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Mobile Robot Kinematics
Autonomous Mobile Robots

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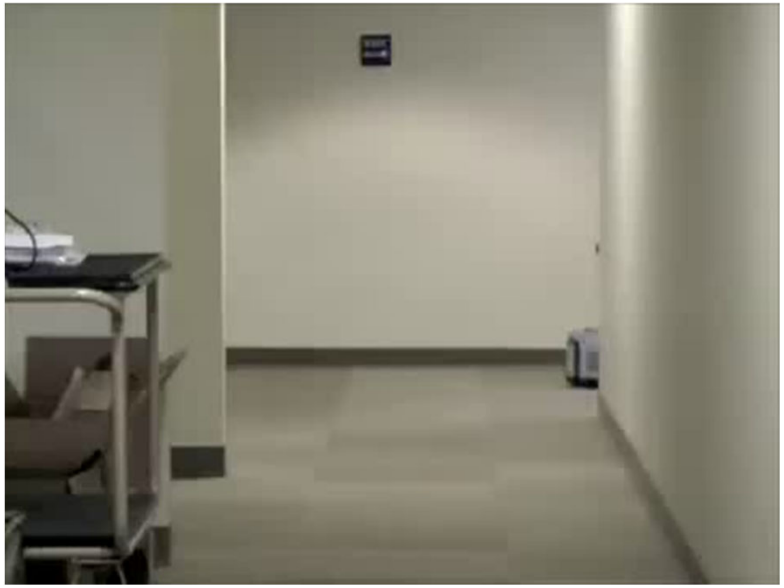
Mobile Robot Kinematics: Overview

- Manipulator arms versus mobile robots
  - Robot arms are fixed to the ground and usually comprised of a single chain of actuated links
  - The motion of mobile robots is defined through rolling and sliding constraints taking effect at the wheel-ground contact points

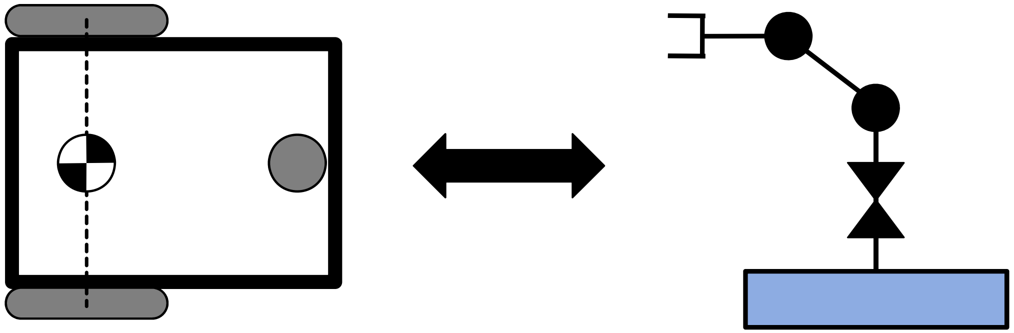
Ride an ABB, https://www.youtube.com/watch?v=bxbjZiKAZP4



C Willow Garage

Mobile Robot Kinematics: Overview

- Manipulator arms versus mobile robots
  - Both are concerned with forward and inverse kinematics
  - However, for mobile robots, encoder values don‘t map to unique robot poses
  - And *mobile robots can move unbound* with respect to their environment
    - There is no direct (=instantaneous) way to measure the robot’s position
    - Position must be integrated over time, depends on path taken
    - Leads to inaccuracies of the position (motion) estimate
  - Understanding mobile robot motion starts with understanding wheel constraints placed on the robot’s mobility

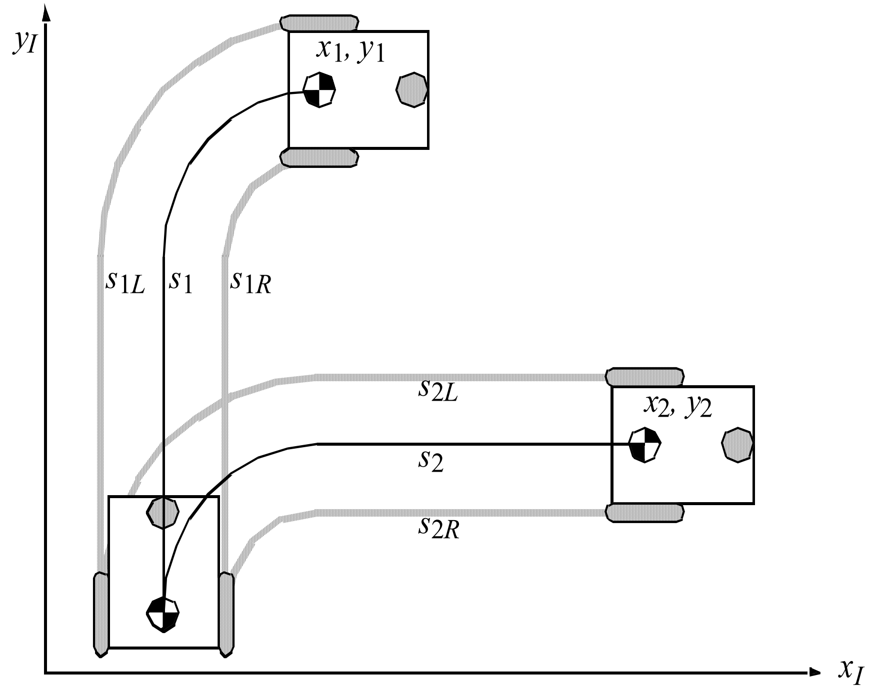


Non-Holonomic Systems

- Non-holonomic systems
  - differential equations are not integrable to the final position.
  - the measure of the traveled distance of each wheel is not sufficient to calculate the final position of the robot. One has also to know how this movement was executed as a function of time.
  - This is in strong contrast to actuator arms

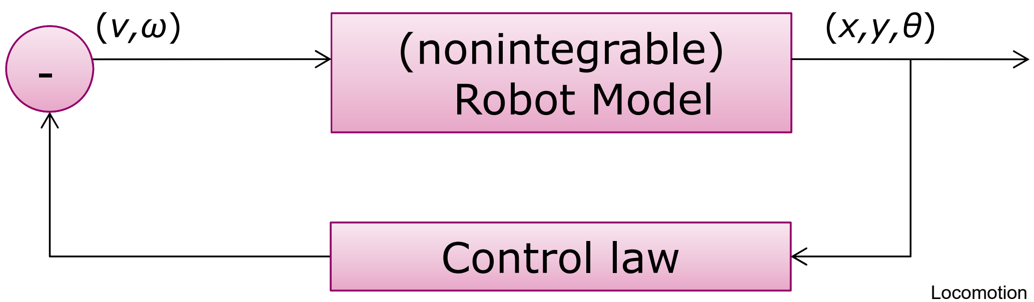
$$s_1 = s_2 , s_{1R} = s_{2R} , s_{1L} = s_{2L}$$

$$x_1 \neq x_2 , y_1 \neq y_2$$



Forward and Inverse Kinematics

- Forward kinematics:
  - Transformation from joint to physical space
- Inverse kinematics
  - Transformation from physical to joint space
  - Required for motion control
- Due to non-holonomic constraints in mobile robotics, we deal with **differential** (inverse) kinematics
  - Transformation between velocities instead of positions
  - Such a differential kinematic model of a robot has the following form:



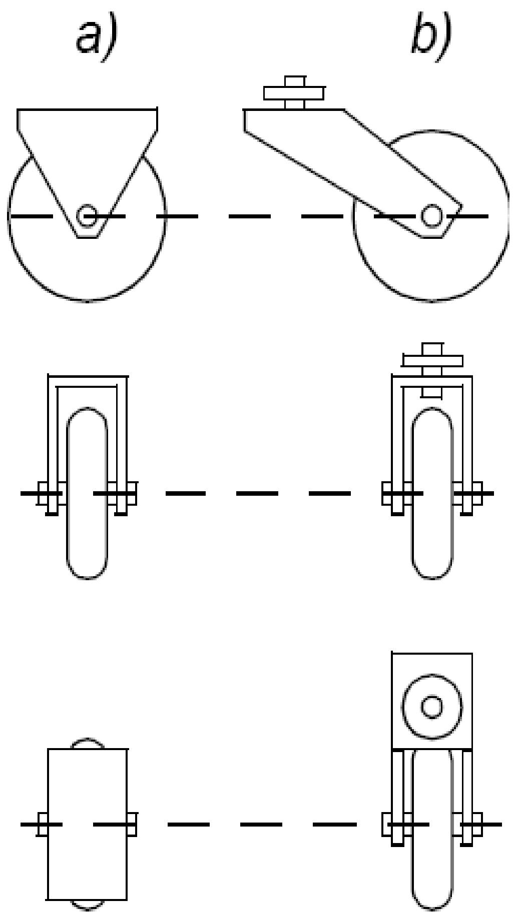
Mobile Robots with Wheels

- Wheels are the most appropriate solution for many applications
- Three wheels are sufficient to guarantee stability
- With more than three wheels an appropriate suspension is required
- Selection of wheels depends on the application

