}$

$\mu=\alpha+i\beta \qquad \underline{p}=\underline{u}+i\underline{v}$

$\bar\mu=\alpha-i\beta \qquad \bar{\underline{p}}=\underline{u}-i\underline{v}$

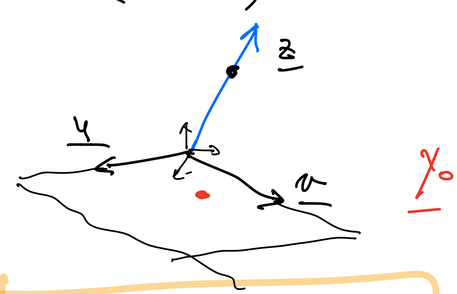
$$\underline{y}(t)=k\,\underline{z}e^{\lambda t}+c(\underline{u}+i\underline{v})e^{\mu t}+\bar{c}(\underline{u}-i\underline{v})e^{\bar\mu t}$$

$\to 0$

$$k=\frac{\underline{w}^T\underline{y}_0}{\underline{w}^T\underline{z}}\overset{!}{=}0$$

$\underline{w}^T\underline{u}=0 \qquad \underline{w}^T\underline{v}=0$

$\underline{y}_0=a\underline{u}+b\underline{v}$



$$\underline{w}^T(\underline{u}+i\underline{v})=0$$

$$\underline{w}^T\underline{\underline{A}}(\underline{u}+i\underline{v})=(\alpha+i\beta)\,\underline{w}^T(\underline{u}+i\underline{v})$$

$$\underline{w}^T\underline{\underline{A}}(\underline{u}+i\underline{v})=\lambda\,\underline{w}^T(\underline{u}+i\underline{v})$$

$$\underline{w}^T\left\{(\alpha \underline{u} - \beta \underline{v}) + i(\beta \underline{u} + \alpha \underline{v})\right\} - \lambda \underline{w}^T(\underline{u} + i\underline{v}) = 0$$

$$\underline{w}^T\left\{\left[(\alpha-\lambda)\underline{u} - \beta\underline{v}\right] + i\left[\beta\underline{u} + (\alpha-\lambda)\underline{v}\right]\right\} = 0$$

$$\underline{w}^T\underline{u} = 0 \qquad \underline{w}^T\underline{v} = 0$$

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real eigenvalues: dynamics consists of linear combination of dynamics along a 1D subspace

a pair of complex eigenvalues: additionally we have a contribution from a 2D dynamics on the plane defined by the complex eigenvectors

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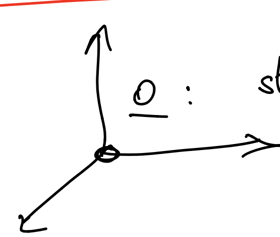
$$e^{\lambda_j t} \xrightarrow[t\to+\infty]{} 0 \quad \text{iff} \quad Re(\lambda_j) < 0$$

$$e^{\lambda_j t} \xrightarrow[t\to+\infty]{} +\infty \quad \text{iff} \quad Re(\lambda_j) > 0$$

$$e^{\lambda_j t} = \text{periodic function} \quad \text{iff} \quad Re(\lambda_j) = 0$$



$\underline{0}$: steady state

① $\lim_{t\to+\infty} \underline{y}(t) = \underline{0}$ iff $Re(\lambda_j)<0 \quad \forall j$

② $\lim_{t\to+\infty} \|\underline{y}(t)\| = +\infty$ if $\exists j : Re(\lambda_j)>0$

③ if $Re(\lambda_j)\le 0 \quad \forall j$ and $\exists j : Re(\lambda_j)<0$

① s.s. / equilibrium is STABLE, asymptotically

② s.s. / eq. is UNSTABLE

③ s.s. / eq. is STABLE

classification of conditions for stability for system of linear ODEs

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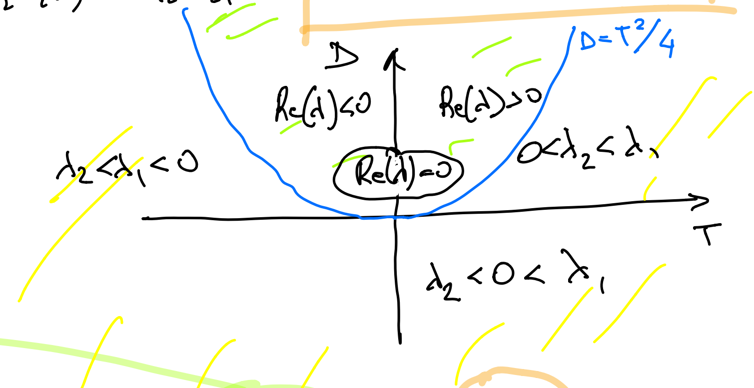
$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ $\quad \underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad |\underline{A}| = D = a_{11}a_{22} - a_{21}a_{21}$

