Modelling and mathematical methods in process engineering

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Revisions of linear algebra

Chapter 1

Definitions - matrix

- (m x n)-matrix $\underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \text{short: } \underline{\underline{A}} = \left\{ a_{ij} \right\}$$

- Square matrix $m = n$

Definitions - Vectors

- Column vector (m x 1)-matrix

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

- Row vector (1 x n)-matrix

$$\underline{y}^T = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

- Notation: $\underline{y}^T$

Definitions - Cofactors

- Cofactor A_{ij} of a matrix element a_{ij}

$$A_{ij} = (-1)^{i+j}\left[\text{complement of } a_{ij}\right]$$

- Complement of a_{ij} is determinant of order (n-1) (Laplace development)

$$\det(A) = \sum_{j=1}^{n} A_{ij}a_{ij} = \sum_{i=1}^{n} A_{ij}a_{ij} = D$$

Cofactors

- Example

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \Rightarrow A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

Definitions - Adjoints

- Adjoint of $\underline{\underline{A}}$

$$adj(\underline{\underline{A}}) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nm} \end{bmatrix}$$

Important properties

- 1^{st} property

$$adj\left(\underline{\underline{A}}\right)\underline{\underline{A}} = \begin{bmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \end{bmatrix} = \det\left(\underline{\underline{A}}\right)\underline{\underline{I}} = D\underline{\underline{I}}$$

- 2^{nd} property

$$\text{If } D \neq 0: \left(\frac{1}{D} adj\left(\underline{\underline{A}}\right)\right)\underline{\underline{A}} = \underline{\underline{I}} = \underline{\underline{A}}^{-1}\underline{\underline{A}} \Rightarrow \underline{\underline{A}}^{-1} = \frac{1}{D} adj\left(\underline{\underline{A}}\right)$$

- 3^{rd} property

If $D = 0$: $\underline{\underline{A}}$ is singular. $adj\left(\underline{\underline{A}}\right)$ is defined, but $\underline{\underline{A}}^{-1}$ isn't defined

Matrix operations

- Transpose (n x m)

$$\underline{\underline{A}}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} = \left\{ a_{ji} \right\}$$

- Product when conformable

$$\underset{(m \times n)}{\underline{\underline{A}}} \quad \underset{(n \times l)}{\underline{\underline{B}}} \quad = \quad \underset{(m \times l)}{\underline{\underline{C}}}$$

More matrix operations

- System of linear equations
  $\underline{\underline{A}}\,\underline{x} = \underline{b}$
- Vector product
  $$\underline{x}^T \underline{z} = \sum_{j=1}^{m} x_j z_j$$
- Norm of a vector
  $$\underline{x}^T \underline{x} = \sum_{j=1}^{m} x_j^2 = \left\|\underline{x}\right\|^2$$

More matrix operations

- Determinant larger 2 x 2

$$\left|\underline{\underline{A}}\right| = \sum_{j=1}^{n} a_{ij} A_{ij}$$

- Example

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\left|\underline{\underline{A}}\right| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24}$$

The eigenvalue problem

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$$

- Only square matrices (n x n)

$$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{x} = \underline{0}$$

$$\underline{\underline{I}}\,\underline{x} = \underline{x}$$

$$\Downarrow$$

$$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{\underline{I}}\,\underline{x} = \underline{0} \quad \Rightarrow \quad \left(\underline{\underline{A}} - \lambda\,\underline{\underline{I}}\right)\underline{x} = \underline{0}$$

Eigenvalue problem

- Where

$$\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

- $\underline{x} \neq 0$: non trivial → $\left| \underline{\underline{A}} - \lambda \underline{\underline{I}} \right| = 0$
- Vectors $\underline{x}$ and $\underline{\underline{A}}\underline{x}$ point in the same direction

Characteristic polynomial and equation

- Developing determinant leads to characteristic polynomial

$$\left|\underline{\underline{A}} - \lambda \underline{\underline{I}}\right| = P_n\left(\lambda\right) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0$$