The Four Basic Wheels Types

- a) Standard wheel: Two degrees of freedom; rotation around the (motorized) wheel axle and the contact point

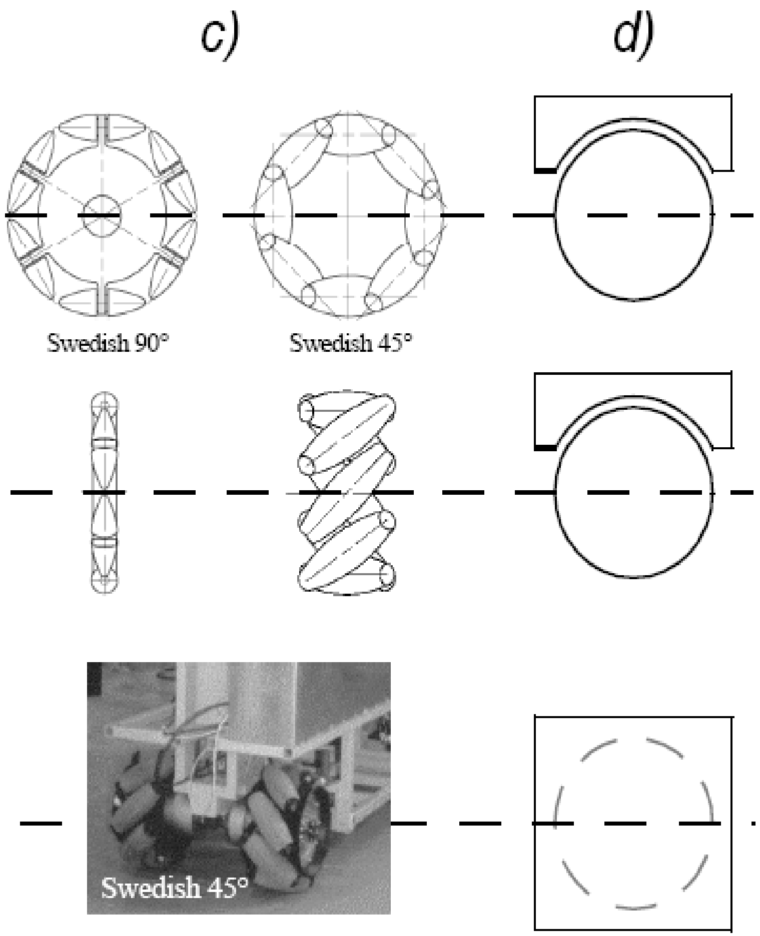
- b) Castor wheel: Three degrees of freedom; rotation around the wheel axle, the contact point and the castor axle



The Four Basic Wheels Types

- c) Swedish wheel: Three degrees of freedom; rotation around the (motorized) wheel axle, around the rollers and around the contact point

- d) Ball or spherical wheel: Suspension technically not solved

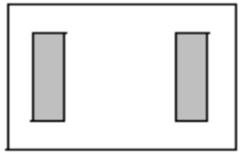


Characteristics of Wheeled Robots and Vehicles

- Stability of a vehicle is be guaranteed with 3 wheels
  - If center of gravity is within the triangle which is formed by the ground contact point of the wheels.
- Stability is improved by 4 and more wheel
  - however, this arrangements are hyper static and require a flexible suspension system.
- Bigger wheels allow to overcome higher obstacles
  - but they require higher torque or reductions in the gear box.
- Most arrangements are non-holonomic (see chapter 3)
  - require high control effort
- Combining actuation and steering on one wheel makes the design complex and adds additional errors for odometry.

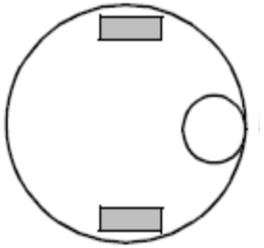
Different Arrangements of Wheels I

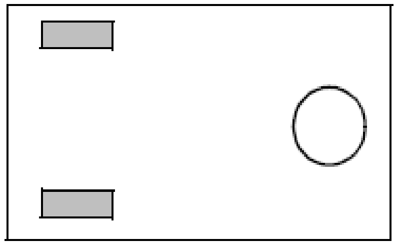
- Two wheels

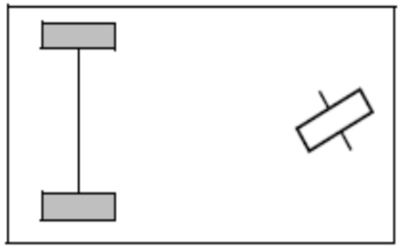


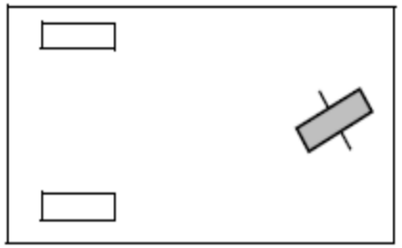
COG below axle

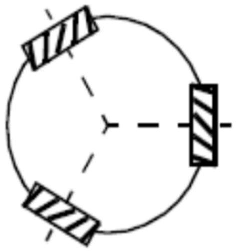
- Three wheels



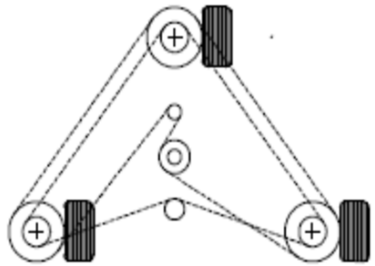








Omnidirectional Drive



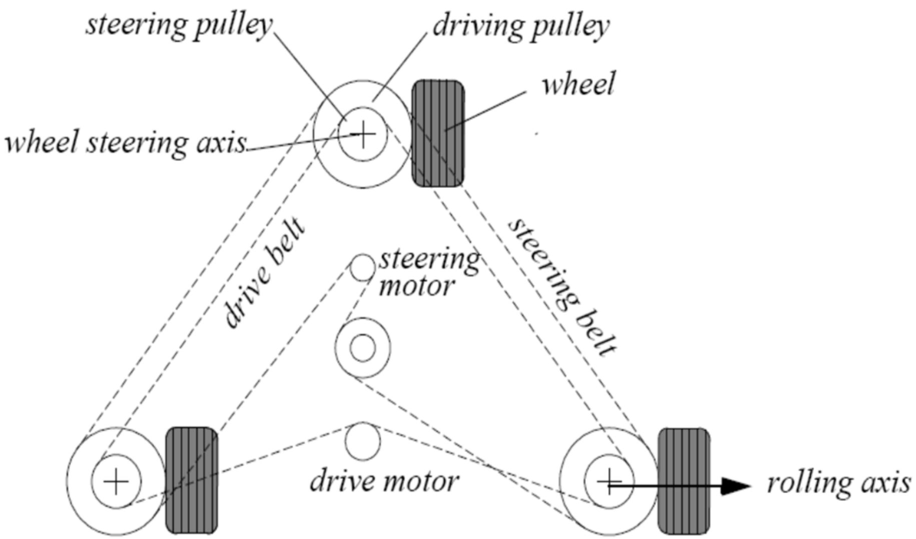
Synchro Drive

Synchro Drive

C J. Borenstein

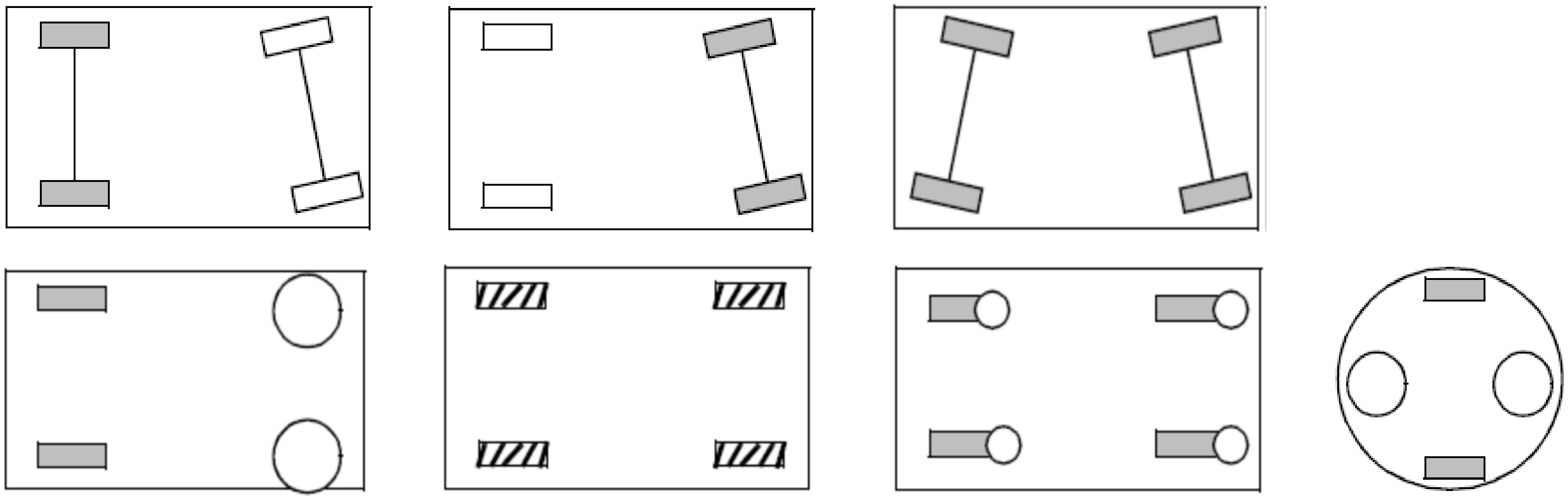
- All wheels are actuated synchronously by one motor
  - defines the speed of the vehicle
- All wheels steered synchronously by a second motor
  - sets the heading of the vehicle

