[I]

Predator - Prey model (Lotka-Volterra)
(P) (N)

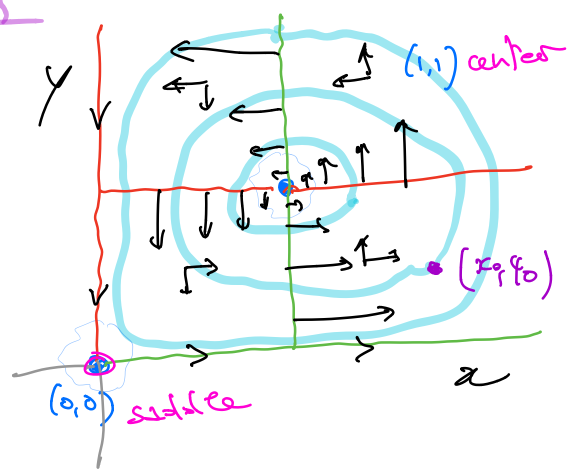
$\dot{P} = -bP$ (when alone) $\qquad \dot{N} = aN$ (when alone)

$a, b, c, d > 0$

$$\boxed{\dot{P} = (dN - b)P \qquad \dot{N} = (a - cP)N}$$

$y = \frac{c}{a} P \qquad x = \frac{d}{b} N$

$$\Rightarrow \boxed{\begin{cases} \dot{x} = ax(1-y) & = \frac{dx}{dt} \\ \dot{y} = by(x-1) & = \frac{dy}{dt} \end{cases}}$$



linearized systems

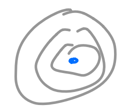
$\underline{J} = \begin{bmatrix} a(1-y) & -ax \\ by & b(x-1) \end{bmatrix}$

$\lambda_1 = a \quad \lambda_2 = -b$

$\underline{J}_{(0,0)} = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} \quad T_{(0,0)} = a - b \quad D_{(0,0)} = -ab < 0 \quad (0,0)$: saddle

$\underline{J}_{(1,1)} = \begin{bmatrix} 0 & -a \\ b & 0 \end{bmatrix} \quad T_{(1,1)} = 0 \quad D_{(1,1)} = ab > 0 \quad (1,1)$: center

$\longrightarrow$ $\lambda_{1,2} = \pm i\sqrt{ab}$ $\longrightarrow$ periodic solutions



$\longrightarrow$ infinite periodic solutions

$N_1: \ \dot{x} = 0 \qquad x = 0 \qquad y = 1$

$N_2: \ \dot{y} = 0 \qquad y = 0 \qquad x = 1$

$$\frac{dx}{dy} = \frac{ax(1-y)}{by(x-1)} \qquad \int_{x_0}^{x} b\,\frac{(x-1)}{x}\,dx = \int_{y_0}^{y} \frac{a(1-y)}{y}\,dy$$

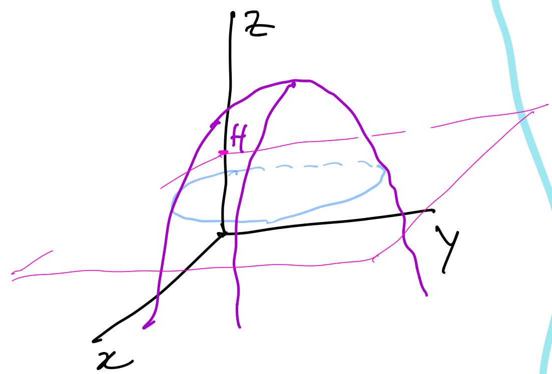
$$b(x - x_0) - b\ln\frac{x}{x_0} = a\ln\frac{y}{y_0} - a(y - y_0)$$

$$\underbrace{f(x,y) = a\ln y - ay - bx + b\ln x}_{} = \underbrace{a\ln y_0 - ay_0 - bx_0 + b\ln x_0}_{} = H$$

$$f_x = -b + \frac{b}{x} \quad (=0 \text{ if } x=1)$$

$$f_y = \frac{a}{y} - a \quad (=0 \text{ if } y=1)$$

$$\left.\begin{aligned} f_{xx} &= -\frac{b}{x^2} \quad (= -b) \\ f_{xy} &= 0 \\ f_{yy} &= -\frac{a}{y^2} \quad (= -a) \end{aligned}\right\} \quad \underline{\underline{H}} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix}$$



$$f(x,y) = f(1,1) + \underbrace{f_x}_{=0}(x-1) + \underbrace{f_y}_{=0}(y-1) + \frac{1}{2} f_{xx}(x-1)^2 +$$