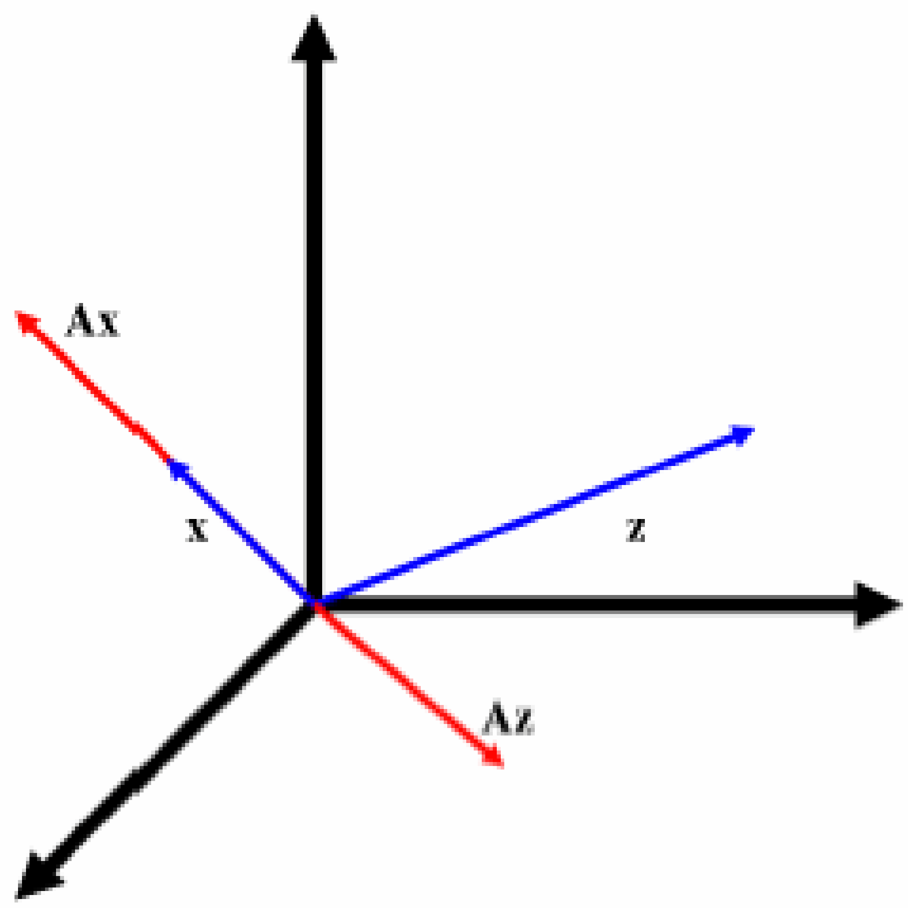
- With the characteristic equation $P_n\left(\lambda\right) = 0$
- n solutions λ_i (i = 1,...,n)

Eigenvalues and -vectors

- λ_i are called eigenvalues
- $\underline{x}_i$ are called eigenvectors
- → non trivial solutions for $\left(\underline{\underline{A}} - \lambda_i \underline{\underline{I}}\right)\underline{x}_i = \underline{0}$
- x_i is any non-zero column of $adj\left(\underline{\underline{A}} - \lambda_i \underline{\underline{I}}\right)$
- Example

$$adj\left(\underline{\underline{A}} - \lambda_i \underline{\underline{I}}\right) = \begin{bmatrix} a & 0 & \alpha a & \beta a \\ b & 0 & \alpha b & \beta b \\ \vdots & \vdots & \vdots & \vdots \\ z & 0 & \alpha z & \beta z \end{bmatrix} \quad \Rightarrow \underline{x}_i = \begin{bmatrix} a \\ b \\ \vdots \\ z \end{bmatrix}$$

Geometrical interpretation



“Eigenrow” problem

- Using eigenrow equation
- Transpose on both sides
- Cancel out to get solution

$$\underline{y}^T \underline{\underline{A}} = \eta \underline{y}^T \Leftrightarrow \left(\underline{y}^T \underline{\underline{A}}\right)^T = \eta \underline{y} \Leftrightarrow \underline{\underline{A}}^T \underline{y} = \eta \underline{y}$$