- The orientation in space of the robot frame will always remain the same
  - It is therefore not possible to control the orientation of the robot frame.

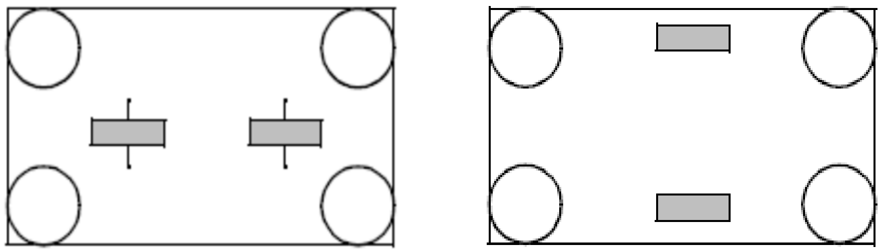


Different Arrangements of Wheels II

- Four wheels

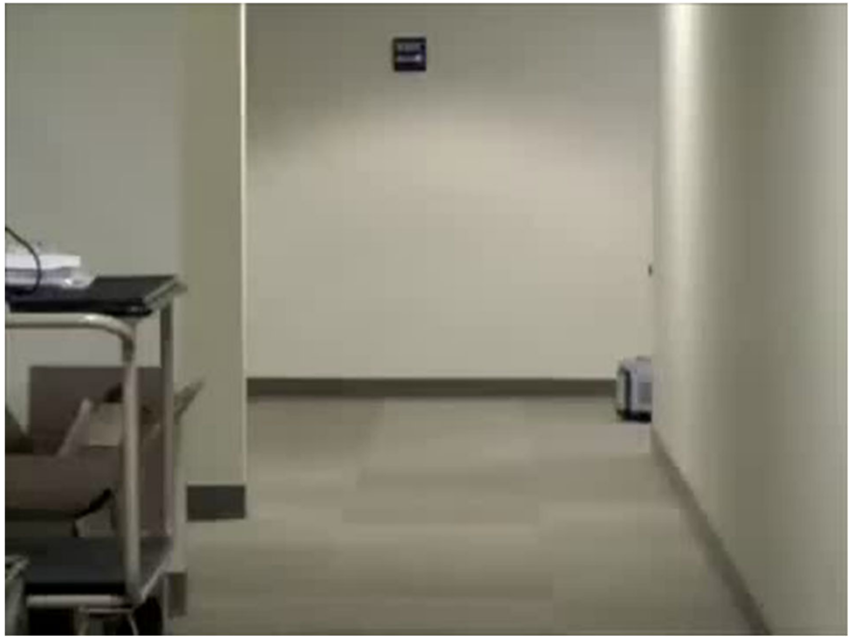


- Six wheels



Case Study: Willow Garage's PR2

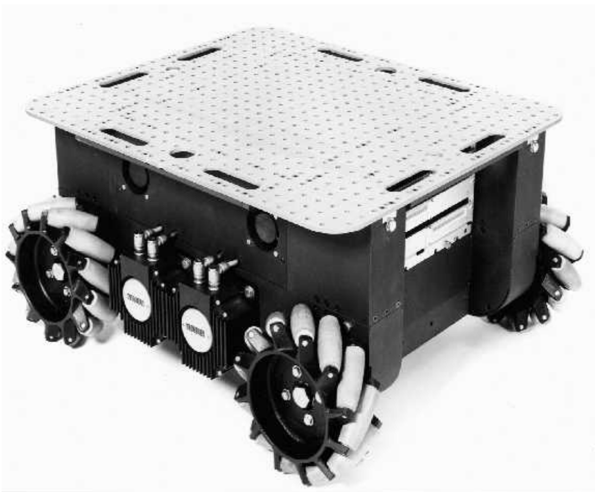
- Four powered and actively steered wheels
- Results in omni-drive-like behavior
- Results in simplified high-level planning (see chapter 6)

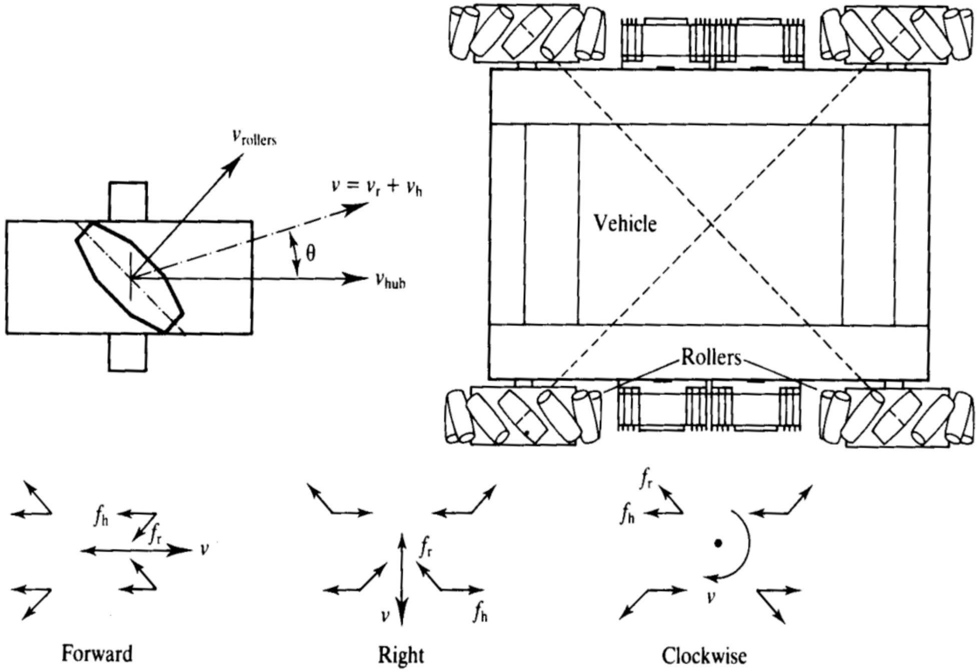


C Willow Garage

CMU Uranus: Omnidirectional Drive with 4 Wheels

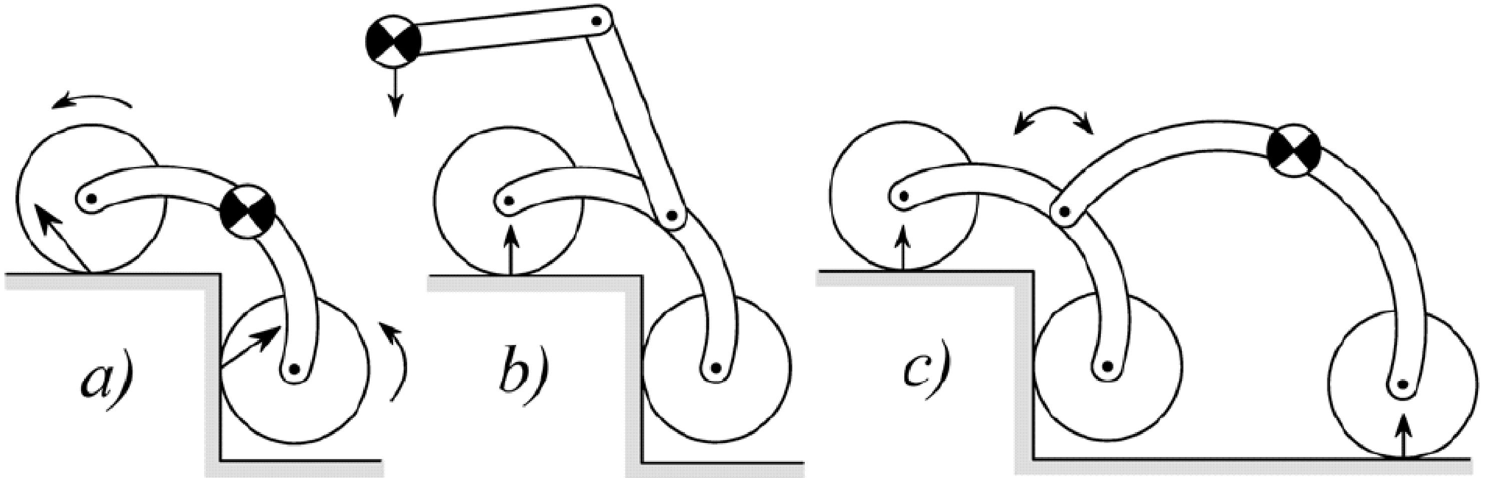
- Movement in the plane has 3 DOF
  - thus only three wheels can be independently controlled
  - It might be better to arrange three Swedish wheels in a triangle





