Behaviour of ODE's

- linear – non-linear
- Qualitative behaviour
- 1. Linear problem:

$$\frac{du}{dt} = -au + b$$

- With a, b > 0 and $u(0)$ = u_0

Behaviour of ODE's - linear

- Examine the sign of the derivative:

- 1. $\dfrac{du}{dt} > 0 \;\rightarrow\; -au + b > 0 \;\rightarrow\; u < \dfrac{b}{a}$

- 2. $\dfrac{du}{dt} < 0$ if $u > \dfrac{b}{a}$

- 3. $\dfrac{du}{dt} = 0$ if $u = \dfrac{b}{a} = u_s$

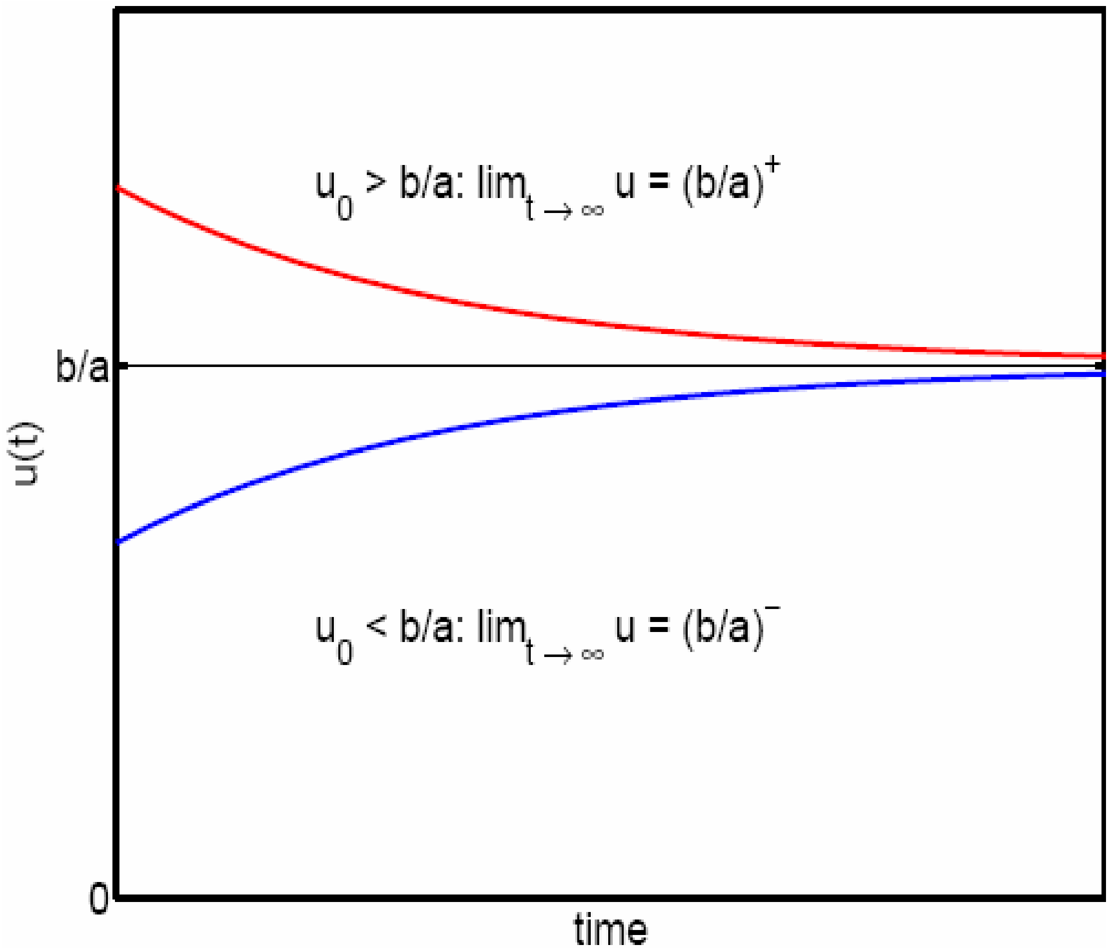
Behaviour of ODE's - linear

- u_s is the equilibrium point or steady state
- solution of the initial value problem is

$$u(t) = \left(u_0 - \frac{b}{a}\right)e^{-at} + \frac{b}{a}$$

- All solutions converge to $\frac{b}{a}$

Behaviour of ODE's - linear



Behaviour of ODE's - linear

- Observations:
  - Only one steady state
  - Initial conditions converge to the steady state (if stable)