"Eigenrow" problem

- Characteristic equation $\left|\underline{\underline{A}}^T - \eta \underline{\underline{I}}\right| = 0$

$$\left|\underline{\underline{A}}^T - \eta \underline{\underline{I}}\right| = \left|\left(\underline{\underline{A}} - \eta \underline{\underline{I}}\right)^T\right| = \left|\underline{\underline{A}} - \eta \underline{\underline{I}}\right| \Rightarrow \eta_i = \lambda_i$$

$$adj\left(\underline{\underline{A}}^T - \eta_i \underline{\underline{I}}\right) = adj\left(\left(\underline{\underline{A}} - \eta_i \underline{\underline{I}}\right)^T\right)$$

$$= \left(adj\left(\underline{\underline{A}} - \eta_i \underline{\underline{I}}\right)\right)^T \text{ and } adj\left(\underline{\underline{A}}^T\right) = \left(adj\,\underline{\underline{A}}\right)^T$$

“Eigenrow” problem

$$adj\left(\underline{\underline{A}} - \lambda_i \underline{\underline{I}}\right) = \begin{bmatrix} & 0 & & 0 \\ & 0 & & 0 \\ & 0 & & 0 \\ & 0 & & 0 \end{bmatrix} \quad \boxed{\underline{y}_i^T}$$

$$\boxed{\underline{x}_i}$$

$$\underline{\underline{A}} \rightarrow \lambda_i \rightarrow \begin{cases} \left\{\underline{x}_i\right\} & \underline{\underline{A}}\,\underline{x}_i = \lambda_i \underline{x}_i \\ \left\{\underline{y}_i\right\} & \underline{y}_i^T \underline{\underline{A}} = \lambda_i \underline{y}_i^T \end{cases}$$

Example

- Calculate the eigenvalues of

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 4 & 10 & -1 \end{bmatrix}$$

Solution

- Characteristic equation

$$\left|\underline{\underline{A}} - \lambda \underline{\underline{I}}\right| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 5-\lambda & -1 \\ 4 & 10 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(5-\lambda)(-1-\lambda)+10\right] - 2\left[2(-1-\lambda)+4\right]$$

$$= (1-\lambda)\left(\lambda^2 - 4\lambda + 1\right) = 0$$

$$\Rightarrow \lambda_1 = 1 \qquad \lambda_{2,3} = 2 \pm \sqrt{3}$$

Example

- Now, find the eigenvector and eigenrow belonging to $\lambda_1 = 1$

Solution

$$\underline{\underline{A}} - \lambda_1 \underline{\underline{I}} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 4 & -1 \\ 4 & 10 & -2 \end{bmatrix}$$

$$adj \begin{bmatrix} 0 & 2 & 0 \\ 2 & 4 & -1 \\ 4 & 10 & -2 \end{bmatrix} = \begin{bmatrix} -8+10 & 4 & -2 \\ -4+4 & 0 & 0 \\ 20-16 & 8 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & 0 \\ 4 & 8 & -4 \end{bmatrix}$$

$$\Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \Rightarrow \underline{y}_1^T = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

Something to prove at home

$$\underline{\underline{A}}\,\underline{x}_1 = \lambda \underline{x}_1$$

Linear independence of vectors

- **Def**: vectors $\underline{x}_1$, $\underline{x}_2$, …, $\underline{x}_n$ are linearly independent if and only if

$$c_1\underline{x}_1 + c_2\underline{x}_2 + ... + c_n\underline{x}_n = \sum_{j=1}^{n} c_j \underline{x}_j = 0 \quad \Leftrightarrow \quad c_j = 0 \ \ \forall \ \ j = 1,...,n$$

- **Theorem**: If the eigenvalues λ_j are distinct, then the set of corresponding eigenvectors $\{\underline{x}_j\}$ forms a linearly independent set of vectors

Linear independence of vectors

- **Proof:**
  - o (i) show $\underline{x}_j \neq c\underline{x}_i$ for $i \neq j$ and $c \neq 0$
    assume: $\underline{x}_j = c\underline{x}_i$ for $i \neq j$ and $c \neq 0$

$$\square \left| \underline{\underline{A}}\,\underline{x} \right. \quad \Leftrightarrow \underline{\underline{A}}\,\underline{x}_j = c\,\underline{\underline{A}}\,\underline{x}_i \quad \Rightarrow \lambda_j \underline{x}_j = c\lambda_i \underline{x}_i \quad (1)$$

$$\square \left| \lambda_j \underline{x} \right. \quad \Leftrightarrow \lambda_j \underline{x}_j = c\lambda_j \underline{x}_i \quad (2)$$

equation (2) - (1):

$$0 = c\left(\lambda_j - \lambda_i\right)\underline{x}_i \overset{\lambda_j \neq \lambda_i}{\Rightarrow} \text{contradiction} \quad \Rightarrow \underline{x}_j \neq c\underline{x}_i$$