Wheeled Rovers: Concepts for Object Climbing

- Purely friction based



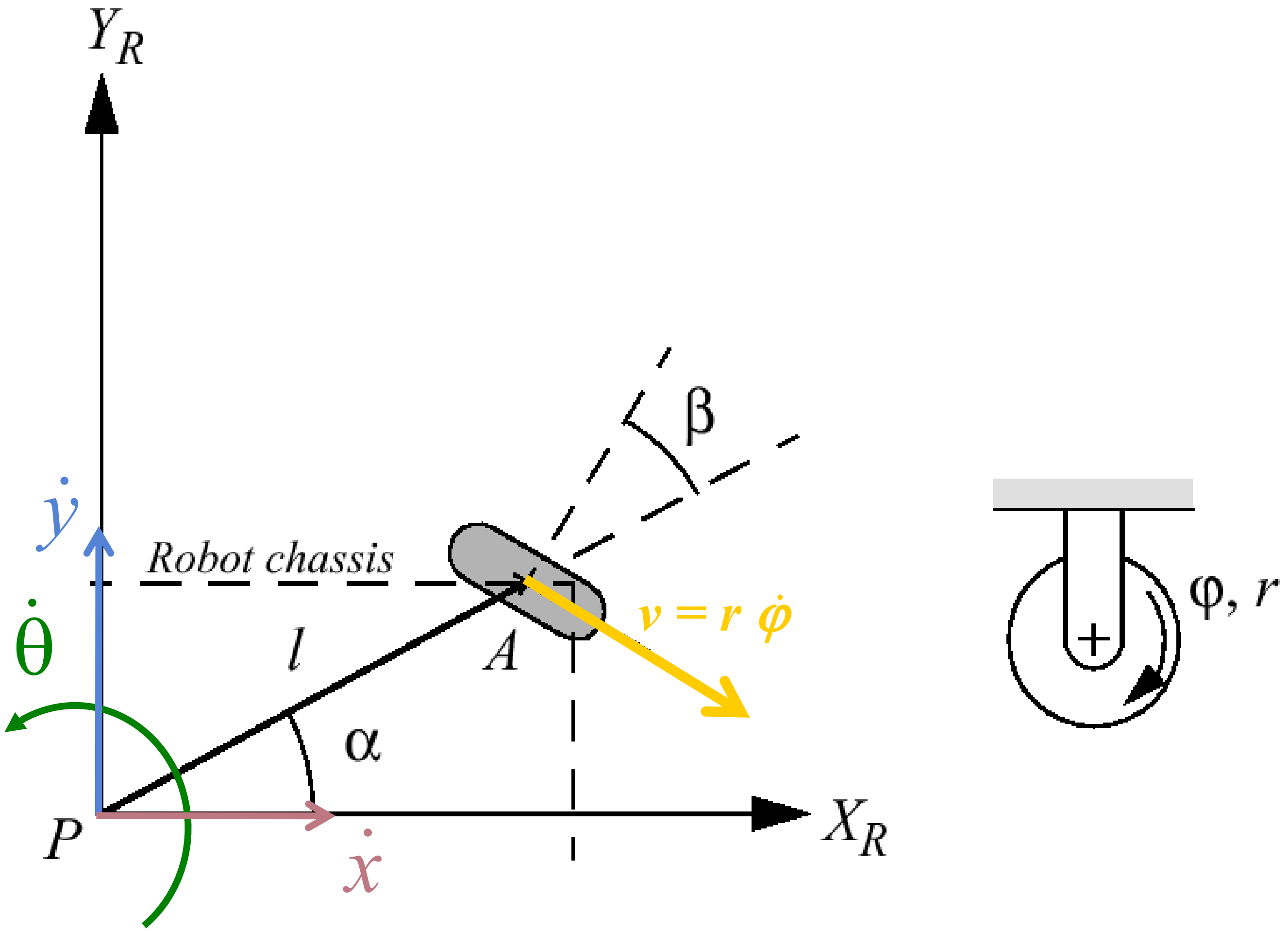
Change of center of gravity (CoG)

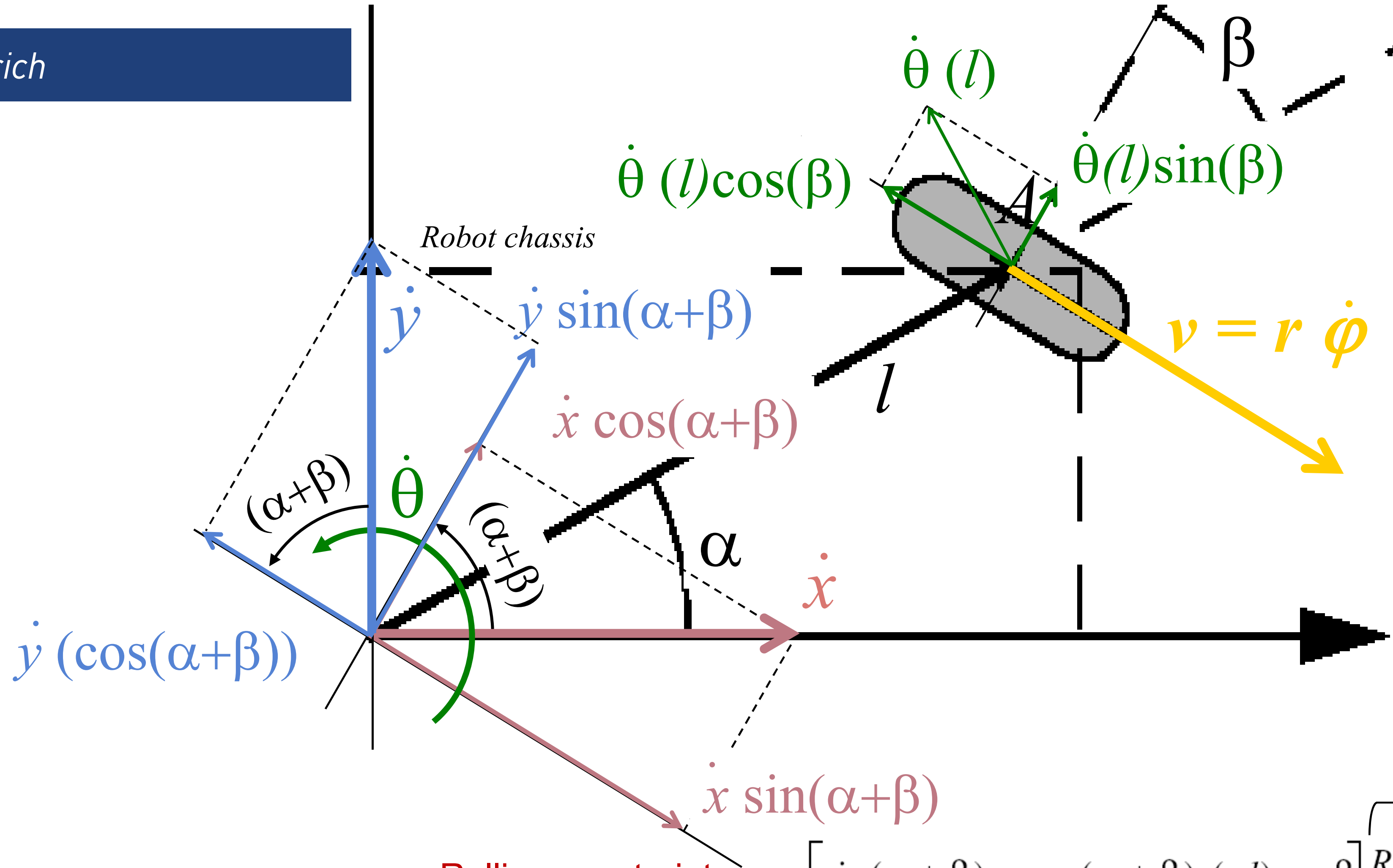
Adapted suspension mechanism with passive or active joints