Behaviour of ODE's – non-linear

- 2. Non-linear problem:

$$\frac{du}{dt} = u\left(\alpha - \beta u\right) = \alpha u - \beta u^2$$

- With $\alpha,\beta > 0$ and $u(0) = u_0$

Behaviour of ODE's – non-linear

- Steady states

$$\frac{du}{dt} = 0 \qquad u(\alpha - \beta u) = 0 \begin{cases} u_{s1} = 0 \\ u_{s2} = \frac{\alpha}{\beta} \end{cases}$$

Behaviour of ODE's – non-linear

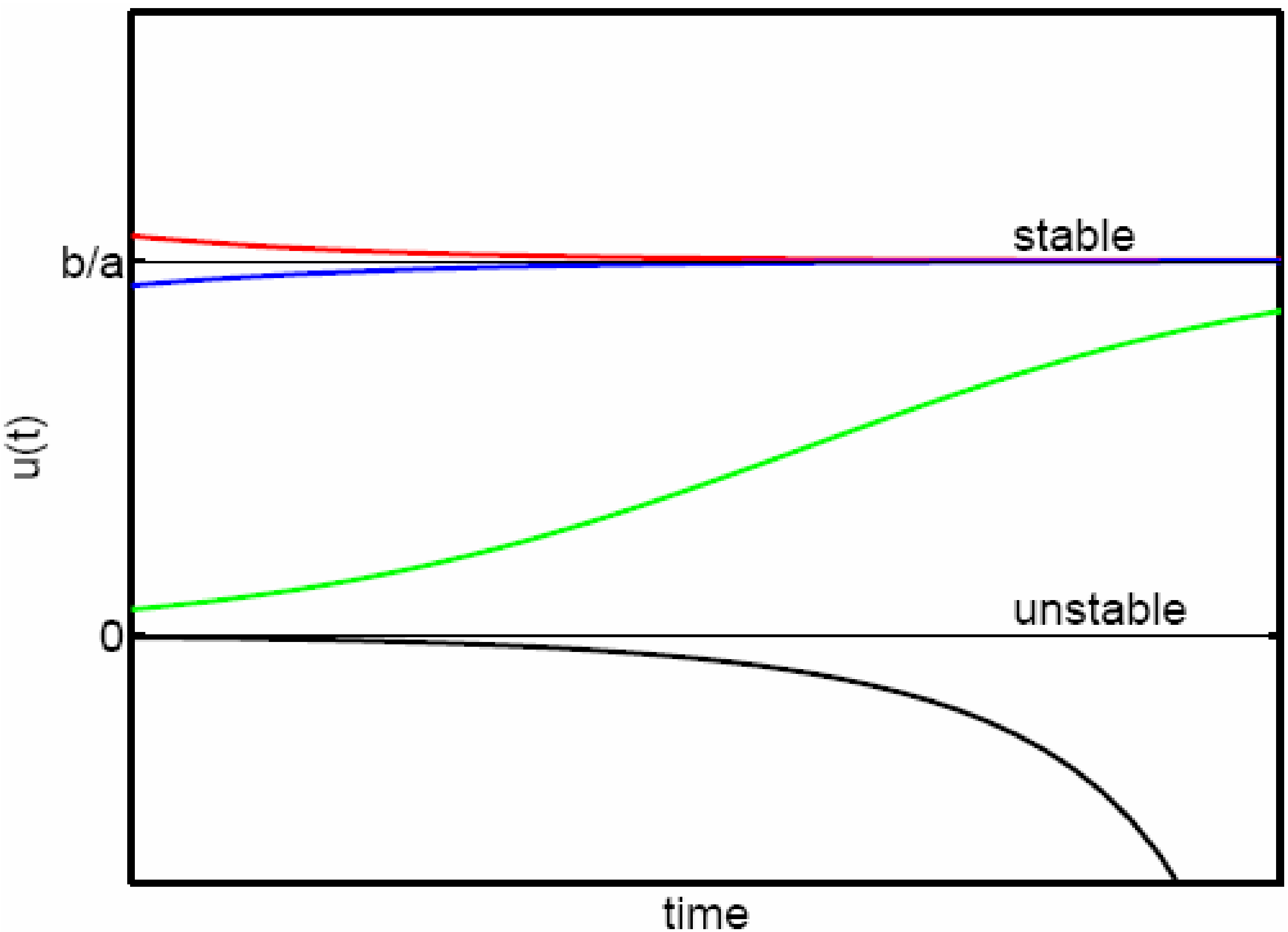
- Examine the sign of the derivative:

- 1. $\frac{du}{dt} > 0$ if $0 < u < \frac{\alpha}{\beta}$

- 2. $\frac{du}{dt} < 0$ if $u > \frac{\alpha}{\beta}$ or $u < 0$

- 3. $\frac{du}{dt} = 0$ if $u = \frac{\alpha}{\beta}$ or $u = 0$

Behaviour of ODE's – non-linear



Behaviour of ODE's – non-linear

- Observations:
  - Multiplicity of steady states
  - Steady states have different stabilities
  - Asymptotic behaviour of u(t) depends on u_0

Linearisation

- Linearisation of non-linear ODE's
- General ODE form

$$\frac{du}{dt} = f(u) \quad u(0) = u_0$$

- System has u_{Si} (i = 1,…,n) steady states where $f(u_{si}) = 0$

Linearisation

- System behaviour in the neighbourhood of the steady state

  → perturbation variable $x(t) = u(t) - u_s$
- Leading to $\dot{x} = \dot{u} = f(x + u_s)$
- x is a small divergence of the system from the steady state

Linearisation

- How does the system evolve in time if $x \neq 0$?
  - The system is stable when it converges to the steady state
  - The system is unstable when it diverges from the steady state

Linearisation – Taylor

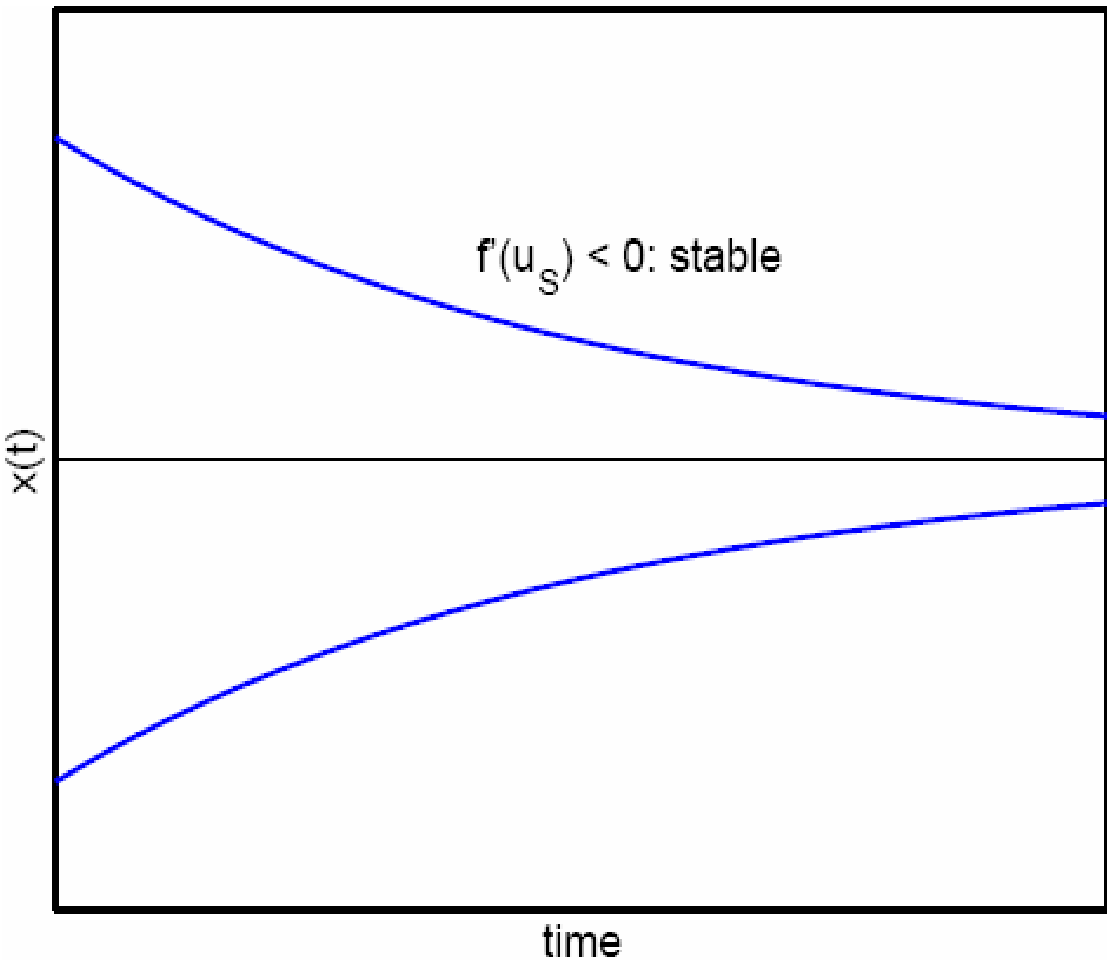
- Taylor expansion of $f(u)$ around u_s:
- immediate neighbourhood around the steady state