Linear independence of vectors

- (ii) all x_j (j = 1,…,n) are independent
  assume: only r < n are independent

$$\underline{x}_{r+1} = c_1 \underline{x}_1 + \ldots + c_r \underline{x}_r = \sum_{j=1}^{r} c_j \underline{x}_j \quad \square \left| \underline{\underline{A}}\,\underline{x} \right. \quad \square \left| \lambda_{r+1} \underline{x} \right.$$

$$\underline{\underline{A}}\underline{x}_{r+1} = \sum_{j=1}^{r} c_j \underline{\underline{A}}\underline{x}_j \quad \Rightarrow \lambda_{r+1} \underline{x}_{r+1} = \sum_{j=1}^{r} c_j \lambda_j \underline{x}_j \quad (1)$$

$$\lambda_{r+1} \underline{x}_{r+1} = \sum_{j=1}^{r} c_j \lambda_{r+1} \underline{x}_j \quad (2)$$

Linear independence of vectors

- Continuing (*ii*):
  equation (2) - (1):

$$0 = \sum_{j=1}^{r} c_j \left(\lambda_{r+1} - \lambda_j\right) \underline{x}_j \overset{\lambda_{r+1} \neq \lambda_j}{\Rightarrow} \text{contradiction}$$

  q.e.d

- → eigenvectors are linearly independent

Linear independence of vectors

- **Summary:**
- (i) $\underline{x}_i \neq c\underline{x}_i$
- (ii) it is not possible that r eigenvectors (r < n) are linearly dependent

Biorthogonality of eigenvectors

- If scalar product of 2 vectors is zero
  → vectors are orthogonal
- Each member of one set is orthogonal to each member of the other set, except for the one with which it has a common eigenvalue

Biorthogonality of eigenvectors

$$\underline{y}_j^T \underline{x}_i \begin{cases} = 0 & i \neq j \\ \neq 0 & i = j \end{cases}$$

- **Proof:**

$$\underline{\underline{A}}\underline{x}_i = \lambda_i \underline{x}_i \quad \Leftrightarrow \underline{y}_j^T \underline{\underline{A}}\underline{x}_i = \lambda_i \underline{y}_j^T \underline{x}_i$$

$$\underline{y}_j^T \underline{\underline{A}} = \lambda_j \underline{y}_j^T \quad \Leftrightarrow \underline{y}_j^T \underline{\underline{A}}\underline{x}_i = \lambda_j \underline{y}_j^T \underline{x}_i$$

$$\Rightarrow 0 = \left(\lambda_i - \lambda_j\right) \underline{y}_j^T \underline{x}_i$$

Biorthogonality – Symmetric matrices

- $a_{ij} = a_{ji}$
- $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad \underline{\underline{A}}^T = \underline{\underline{A}}$
- → eigenvectors = eigenrows

$\underline{x}_j = \underline{y}_j$

$\underline{x}_j^T \underline{x}_i \begin{cases} = 0 & i \neq j \\ \neq 0 & i = j \end{cases}$

Expansion of an arbitrary vector

- $\underline{\underline{A}}$ with distinct λ_j (j = 1,...,n) → {$\underline{x}_j$} , {$\underline{y}_j^T$}
- Take z of order n
  - (i) $\underline{z} = \alpha_1 \underline{x}_1 + \ldots + \alpha_n \underline{x}_n = \sum_{j=1}^{n} \alpha_j \underline{x}_j$

Proof:

$$\underline{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \underline{\underline{X}} = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \end{bmatrix} \quad \underline{z} = \underline{\underline{X}}\,\underline{a}$$

Expansion of an arbitrary vector

- $\left|\underline{\underline{X}}\right| \neq 0$ → eigenvectors are all linearly independent → $\underline{a}$ exists and is unique

  - $(ii)\ \alpha_j = \dfrac{\underline{y}_j^T \underline{z}}{\underline{y}_j^T \underline{x}_j}$