Climbing with Legs: Shrimp (ASL EPFL/ETH)

- Passive locomotion concept
- 6 wheels
  - two boogies on each side
  - fixed wheel in the rear
  - front wheel with spring suspension
- Dimensions
  - length: 60 cm
  - height: 20 cm
- Characteristics
  - highly stable in rough terrain
  - overcomes obstacles up to 2 times its wheel diameter

Kinematic Constraints: Fixed Standard Wheel





$$\dot\xi_R = [\dot x \quad \dot y \quad \dot\theta]$$

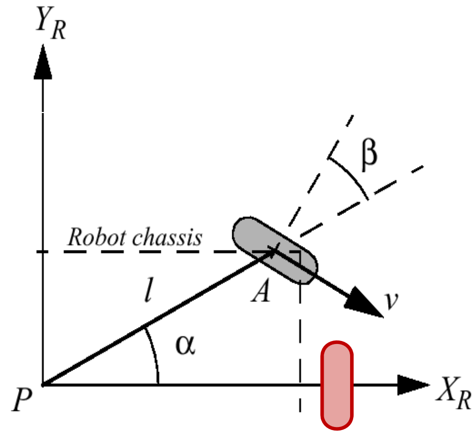
Rolling constraint → $\left[\sin(\alpha+\beta)\ \ -\cos(\alpha+\beta)\ \ (-l)\cos\beta\right] R(\theta)\dot\xi_I - r\dot\varphi = 0$

Sliding constraint → $\left[\cos(\alpha+\beta)\ \ \sin(\alpha+\beta)\ \ l\sin\beta\right] R(\theta)\dot\xi_I = 0$

Example

$$\left[\sin(\alpha+\beta)\ \ -\cos(\alpha+\beta)\ \ (-l)\cos\beta\right] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta)\ \ \sin(\alpha+\beta)\ \ l\sin\beta\right] R(\theta)\dot{\xi}_I = 0$$



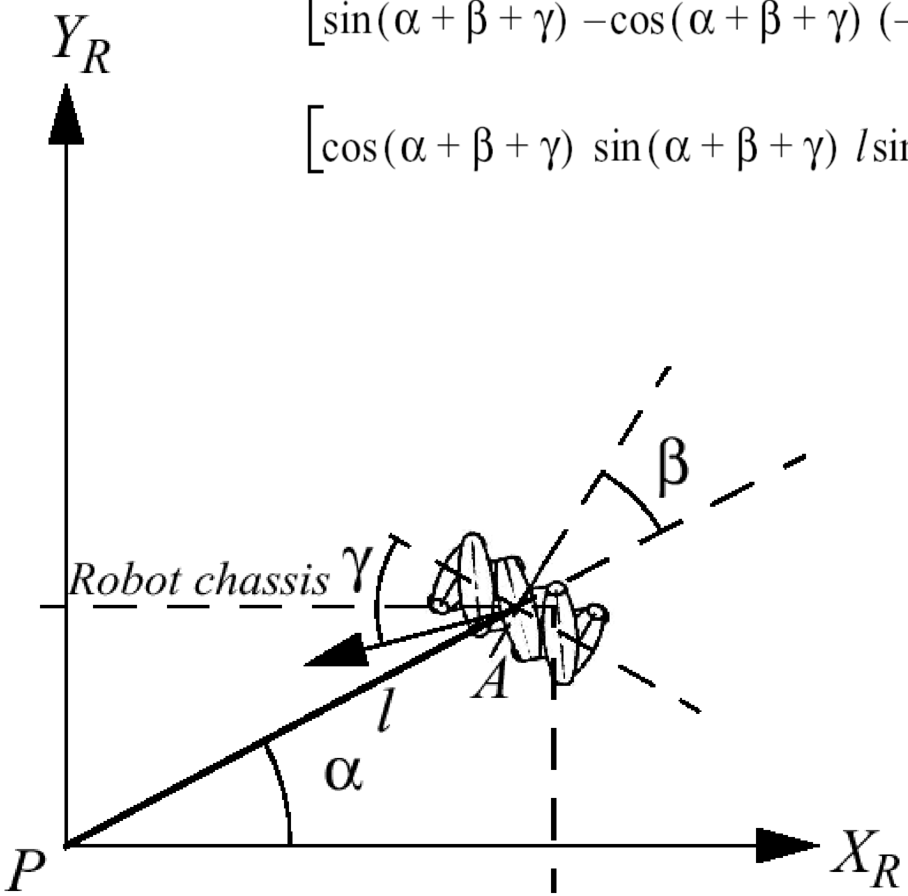
- Suppose that the wheel *A* is in position such that α **= 0 and** β **= 0**
- This would place the contact point of the wheel on X_I with the plane of the wheel oriented parallel to Y_I. If $\theta = 0$, then the **sliding constraint** reduces to:

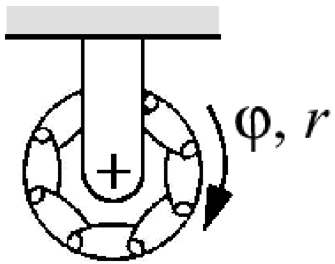
$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{\theta}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{\theta}\end{bmatrix} = 0$$

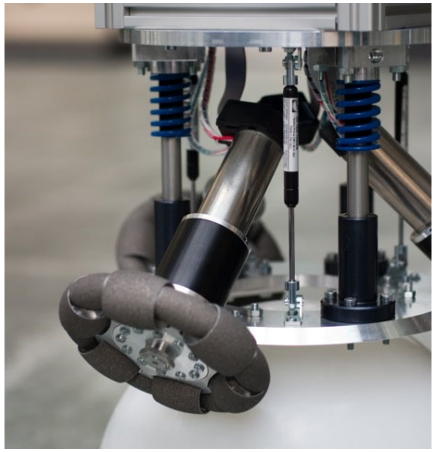
Kinematic Constraints:

$$\left[\sin(\alpha+\beta+\gamma)\ -\cos(\alpha+\beta+\gamma)\ (-l)\cos(\beta+\gamma)\right]R(\theta)\dot{\xi}_I - r\dot{\varphi}\cos\gamma = 0$$

$$\left[\cos(\alpha+\beta+\gamma)\ \sin(\alpha+\beta+\gamma)\ l\sin(\beta+\gamma)\right]R(\theta)\dot{\xi}_I - r\dot{\varphi}\sin\gamma - r_{sw}\dot{\varphi}_{sw} = 0$$







Kinematic Constraints: Complete Robot

- Given a robot with *M* wheels
  - each wheel imposes zero or more constraints on the robot motion
  - **only fixed and steerable standard wheels impose constraints**
- Suppose we have a total of $N=N_f+N_s$ standard wheels
  - We can develop the equations for the constraints in matrix forms:
  - Rolling

$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_1\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}_{(N_f+N_s)\times 1} \qquad J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3} \qquad J_2 = diag(r_1 \cdots r_N)$$

$$\begin{bmatrix} \sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta \end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

  - Lateral movement

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0 \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3}$$

$$\begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta \end{bmatrix} R(\theta)\dot{\xi}_I = 0$$

Mobile Robot Maneuverability

- The maneuverability of a mobile robot is the combination
  - of the mobility available based on the sliding constraints
  - plus additional freedom contributed by the steering

- Three wheels is sufficient for static stability
  - additional wheels need to be synchronized
  - this is also the case for some arrangements with three wheels

- It can be derived using the equation seen before
  - Degree of mobility δ_m
  - Degree of steerability δ_s
  - Robots maneuverability $\delta_M = \delta_m + \delta_s$

Mobile Robot Maneuverability: Degree of Mobility

- To avoid any lateral slip the motion vector $R(\theta)\dot{\xi}_I$ has to satisfy the following constraints:

$$C_{1f} R(\theta)\dot{\xi}_I = 0$$

$$C_{1s}(\beta_s) R(\theta)\dot{\xi}_I = 0$$

$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

- Mathematically:
  - $R(\theta)\dot{\xi}_I$ must belong to the *null space* of the projection matrix $C_1(\beta_s)$
  - *Null space* of $C_1(\beta_s)$ is the space N such that for any vector n in N