$\operatorname{tr} \underline{A} = T = a_{11} + a_{2}$

$$0 = |\underline{A} - \lambda \underline{\underline{I}}| = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = \boxed{\lambda^2 - T\lambda + D = 0}$$

$$\Delta = T^2 - 4D \begin{cases} >0 \\ =0 \\ <0 \end{cases}$$



$\lambda_1 + \lambda_2 = T$ $\quad \lambda = \alpha \pm i\beta$

$\lambda_1 \lambda_2 = D$

$$0 = \lambda^2 - T\lambda + D = (\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$$

$\lambda = \alpha \pm i\beta$

$\alpha < 0$ $\qquad$ $\alpha = 0$ $\qquad$ $\alpha > 0$

e.s. $\qquad$ asymptotically stable focus $\qquad$ centre st. $\qquad$ unstable focus

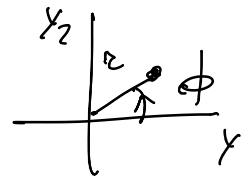
$\dot{\phi} > 0$ $\qquad a_{21} > 0 > a_{12}$ counterclockwise

$a_{21} < 0 < a_{12}$: clockwise $\quad \dot{\phi} < 0$

$$\begin{cases} \dot{y}_1 = a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \end{cases}$$

Hyp

$$T^2 - 4\Delta = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) < 0$$



$$\begin{cases} r^2 = y_1^2 + y_2^2 \\ \phi = \operatorname{arctg}\left(\frac{y_2}{y_1}\right) \end{cases}$$

$$r\dot{r} = y_1 \dot{y}_1 + y_2 \dot{y}_2 = a_{11} y_1^2 + y_1 y_2 (a_{12} + a_{21}) + a_{22} y_2^2 = Q_r$$

$$\dot{\phi} = \frac{1}{1 + \left(\frac{x_2}{x_1}\right)^2} \frac{\dot{y}_2 y_1 - \dot{y}_1 y_2}{y_1^2} = \frac{a_{21} y_1^2 + (a_{22} - a_{11}) y_1 y_2 - a_{12} y_2^2}{y_1^2 + y_2^2} = \frac{Q_\phi}{y_1^2 + y_2^2}$$

$$f(x) = a x^2 + bx + c \; (= 0)$$

$$Q(x,y) = ax^2 + bxy + cy^2 = y^2 \underbrace{\left[a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c \right]}$$

$Q(x,y) > 0$ iff $\Delta < 0$ $\quad a > 0$

$Q(x,y) < 0$ iff $\Delta < 0$ $\quad a < 0$

$a_{21} \gtrless 0$

$$\Delta_\phi = (a_{22} - a_{11})^2 + 4 a_{21} a_{12} + 4 a_{11} a_{22} - 4 a_{11} a_{22} =$$

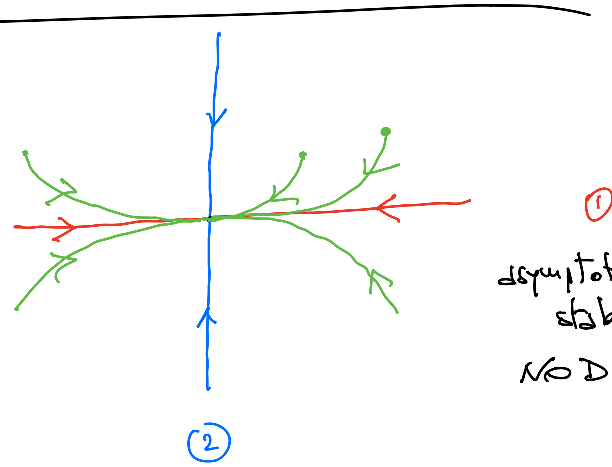
$$= (a_{22} + a_{11})^2 - 4(a_{11} a_{22} - a_{21} a_{12}) < 0$$

$$(a_{22} - a_{11})^2 < -4 a_{21} a_{12} \qquad a_{21} a_{12} < 0$$

$\lambda_2 < \lambda_1 < 0$

$\underline{y}(t) = c_1 \underline{z}_1 e^{\lambda_1 t} + c_2 \underline{z}_2 e^{\lambda_2 t}$