Proof: $\underline{y}_i^T \underline{z} = \sum_{j=1}^{n} \alpha_j \underline{y}_i^T \underline{x}_j \overset{\text{biorthogonality}}{=} \alpha_i \underline{y}_i^T \underline{x}_i$

$$\underline{z} = \sum_{j=1}^{n} \frac{\underline{y}_j^T \underline{z}}{\underline{y}_j^T \underline{x}_j} \underline{x}_j$$

Properties of char. polynomial

- Square matrix of order *n* $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$
- Characteristic polynomial

  $$\left| \underline{\underline{A}} - \lambda \underline{\underline{I}} \right| = P_n(\lambda) = 0$$
- Polynomial of order n

  $$P_n(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0$$
- where

  $$\alpha_0 = P_n(\lambda = 0) = \left| \underline{\underline{A}} - 0\underline{\underline{I}} \right| = \left| \underline{\underline{A}} \right|$$

Properties of char. polynomial

- correlations

$$\left|\underline{\underline{A}}-\lambda \underline{\underline{I}}\right| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{24} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) - a_{12}P_{n-2}(\lambda) + \ldots =$$

$$= (-1)^n \lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\ldots+a_{nn})\lambda^{n-1} + R_{n-2}(\lambda) =$$

$$= (-1)^n \lambda^n + (-1)^{n-1}\sum_{j=1}^{n} a_{jj}\lambda^{n-1} + R_{n-2}(\lambda) =$$

$$= (-1)^n \lambda^n + (-1)^{n-1} Tr(\underline{\underline{A}})\lambda^{n-1} + R_{n-2}(\lambda)$$

Properties of char. polynomial

- Coefficient comparison

$$\alpha_n = (-1)^n \qquad \alpha_{n-1} = (-1)^{n-1} Tr\left(\underline{\underline{A}}\right)$$

- Characteristic polynomial is also:

$$P_n(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) = \prod_{j=1}^{n} (\lambda_j - \lambda)$$

- Which implies

$$\alpha_0 = P_n(0) = \boxed{\prod_{j=1}^{n} \lambda_j = \left|\underline{\underline{A}}\right|} \Rightarrow \left|\underline{\underline{A}}\right| \neq 0 \Leftrightarrow \text{ all } \lambda_j \neq 0$$

$$\alpha_{n-1} = (-1)^{n-1}(\lambda_1 + \lambda_2 + \ldots + \lambda_n) \Rightarrow \boxed{Tr\left(\underline{\underline{A}}\right) = \sum_{j=1}^{n} \lambda_j}$$

Properties of char. polynomial

- Example: n = 2

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\alpha_2 = (-1)^2 = 1$$

$$\alpha_1 = (-1)^{n-1} Tr\left(\underline{\underline{A}}\right) = -\left(a_{11} + a_{22}\right)$$

$$\alpha_0 = \left|\underline{\underline{A}}\right| = a_{11}a_{22} - a_{12}a_{21}$$

Systems of linear equations

- Assumptions:

$\left| \underline{\underline{A}} \right| \neq 0$

λ_j are distinct $\quad \lambda_j \neq 0$

$\left\{ \underline{x}_j \right\}$ set of eigenvectors

$\left\{ \underline{y}_j^T \right\}$ set of eigenrows

vector b is known

Systems of linear equations

- Linear system $\underline{\underline{A}}\,\underline{z} = \underline{b} \;\Rightarrow\; \underline{z} = \underline{\underline{A}}^{-1}\underline{b}$

$$\underline{z} = \sum_{j=1}^{n} \alpha_j \underline{x}_j \qquad \sum_{j=1}^{n} \alpha_j \underline{\underline{A}}\,\underline{x}_j = \underline{b} = \sum_{j=1}^{n} \alpha_j \lambda_j \underline{x}_j$$

$$\alpha_j \lambda_j = \frac{\underline{y}_j^T \underline{b}}{\underline{y}_j^T \underline{x}_j} \qquad \Rightarrow \alpha_j = \frac{\underline{y}_j^T \underline{b}}{\underline{y}_j^T \underline{x}_j} \frac{1}{\lambda_j}$$

$$\underline{z} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \frac{\underline{y}_j^T \underline{b}}{\underline{y}_j^T \underline{x}_j} \underline{x}_j$$