Taylor expansion

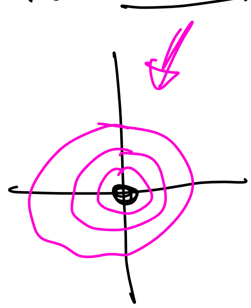
$$+ \underbrace{f_{xy}}_{=0}(x-1)(y-1) + \frac{1}{2} f_{yy}(y-1)^2 + \ldots =$$

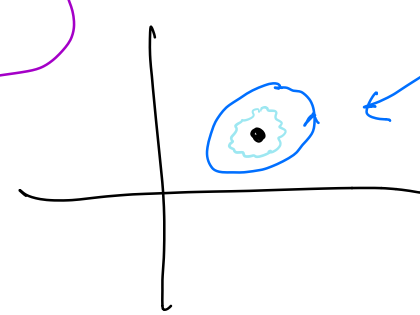
$$= -(a+b) \underbrace{- \frac{b}{2}(x-1)^2 - \frac{a}{2}(y-1)^2}_{}$$

quadratic

(II) Hopf bifurcation $\longrightarrow$ limit cycles

linear system 1 non linear system 2





$$\begin{cases} \dot{x} = -y + x\left(\mu + a\left(x^2+y^2\right)\right) \\ \dot{y} = x + y\left(\mu + a\left(x^2+y^2\right)\right) \end{cases}$$

$\mu \in \mathbb{R}$, $a = \pm 1$

s.s. $(0,0)$

$$\underline{\underline{J}} = \begin{bmatrix} \mu + a(x^2+y^2) + 2ax^2 & -1 + 2axy \\ 1 + 2axy & \mu + a(x^2+y^2) + 2ay^2 \end{bmatrix} = \underline{\underline{J}}\Big|_{0,0} = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

$T = 2\mu \gtrless 0$

$D = \mu^2 + 1 > 0$

$$D\left(\underline{\underline{J}}\Big|_{0,0} - \lambda \underline{\underline{I}}\right) = \begin{vmatrix} \mu-\lambda & -1 \\ 1 & \mu-\lambda \end{vmatrix} = (\lambda-\mu)^2 + 1 = 0$$

$(\lambda-\mu)^2 = -1$

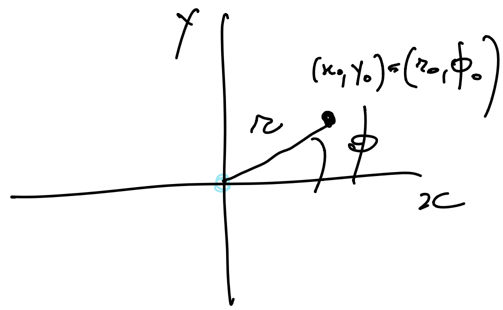
$\lambda - \mu = \pm i$

$$\boxed{\lambda_{1,2} = \mu \pm i}$$

	linearized system $\underline{\underline{J}}\big\vert_{0,0}$	non linear system
$\mu<0$	asymptotically stable focus	
$\mu=0$	center	???
$\mu>0$	unstable focus	

Hopf bifurcation theorem

$$\left.\begin{aligned} r^2 &= x^2+y^2 \\ \operatorname{tg}\phi &= y/x \end{aligned}\right\} \text{ polar coordinates}$$



$$\begin{cases} x = r\cos\phi = rC \\ y = r\sin\phi = rS \end{cases} \Rightarrow$$

$$\Rightarrow \quad \boxed{\begin{aligned} \dot{x} &= \dot{r}C - rS\dot{\phi} = -rS + rC\left(\mu + ar^2\right) \quad &(1) \\ \dot{y} &= \dot{r}S + rC\dot{\phi} = rC + rS\left(\mu + ar^2\right) \quad &(2) \end{aligned}}$$