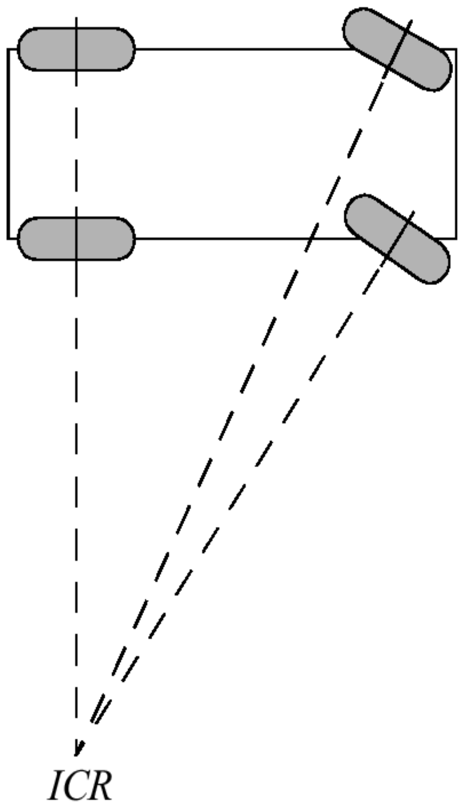
$$C_1(\beta_s) \cdot n = 0$$

  - Geometrically this can be shown by the *Instantaneous Center of Rotation (ICR)*

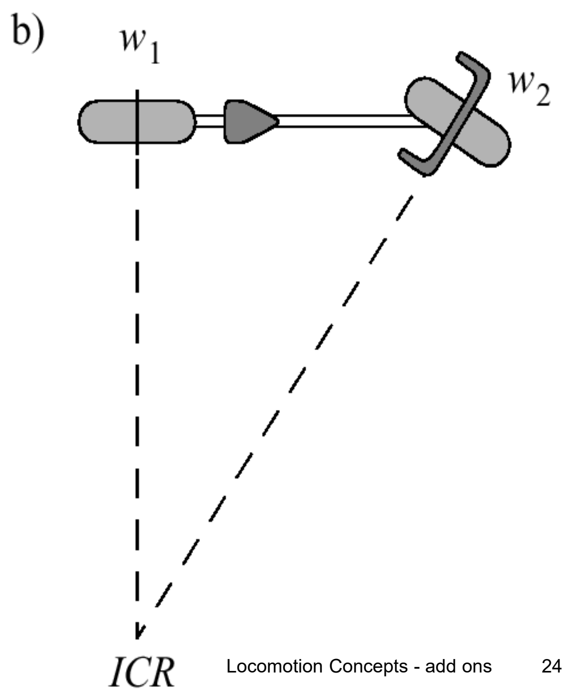
Mobile Robot Maneuverability: ICR

- Instantaneous center of rotation (ICR)

Ackermann Steering



Bicycle



Mobile Robot Maneuverability: More on Degree of Mobility

- Robot chassis kinematics is a function of the set of independent constraints

$$rank\left[C_1(\beta_s)\right] \qquad C_1(\beta_s)=\begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix} \qquad \begin{aligned} C_{1f}R(\theta)\dot{\xi}_I &= 0 \\ C_{1s}(\beta_s)R(\theta)\dot{\xi}_I &= 0 \end{aligned}$$

  - the greater the rank of $C_1(\beta_s)$ the more constrained is the mobility

$$\delta_m = \dim N\left[C_1(\beta_s)\right] = 3 - rank\left[C_1(\beta_s)\right] \qquad 0 \le rank\left[C_1(\beta_s)\right] \le 3$$

- Mathematically
    - no standard wheels $rank\left[C_1(\beta_s)\right]=0$
    - all direction constrained $rank\left[C_1(\beta_s)\right]=3$
- Examples:
  - Unicycle: One single fixed standard wheel
  - Differential drive: Two fixed standard wheels
    - wheels on same axle
    - wheels on different axle

Mobile Robot Maneuverability: Degree of Steerability

- Indirect degree of motion

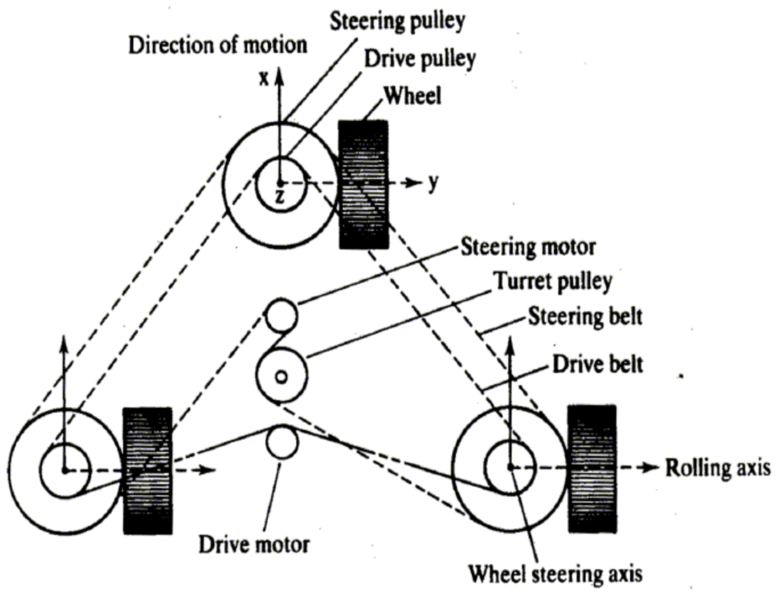
$$\delta_s = rank\left[C_{1s}(\beta_s)\right]$$

  - The particular orientation at any instant imposes a kinematic constraint
  - However, the ability to change that orientation can lead additional degree of maneuverability
- Range of δ_s : $0 \le \delta_s \le 2$

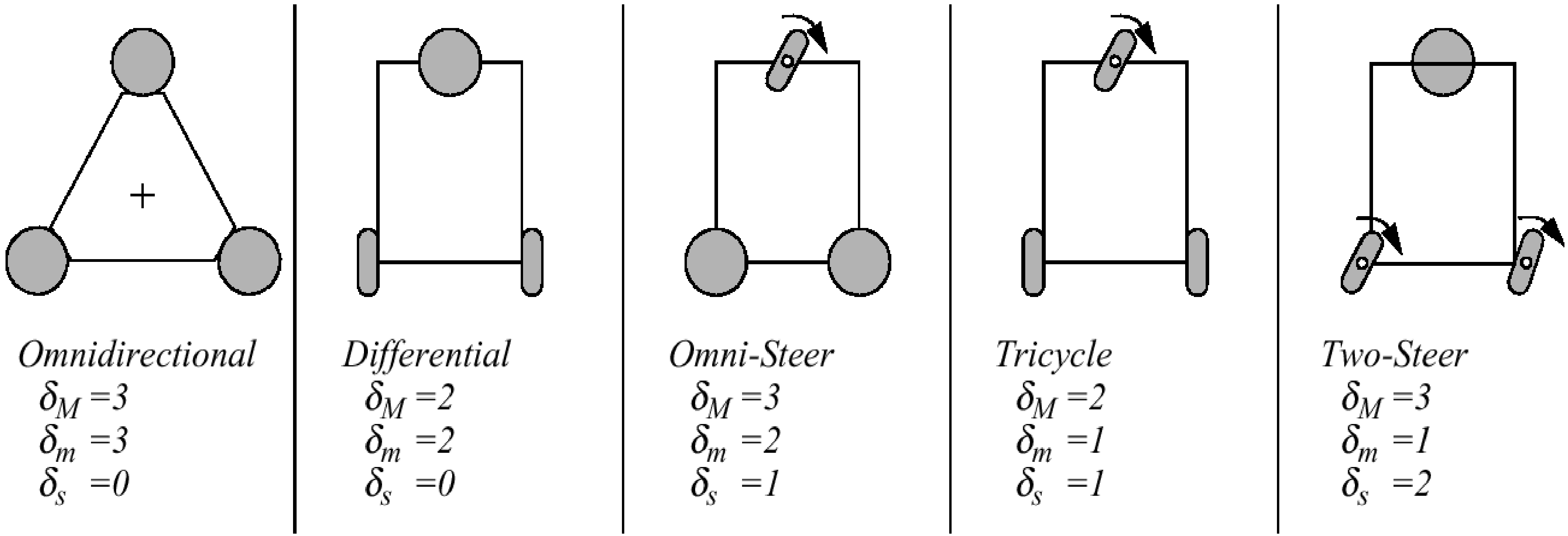
- Examples:
  - one steered wheel: Tricycle
  - two steered wheels: No fixed standard wheel
  - car (Ackermann steering): $N_f = 2$, $N_s=2$ -> common axle

Mobile Robot Maneuverability: Robot Maneuverability

- Degree of Maneuverability $\delta_M = \delta_m + \delta_s$
  - Two robots with same δ_M are not necessary equal
  - Example: Differential drive and Tricycle (see MOOC video segment or book)
  - For any robot with $\delta_M = 2$ the ICR is always constrained to *lie on a line*
  - For any robot with $\delta_M = 3$ the ICR is not constrained and can *be set to any point on the plane*
- The Synchro Drive example: $\delta_M = \delta_m + \delta_s = 1 + 1 = 2$