Systems of ordinary differential equations (ODE's)

Chapter 2

Introduction

- Study of ODE’s
- Interested in
  - qualitative behaviour
  - Equilibrium points
  - Stability
- Nonlinear systems using linearisation

Systems of ODE's

- Consider:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1 \qquad x_1(0) = x_{10}$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_2 \qquad x_2(0) = x_{20}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_n \qquad x_n(0) = x_{n0}$$

Systems of ODE's

- a_{ij} and b_i are constants
- $\underline{x}_j(0)$ are the initial conditions where j = 1,…,n
- Find how unknowns $\underline{x}$ vary with independent variable t

  → find a solution $\underline{x}(t)$ of the vector-form eq.

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{b} \qquad \underline{x}(0) = \underline{x}_0$$

Systems of ODE's

- Assumptions: λ_j distinct, $\left|\underline{\underline{A}}\right| \neq 0$

- Notation:
  - $\underline{z}_j$ eigenvectors
  - $\underline{w}_j$ eigenrows
  - derivative $\frac{d\underline{x}}{dt} = \dot{\underline{x}}$

Systems of ODE's – 1.

- Solution of homogeneous equation $\dot{\underline{x}}_h = \underline{\underline{A}}\,\underline{x}_h$
- Assume: $\underline{x}_h = \underline{z}e^{\lambda t}$ $\underline{z}, \lambda$ unknowns
- → $\underline{\underline{A}}\,\underline{x_h} = \underline{\underline{A}}\,\underline{z}e^{\lambda t}$

$$\dot{\underline{x}}_h = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{z}e^{\lambda t}\right) = \lambda \underline{z}e^{\lambda t}$$

$$\left.\right\} \underline{\underline{A}}\,\underline{z}\boxed{e^{\lambda t}} = \lambda \underline{z}\boxed{e^{\lambda t}}$$

$$e^{\lambda t} \neq 0 \;\Rightarrow\; \underline{\underline{A}}\,\underline{z} = \lambda \underline{z}$$

- $\underline{z} \neq 0$ → λ is an eigenvalue of $\underline{\underline{A}}$
- $\underline{\underline{A}}$ of order n → n eigenvalues and eigenvectors

Systems of ODE's – 1.

- General solution is linear combination of eigenvectors of homogeneous problem

$$\underline{x}_h = \sum_{j=1}^{n} c_j \underline{z}_j e^{\lambda_j t}$$

- Where c_j are arbitrary constants

Systems of ODE's – 2.

- Particular solution obtained by assuming

$$\frac{d\underline{x}}{dt} = 0 = \underline{\underline{A}}\,\underline{x}_p + \underline{b}$$

$$\Rightarrow \underline{x}_p = -\underline{\underline{A}}^{-1}\underline{b}$$

Systems of ODE's – 3.

- General solution of the non-homogeneous equation is a superposition of the particular and homogeneous solution

$$\underline{x}(t) = \sum_{j=1}^{n} c_j \underline{z}_j e^{\lambda_j t} - \underline{\underline{A}}^{-1} \underline{b}$$

Systems of ODE's – 4.

- Initial value problem $\underline{x}(t=0)\overset{!}{=}\underline{x}_0$

$$\underline{x}(0)=\sum_{j=1}^{n}c_j\underline{z}_j-\underline{\underline{A}}^{-1}\underline{b}\overset{!}{=}\underline{x}_0$$

$$\Rightarrow\underbrace{\sum_{j=1}^{n}c_j\underline{z}_j}_{\text{expansion of a vector}}=\underbrace{\underline{x}_0+\underline{\underline{A}}^{-1}\underline{b}}_{\text{vector}}\Rightarrow c_j=\frac{\underline{w}_j^T\left(\underline{x}_0+\underline{\underline{A}}^{-1}\underline{b}\right)}{\underline{w}_j^T\underline{z}_j}$$

Systems of ODE's – 4.