$t \to +\infty \quad \underline{y}(t) \propto \underline{z}_1 e^{\lambda_1 t}$

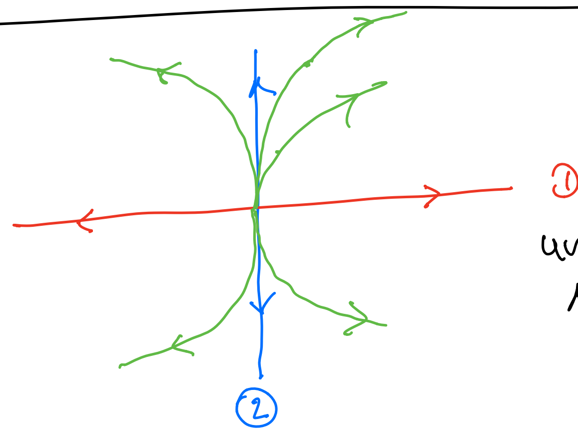


asymptotically stable NODE

$0 < \lambda_2 < \lambda_1$

$t \to +\infty \quad \underline{y}(t) \propto \underline{z}_1 e^{\lambda_1 t}$

$t \to -\infty \quad \underline{y}(t) \propto \underline{z}_2 e^{\lambda_2 t}$



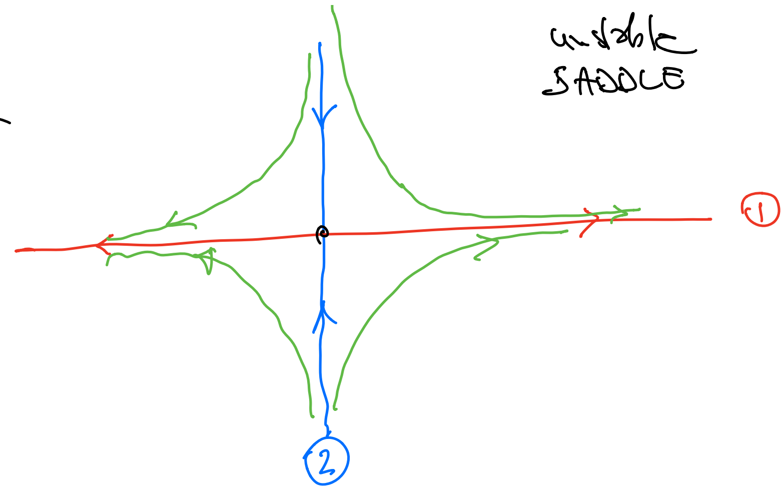
unstable NODE

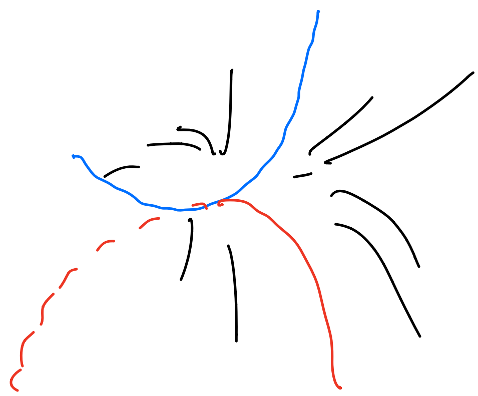
$\lambda_2 < 0 < \lambda_1$

$\underline{x}(t) = c_1 \underline{z}_1 e^{\lambda_1 t} + c_2 \underline{z}_2 e^{\lambda_2 t}$

$t \to +\infty \quad \underline{x}(t) \propto \underline{z}_1 e^{\lambda_1 t}$

$t \to -\infty \quad \underline{x}(t) \propto \underline{z}_2 e^{\lambda_2 t}$





nonlinear system
autonomous systems
stability
linearization