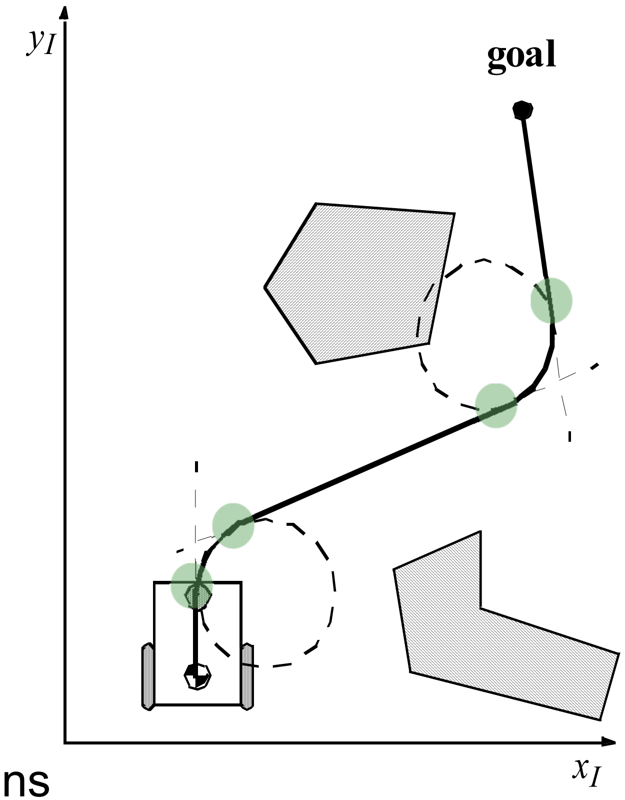
Five Basic Types of Three-Wheel Configurations



Wheeled Mobile Robot Motion Control: Overview

- The objective of a kinematic controller is to follow a trajectory described by its position and/or velocity profiles as function of time.
- Motion control is not straight forward because mobile robots are typically non-holonomic and MIMO systems.
- Most controllers (including the one presented here) are not considering the dynamics of the system

Motion Control: Open Loop Control



- Trajectory (path) divided in motion segments of clearly
- Defined shape:
  - straight lines and segments of a circle
  - Dubins car, and Reeds-Shepp car
- Control problem:
  - pre-compute a smooth trajectory
    based on line, circle (and clothoid) segments
- Disadvantages:
  - It is not at all an easy task to pre-compute a feasible trajectory
  - limitations and constraints of the robots velocities and accelerations
  - does not adapt or correct the trajectory if dynamical changes
    of the environment occur.
  - The resulting trajectories are usually not smooth (in acceleration, jerk, etc.)

Motion Control: Feedback Control

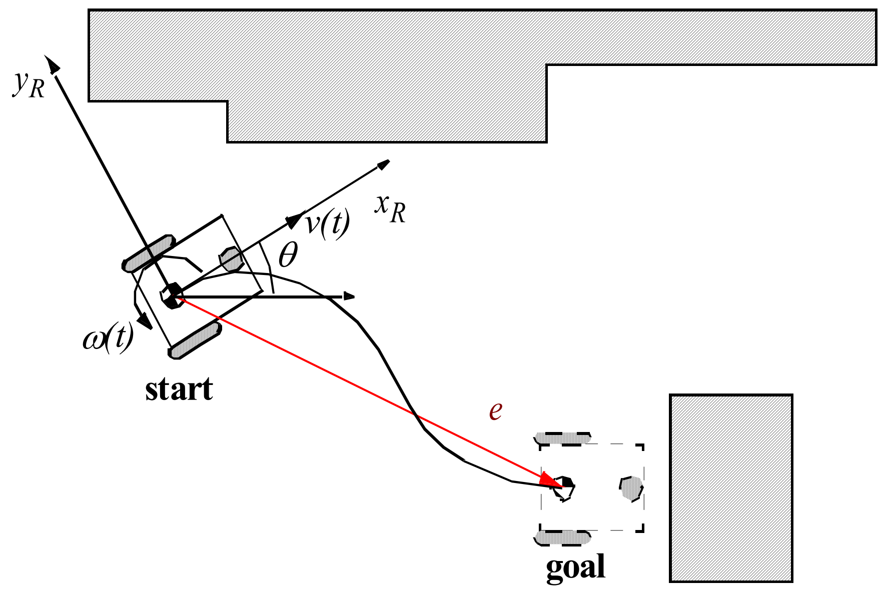
- Find a control matrix K, if exists

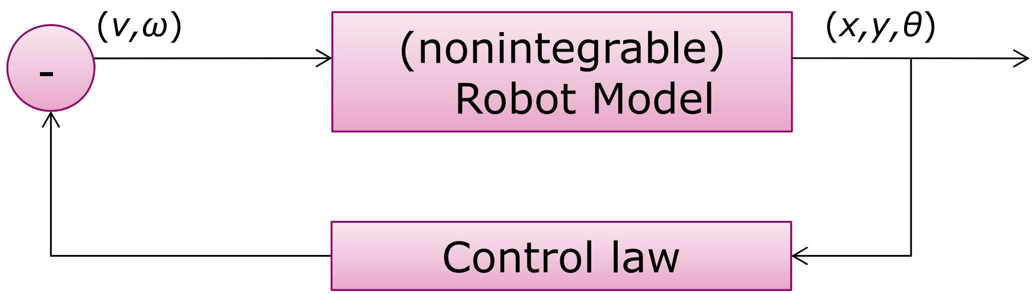
$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$$ with $k_{ij}=k(t,e)$

- such that the control of $v(t)$ and $\omega(t)$

$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = K \cdot e = K \cdot {}^{R}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

- drives the error e to zero $\lim_{t \to \infty} e(t) = 0$
- MIMO state feedback control

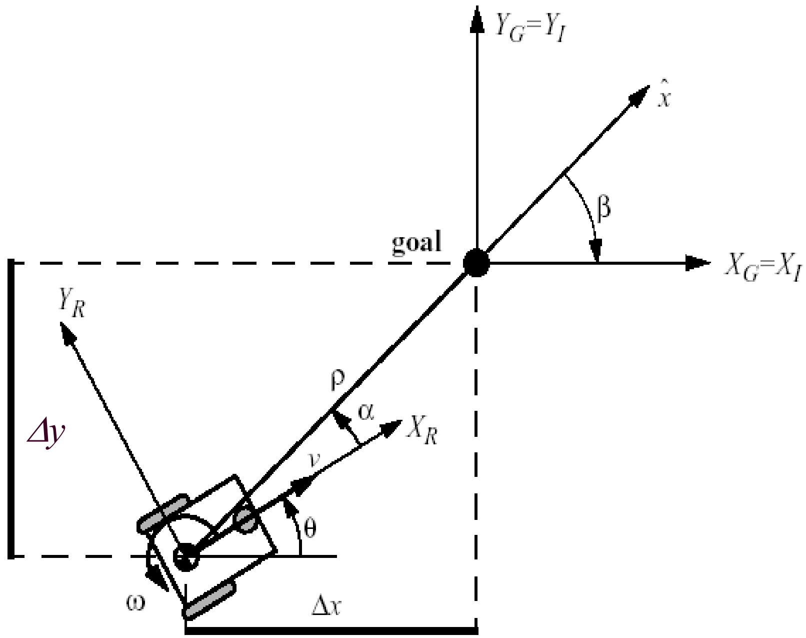




Motion Control: Kinematic Position Control

- The kinematics of a differential drive mobile robot described in the inertial frame $\{ x_I, y_I, \theta \}$ is given by,

$$^{I}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} cos\theta & 0 \\ sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$



- where $\dot{x}$ and $\dot{y}$ are the linear velocities in the direct of the x_I and y_I of the inertial frame.
- Let α denote the angle between the x_R axis of the robots reference frame and the vector connecting the center of the axle of the wheels with the final position.