→ very small perturbation variable x << 1

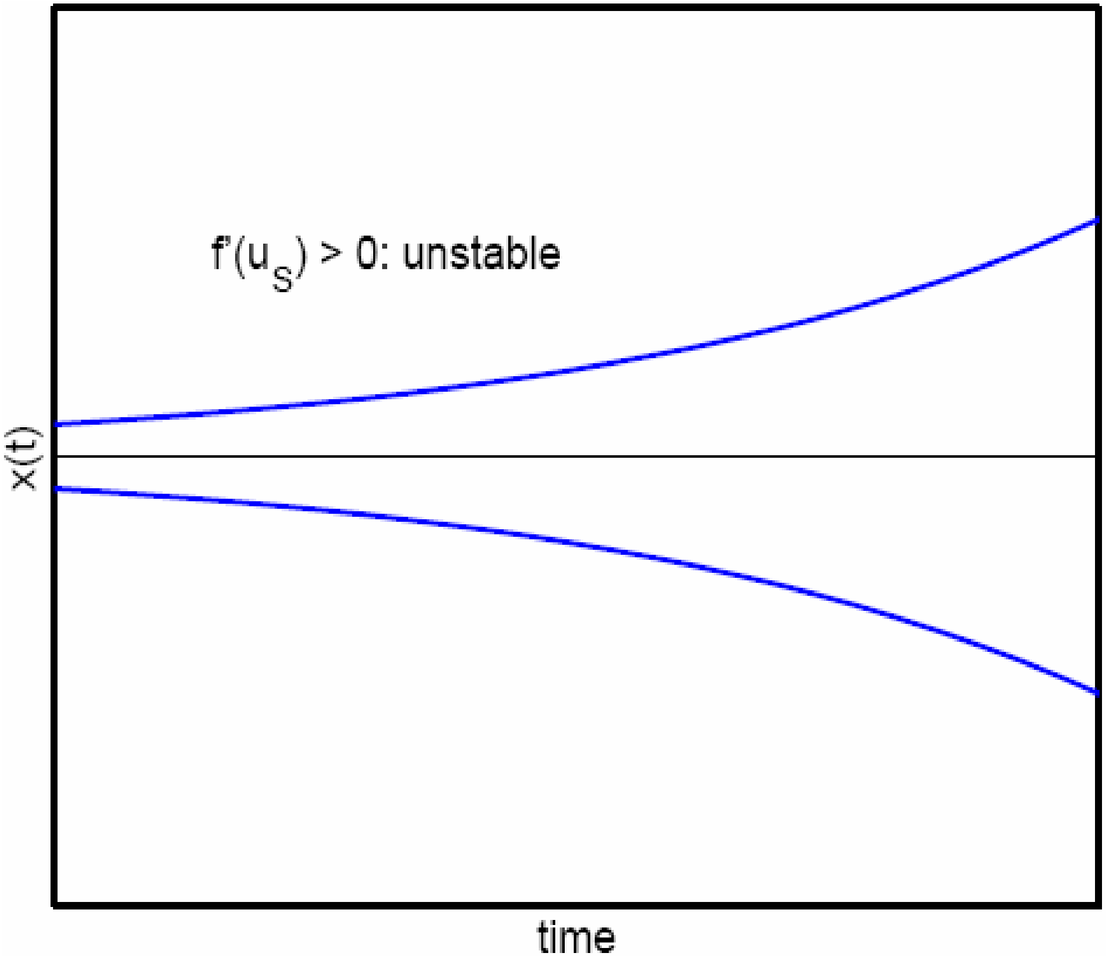
$$f(x+u_s) = f(u_s) + f'(u_s)(x+u_s-u_s) + \frac{1}{2} f''(u_s)x^2 + \ldots$$

$$\cong f'(u_s)x$$

Linearisation – Taylor



Linearisation – Taylor



Linearisation – Taylor

- Example:

$$f\left(u\right)=\alpha u-\beta u^2 \quad \Rightarrow f'\left(u\right)=\alpha-2\beta u$$

- (1) $u_s=u_{s1}=0 \qquad f'\left(0\right)=\alpha \qquad >0$

  $\Rightarrow$ unstable

- (2) $u_s=u_{s2}=\frac{\alpha}{\beta} \qquad f'\left(\frac{\alpha}{\beta}\right)=-\alpha \qquad <0$

  $\Rightarrow$ stable

- Local behaviour of $u(t)$