$\dot{r} = F(r,\phi)$

$\dot{\phi} = G(r,\phi)$

$-S(1) + C(2)$

$C(1) + S(2)$

$$\begin{cases} r\dot{\phi} = r & r \neq 0 \quad \boxed{\dot{\phi} = 1} \\ \boxed{\dot{r} = r\left(\mu + ar^2\right)} \end{cases}$$

$$\boxed{\phi(t) = \phi_0 + t}$$

decoupled equations

s.s. $\dot{r} = 0$

$r = 0$ or $\mu + ar^2 = 0$

3

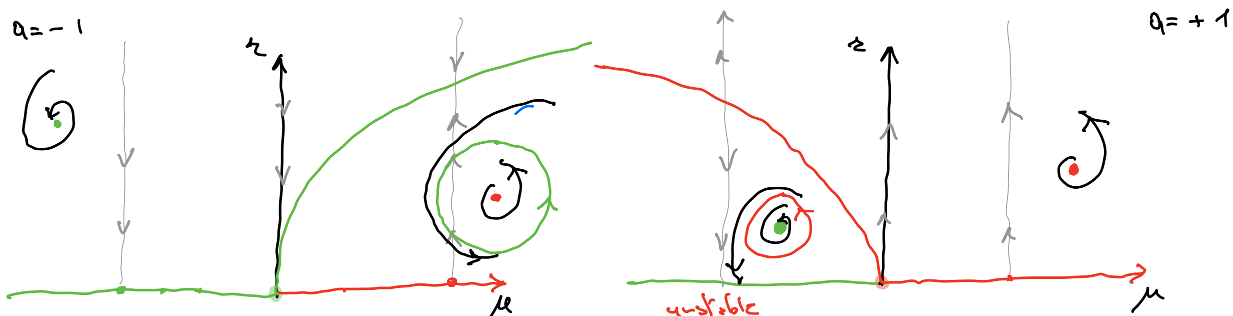
$r^2 = \mu > 0$

$r = \sqrt{\mu}$ $\quad \mu > 0$

$$\boxed{\phi(t) = \phi_0 + t}$$

$r^2 = -\mu > 0$

$r = \sqrt{-\mu}$ $\quad \mu < 0$



$\dot{r} = r(\mu - r^2)$

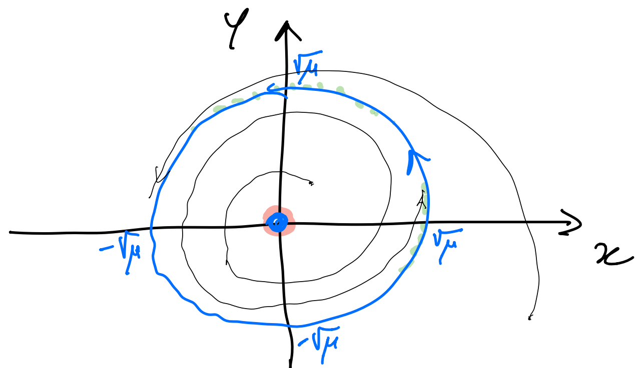
limit cycle asymptotically stable

unstable limit cycle

$\dot{r} = r(\mu + r^2)$

$\mu = 0 \Rightarrow \dot{r} = -r^3 < 0$

$\mu = 0 \Rightarrow \dot{r} = r^3 > 0$



Hopf bifurcation theorem

$\underline{\dot{x}} = \underline{f}(\underline{x}, \mu) \qquad \underline{x}, \underline{f} \in \mathbb{R}^n \qquad \mu \in \mathbb{R}$

Hyp.

(1) $\underline{f}(\underline{x_0}, \mu_0) = 0$ $\qquad$ $\underline{x_0}$ is steady state corresponding to μ_0

$\underline{x_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall \mu$

(2) $\underline{J}\Big|_{\underline{x}, \mu_0}$ has a pair of purely imaginary eigenvalues

$\mu_0 = 0 \Leftrightarrow Re(\lambda)\Big|_{\mu_0} = 0$

(3) $\dfrac{d}{d\mu} Re(\lambda)\Big|_{\mu = \mu_0} \neq 0$

$\lambda = \mu \pm i \Rightarrow Re(\lambda) = \mu$

$\dfrac{d}{d\mu} Re(\lambda) = 1 \neq 0$

Thesis

limit cycles arise as μ crosses μ_0 (emergence on one side, and disappearance on the other)

$$T_0 = \frac{2\pi}{\omega_0} \quad !!$$