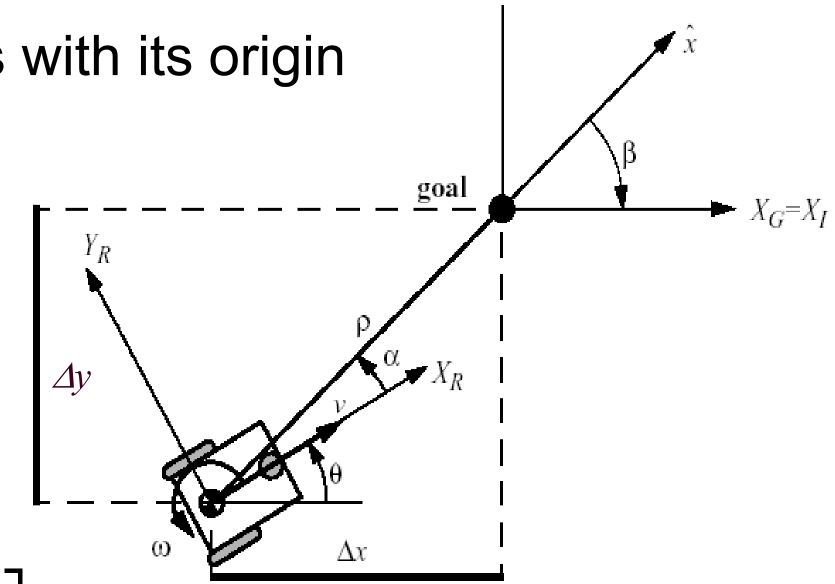
Kinematic Position Control: Coordinates Transformation

- Coordinates transformation into polar coordinates with its origin at goal position:

$$\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x)$$

$$\beta = -\theta - \alpha$$



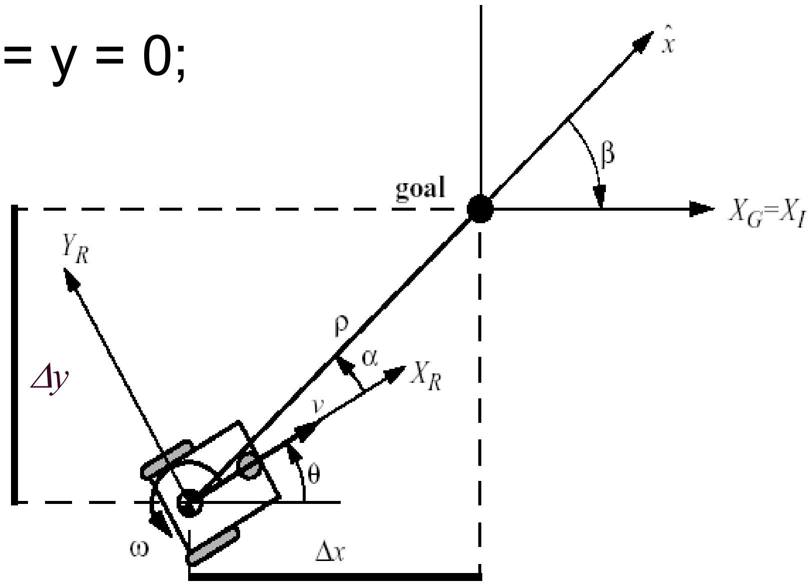
- System description, in the new polar coordinates

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos\alpha & 0 \\ \frac{\sin\alpha}{\rho} & -1 \\ -\frac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \quad \textit{for } I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ -\frac{\sin\alpha}{\rho} & 1 \\ \frac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \quad \textit{for } I_2 = (-\pi, -\pi/2] \cup (\pi/2, \pi]$$

Kinematic Position Control: Remarks

- The coordinates transformation is not defined at x = y = 0;



- For $\alpha \in I_1$ the forward direction of the robot points toward the goal, for $\alpha \in I_2$ it is the backward direction.

$$\alpha \in I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- By properly defining the forward direction of the robot at its initial configuration, it is always possible to have $\alpha \in I_1$ at t=0. However this does not mean that α remains in I_1 for all time t.

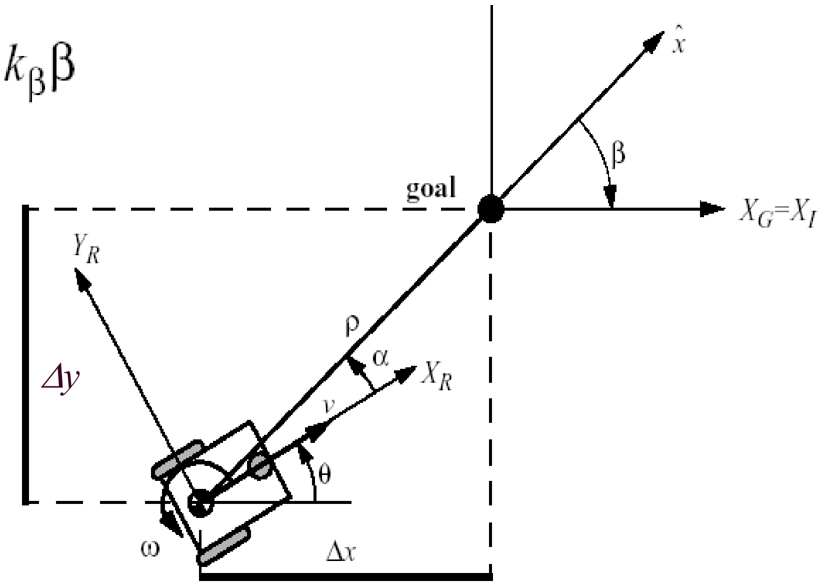
Kinematic Position Control: The Control Law

- It can be shown, that with $v = k_\rho \rho \qquad \omega = k_\alpha \alpha + k_\beta \beta$

  the feedback controlled system

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix}$$

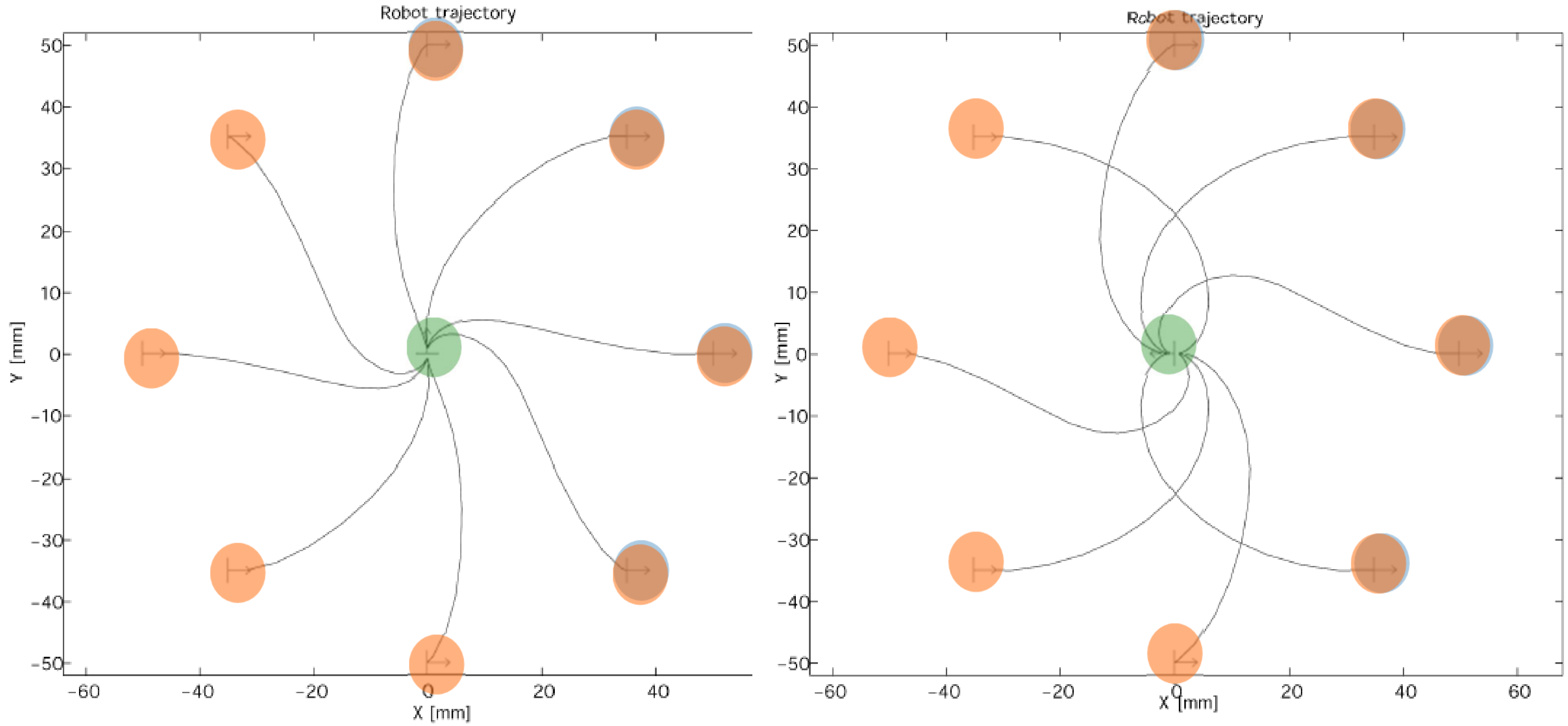
  will drive the robot to $(\rho, \alpha, \beta) = (0, 0, 0)$



- The control signal v has always constant sign,
  - the direction of movement is kept positive or negative during movement
  - parking maneuver is performed always in the most natural way and without ever inverting its motion.

Kinematic Position Control: Resulting Path

- The goal is in the center and the initial position on the circle.



$$k = (k_\rho, k_\alpha, k_\beta) = (3, 8, -1.5)$$