Stokes flow at low Reynolds (Re) number

- Show that the Stokes flow is a simplification of the Navier-Stokes equation at low Re. (non-dimensionalized equations can be used)

- Sketch the Stokes flow profile around a sphere. What happens if a star-like structure is used instead?

- Why do we have to consider Stokes flow when working with micro robots?

- How can fluid mechanics of micro robots be simulated at a macroscopic scale?
Show that the Stokes Flow is a simplification of the Navier-Stokes equation at low Re. (non-dimensionalized equations can be used)

Reynolds number $<<1$

\[
\frac{\rho v_s L}{\mu} \left( \frac{\partial \vec{\nu}}{\partial t} + (\vec{\nu} \cdot \nabla)\vec{\nu} \right) = -\nabla \tilde{p} + \nabla^2 \tilde{\nu}
\]

\[
\Rightarrow 0 = -\nabla \tilde{p} + \nabla^2 \tilde{\nu}
\]

(Non-dimensional Stokes flow)

Left side of the equation is very small compared to the rest. This term is negligible at low Re

Filling the dimensional variables back in using

\[
\tilde{p} = \frac{pL}{\mu v_s}, \quad \tilde{\nu} = \frac{\nu}{v_s}, \quad \nabla = L \cdot \nabla
\]

\[
\Rightarrow \nabla p = \mu \nabla^2 \nu \quad \text{(Stokes flow)}
\]
Problem 1

- Sketch the Stokes flow profile around a sphere.

- What happens if a star-like structure is used instead?
  The flow will still be non-turbulent, since the Stokes flow (which occur at low Re) is always laminar, independent on the shape of the object. The liquid will “creep” past the star-like structure.
Why do we have to consider Stokes flow when working with micro robots?

Micro robots have a small characteristic length \( L \) and a small characteristic velocity \( v_s \), this leads to a small Reynolds number and Stokes flow.

\[
Re = \frac{\rho v_s L}{\mu} \ll 1
\]

How can fluid mechanics of micro robots be simulated at a macroscopic scale?

A small Reynolds number at macroscopic scale can be achieved by using a fluid with a high viscosity e.g. honey, corn syrup, tar.

\[
Re = \frac{\rho v_s L}{\mu} \ll 1
\]
Problem 2

Which of the following motions lead to a net displacement in the low Re number regime?

a) ![Diagram a]

b) ![Diagram b]
Why?

- At low Re, the Navier-Stokes Equation becomes time independent

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} \]

- “The pattern of motion is the same, whether slow or fast, whether forward or backward in time”

  Life at low Reynolds number, E.M. Purcell, American Journal of Physics, Vol. 45, No 1, 1977

- A shape change generates a motion. When the shape is changed back to its original configuration with the exact reversed sequence (no matter how fast or slow), the body is moved back to its initial position. This is called reciprocal motion.

- A micro-swimmer must generate non-reciprocal motion in order to produce a net displacement
Consider a spherical micro object of 0.1 mm in diameter in a cubic container (10cm x 10cm x 10cm) filled with a Newtonian fluid.

### 3.1 Calculate the velocity of the micro object (m.o.).

- **Given are:** \( D_{\text{m.o.}} = 10^{-4} \text{ m}, \rho_{\text{fluid}} = 1000 \text{ kg/m}^3, \rho_{\text{m.o.}} = 1100 \text{ kg/m}^3 \) and \( \mu_{\text{fluid}} = 5 \text{ Pa} \cdot \text{s} \)

- \( F_{\text{drag}} \) depends on the velocity and is zero at the beginning. The difference between \( F_G \) (gravitation) and \( F_B \) (bouyancy) accelerates the micro robot. With increasing velocity, \( F_{\text{drag}} \) increases until all the forces cancel each other out and a steady velocity is reached.

\[
\vec{F}_G + \vec{F}_B + \vec{F}_{\text{drag}} = 0
\]

**In scalars**

\[
F_G - F_B - F_{\text{drag}} = 0
\]

\[
\Rightarrow F_G - F_B = F_{\text{drag}}
\]
Problem 3.1

\[ \vec{F}_G - \vec{F}_B = \vec{F}_{drag} \]

\[ \rho_{m.o.} V_{m.o.} g - \rho_{fluid} V_{m.o.} g = 6\pi \mu R v \]

- with \( V = \frac{4}{3} \pi R^3 \) and \( R = \frac{D}{2} \), the velocity is given by

\[ v = \frac{\rho_{m.o.} \cdot \frac{4}{3} \pi R^3 \cdot g - \rho_{fluid} \cdot \frac{4}{3} \pi R^3 \cdot g}{6\pi \mu R} \approx 0.11 \mu m/s \]
3.2 Give an approximation of the ratio of inertial forces to viscous forces.

\[
\frac{\text{Inertial forces}}{\text{Viscous forces}} \triangleq Re = \frac{\rho_{\text{fluid}} v_s L}{\mu_{\text{fluid}}} \approx 2.2 \cdot 10^{-9}
\]

Characteristic length \( \approx D_{\text{m.o.}} \)
Stokes flow at low Reynolds (Re) number

- **Diffusion increases with increasing viscosity** \( \times \)
- **Diffusion increases with increasing temperature** \( \checkmark \)
- **Brownian motion is larger for small particles** \( \checkmark \)
- **Brownian motion is independent of viscosity** \( \times \)

**Fick’s 1st Law 1D**
(Macroscopic Diffusion)

\[
J = -D \frac{dc}{dx}
\]

**Brownian Motion 1D**
(Mean squared displacement)

\[
\langle r^2 \rangle = 2Dt
\]

- Diffusion and Brownian motion depend on the diffusivity \( D \) given by

\[
D = \frac{kT}{6\pi R \mu}
\]

- Increasing temperature, decreasing viscosity or decreasing the radius leads to a larger diffusivity
A particle in a temperature gradient moves from cold to warm

A particle in a temperature gradient experiences a force towards the lower temperature