- Solution of the initial value problem

$$\underline{x} = \sum_{j=1}^{n} \frac{\underline{w}_j^T \left(\underline{x}_0 + \underline{\underline{A}}^{-1} \underline{b} \right)}{\underline{w}_j^T \underline{z}_j} \underline{z}_j e^{\lambda_j t} - \underline{\underline{A}}^{-1} \underline{b}$$

Systems of ODE's – Example

- Example

$$\frac{dx_1}{dt} = +2x_1 + x_2 - 3x_3 + 4 \qquad x_1(0) = -\tfrac{1}{2}$$

$$\frac{dx_2}{dt} = -2x_2 + 3x_3 - 2 \qquad x_2(0) = 0$$

$$\frac{dx_3}{dt} = 2x_2 - 1x_3 + 2 \qquad x_3(0) = 2$$

$$\Leftrightarrow \quad \underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b} \qquad \underline{x}(0) = \underline{x}_0$$

Systems of ODE's – Example

- with

$$\underline{\dot{x}} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} \quad \underline{\underline{A}} = \begin{bmatrix} 2 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & -1 \end{bmatrix} \quad \underline{x}_0 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 2 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

Systems of ODE's – Example

- Characteristic equation:

$$(2-\lambda)\begin{vmatrix} -2-\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix} = (2-\lambda)\left\{(\lambda+2)(\lambda+1)-6\right\} =$$

$$=(2-\lambda)\left(\lambda^2+3\lambda-4\right)$$

$$=(2-\lambda)(1-\lambda)(4+\lambda)=0$$

- Eigenvalues:

λ_1 = 2

λ_2 = 1

λ_3 = -4

Systems of ODE's – Example

- Corresponding eigenvectors
- Corresponding eigenrows

$$\underline{z}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{z}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \underline{z}_3 = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$$

$$\underline{w}_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \quad \underline{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \quad \underline{w}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Systems of ODE's – Example

- The homogeneous solution:

$$\underline{x}_h(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} e^{-4t}$$

- with

$$\underline{\underline{A}}^{-1} = \frac{1}{8} \begin{bmatrix} 4 & 5 & 3 \\ 0 & 2 & 6 \\ 0 & 4 & 4 \end{bmatrix} \qquad \underline{\underline{A}}^{-1}\underline{b} = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \qquad \left(\underline{x}_0 + \underline{\underline{A}}^{-1}\underline{b}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Systems of ODE's – Example

- The particular solution:

$$\underline{x}_p = -\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \quad \rightarrow c_1 = \frac{\underline{w}_1^T \left(\underline{x}_0 + \underline{\underline{A}}^{-1} \underline{b} \right)}{\underline{w}_1^T \underline{z}_1} = -\frac{5}{2}$$

$$c_2 = \frac{8}{5}$$

$$c_3 = \frac{1}{10}$$

Systems of ODE's – Example

- Whole solution of initial value problem

$$\underline{x}(t) = -\frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \frac{8}{5}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{t} + \frac{1}{10}\begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} e^{-4t} - \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

Reduction of non-homogeneous problems

- Non-homogeneous problem:

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b}$$

- with $\underline{\underline{A}} \quad \rightarrow \quad \lambda_j,\ \left\{\underline{z}_j\right\},\ \left\{\underline{w}_j\right\}$

$$\left|\underline{\underline{A}}\right| \neq 0 \qquad \underline{x}(0) = \underline{x}_0$$

Reduction of non-homogeneous problems

- Consider:

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{I}}\,\underline{b} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{A}}\,\underline{\underline{A}}^{-1}\underline{b} = \underline{\underline{A}}\left(\underline{x} + \underline{\underline{A}}^{-1}\underline{b}\right)$$

- Substitution: $\underline{y} = \underline{x} + \underline{\underline{A}}^{-1}\underline{b}$

$$\left.\begin{aligned} \underline{y}(0) &= \underline{x}(0) + \underline{\underline{A}}^{-1}\underline{b} \\ \dot{\underline{y}} &= \dot{\underline{x}} + \frac{d}{dt}\left(\underline{\underline{A}}^{-1}\underline{b}\right) = \dot{\underline{x}} \end{aligned}\right\} \quad \dot{\underline{y}} = \underline{\underline{A}}\,\underline{y}$$

- → homogeneous: $\underline{y}(t) = \sum_{j=1}^{n} \frac{\underline{w}_j^T \underline{y}_0}{\underline{w}_j^T \underline{z}_j} \underline{z}_j e^{\lambda_j t}$